CS 565

Programming Languages (graduate)

Spring 2024

Week 10

Axiomatic Semantics and Hoare Logic

Semantics

Before Break (Programming Language Foundations):

- Operational Semantics
  - Simple abstract machine shows *how* to evaluate expression

Can Prove:

- Determinism of Evaluation
- Soundness of Program Transformations
- Program Equivalence

Metatheoretic Properties

Axiomatic Semantics

Axiomatic Semantics

- Meaning given by proof rules
- Useful for reasoning about properties of *specific* programs

- <u>Step 1</u>: Define a language of claims
- <u>Step 2</u>: Define a set of rules (axioms) to build proofs of claims
- <u>Step 3</u>: Verify specific programs

Assertions

- Not unusual to see pre- and post-conditions in code comments:

```
/*Precondition: 0 <= i <= A.length
  Postcondition: returns A[i]*/

public int get(int i) {
  return A[i]
}
```

- Step 1A: Define a language of assertions to capture these sorts of claims

Assertions

- Step 1A: Define a language of *assertions* to capture these claims about states
- Examples:
  - ★ The value of the variable X is greater than 4
  - ★ The variable Y holds an even number
  - ★ The value of X is half of the value of Z
- Formalize claims in some logic with variables
  - ★ Coq (Software Foundations)
  - ★ smt-lib (many automated verifiers)
  - ★ First-order logic: $\forall$, $\exists$, $\wedge$, $\rightarrow$, $X = Y$

Hoare Triple

- Step 1B: Define a judgement for claims about programs involving assertions
- Partial Correctness Triple:

$$\{P\}\ c\ \{Q\}$$

Hoare Triple

An Axiomatic Basis for Computer Programming

C. A. R. Hoare
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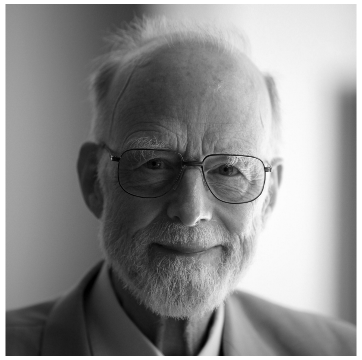
In this paper an attempt is made to explore the logical foundations of computer programming by use of techniques which were first applied in the study of geometry and have later been extended to other branches of mathematics. This involves the elucidation of sets of axioms and rules of inference which can be used in proofs of the properties of computer programs. Examples are given of such axioms and rules, and a formal proof of a simple theorem is displayed. Finally, it is argued that important advantages, both theoretical and practical, may follow from a pursuance of these topics.

KEY WORDS AND PHRASES: axiomatic method, theory of programming' proofs of programs, formal language definition, programming language design, machine-independent programming, program documentation
CR CATEGORY: 4.0, 4.21, 4.22, 5.20, 5.21, 5.23, 5.24



of axioms it is possible to deduce such simple theorems as:

$$x = x + y \times 0$$

$$y \leqslant r \supset r + y \times q = (r - y) + y \times (1 + q)$$

The proof of the second of these is:

$$\begin{array}{lll} \text{A5} & (r - y) + y \times (1 + q) & \\ & & = (r - y) + (y \times 1 + y \times q) \\ \text{A9} & & = (r - y) + (y + y \times q) \\ \text{A3} & & = ((r - y) + y) + y \times q \\ \text{A6} & & = r + y \times q \quad \text{provided } y \leqslant r \end{array}$$

The axioms A1 to A9 are, of course, true of the traditional infinite set of integers in mathematics. However, they are also true of the finite sets of "integers" which are manipulated by computers provided that they are confined to *nonnegative* numbers. Their truth is independent of the size of the set; furthermore, it is largely independent of the choice of technique applied in the event of "overflow"; for example:

(1) Strict interpretation: the result of an overflowing operation does not exist; when overflow occurs, the offending program never completes its operation. Note that in this case, the equalities of A1 to A9 are strict, in the sense that both sides exist or fail to exist together.

(2) Firm boundary: the result of an overflowing operation is taken as the maximum value represented.

C. A. R. Hoare. 1969. An axiomatic basis for computer programming. *Commun. ACM* 12, 10 (Oct. 1969), 576–580.

Hoare Triple

- Step 1B: Define a judgement for claims about programs involving assertions
- Partial Correctness Triple:

$$\{P\}\ c\ \{Q\}$$

- Total Correctness Triple:

$$[P]\ c\ [Q]$$

- A triple that makes a true claim is said to be *valid*

Hoare Triples

What *should* these mean:

```
{True} c {X = 5}
∀m. {X = m} c {X = m + 5}
[X <= Y] c [Y <= X]
```

Concept Check

Which of these should be valid?

```
{X = 2} X := X + 1 {X = 3}
{X = 2} X := 5; Y := 3 {X = 5}
{False} skip {True}
[Y = 5] X := Y + 3 [X = 5]
{True} while true do SKIP end {False}
[True] while true do SKIP end [False]
[True] while true do SKIP end [True]
```

Axiomatic Semantics

- Step 1: Define a language of claims
- Step 2: Define a set of rules (axioms) to build proofs of claims
- Step 3: Verify specific programs

Imp Assertions

One assertion language for Imp commands is:

$$X \in \mathtt{Id} \qquad\qquad N \in \mathbb{N}$$

$$A ::= N \mid A + A \mid A - A \mid A * A \mid X$$

$$P, Q ::= T \mid \bot \mid A < A \mid A = A \mid P \wedge Q \mid P \vee Q \mid \neg P$$

Examples Assertions:

The value of the variable X is greater than 4
The variable Y holds an even number
The value of X is half of the value of Z

Satisfiability

- We define a semantics for this language to identify when a state σ *satisfies* an assertion P:

$$\sigma \models P$$

$$\frac{}{\sigma \models T}$$

$$\frac{\sigma, a_1 \Downarrow v_1 \qquad \sigma, a_2 \Downarrow v_2 \qquad v_1 <_{\mathbb{N}} v_2}{\sigma \models a_1 < a_2}$$

$$\frac{\sigma, a_1 \Downarrow v_1 \qquad \sigma, a_2 \Downarrow v_2 \qquad v_1 =_{\mathbb{N}} v_2}{\sigma \models a_1 = a_2}$$

Satisfability

We define a semantics for this language to identify when a state σ *satisfies* an assertion P:

$$\sigma \models P$$

$$\frac{\sigma \models P \qquad \sigma \models Q}{\sigma \models P \wedge Q}$$

$$\frac{\sigma \models P}{\sigma \models P \vee Q} \qquad \frac{\sigma \models Q}{\sigma \models P \vee Q} \qquad \frac{\sigma \not\models P}{\sigma \models \neg P}$$

Validity

We can now precisely define when a partial Hoare Triple is valid:

If we start in a state satisfying P

$$\{P\}\ c\ \{Q\} =$$

$$\forall \sigma.\ \sigma \models P \rightarrow \forall \sigma'.\ \sigma, c \Downarrow \sigma' \rightarrow \sigma' \models Q$$

And c terminates in a st

then that final state satisfies Q

$$\sigma \models P$$

VALIDITY

Proving Validity

- That gives us the first part of axiomatic semantics
  - ★ Step 1: Define a language of claims
- How to prove that {P} c {Q} is valid?
  - ★ Could reason directly about the semantics of c
  - ★ Step 2: Define a set of rules (axioms) to build proofs of claims without reasoning directly about states and executions

$$\boxed{\vdash \{P\}\ c\ \{Q\}}$$

Proof Rules

How to prove that {P} c {Q} is valid?

- Could reason directly about the semantics of c
- Step 2: Define a set of rules (axioms) to build proofs of claims without reasoning directly about states and executions

$$\boxed{\vdash \{P\}\ c\ \{Q\}}$$

Hoare Skip

Use our intuition about what we want to be able to prove to guide definition of rules

$$\{P\}\, c\, \{Q\} \equiv$$
$$\forall \sigma.\ \sigma \models P \rightarrow \ \forall \sigma'.\ \sigma, c \Downarrow \sigma' \rightarrow \sigma' \models Q$$

Hoare Skip?

$\{?\}$ skip $\{Q\} \equiv$

$\forall \sigma.\ \sigma \models ? \rightarrow \ \forall \sigma'.\ \sigma, \text{skip} \Downarrow \sigma' \rightarrow \sigma' \models Q$

$$\frac{}{\vdash \{?\}\ \text{skip}\ \{Q\}}$$

Hoare Skip!

$\{Q\}\ \mathtt{skip}\ \{Q\} \equiv$

$\forall \sigma.\ \sigma \models Q \rightarrow \forall \sigma'.\ \sigma, \mathtt{skip} \Downarrow \sigma' \rightarrow \sigma' \models Q$

$$\frac{}{\vdash \{Q\}\ \mathtt{skip}\ \{Q\}}\ \textsc{HLSkip}$$

Hoare Assign?

$$\{\ ??\ \}\ X := a\ \{Q\} \equiv$$

$$\forall \sigma.\ \sigma \models\ ??\ \rightarrow$$

$$\forall \sigma'.\ \sigma, X := a \Downarrow \sigma' \rightarrow \sigma' \models Q$$

$$\frac{}{\vdash \{\ ??\ \}\ X := a\ \{Q\}}$$

Hoare Assign!

$$\{[X{:=}a]Q\}\ X{:=}a\ \{Q\} \equiv$$

$$\forall\sigma.\ \sigma \models [X{:=}a]Q \rightarrow$$

$$\quad \forall\sigma'.\ \sigma,\ X{:=}a \Downarrow \sigma' \rightarrow \sigma' \models Q$$

$$\frac{}{\vdash \{[X{:=}a]Q\}\ X{:=}a\ \{Q\}}\ \textsc{HLAssign}$$

Hoare Assign^{bad}

- Why not this “forward” rule?

$$\frac{}{\vdash \{P\}\ X := a\ \{[X := a]P\}}$$

Hoare Assign!

$$\{[X \coloneqq a]Q\}\ X \doteq a\ \{Q\} \equiv$$
$$\forall \sigma.\ \sigma \models [X \coloneqq a]Q \rightarrow$$
$$\quad \forall \sigma'.\ \sigma, X \doteq a \Downarrow \sigma' \rightarrow \sigma' \models Q$$

$$\frac{}{\vdash \{[X \coloneqq a]Q\}\ X \doteq a\ \{Q\}}\ \text{HLASSIGN}$$

Hoare Seq?

$$\{\ ?\ \}\ c_1; c_2\ \{Q\} \equiv$$

$$\forall \sigma.\ \sigma \models\ ?\ \rightarrow$$

$$\forall \sigma'.\ \sigma, c_1; c_2 \Downarrow \sigma' \rightarrow \sigma' \models Q$$

$$\frac{}{\vdash \{\ ?\ \}\ c_1; c_2\ \{Q\}}$$

Hoare Seq?

$\{\ ?\ \}\ c_1; c_2\ \{Q\} \equiv$

$\forall \sigma_1.\ \sigma_1 \models ? \rightarrow \forall \sigma_3.$

$(\exists \sigma_2.\ \sigma, c_1 \Downarrow \sigma_2 \wedge \sigma, c_2 \Downarrow \sigma_3) \rightarrow$

$\sigma_3 \models Q$

$$\frac{}{\vdash \{\ ?\ \}\ c_1; c_2\ \{Q\}}$$

Hoare Seq?

$\{\ ?_1\ \}\ c_1 ; c_2\ \{Q\} \equiv$

$\forall \sigma_1.\ \sigma_1 \models\ ?_1 \rightarrow\ \forall \sigma_3.$

$(\exists \sigma_2.\ \sigma, c_1 \Downarrow \sigma_2 \wedge \sigma, c_2 \Downarrow \sigma_3) \rightarrow$

$\sigma_3 \models Q$

$$\frac{\vdash \{?_1\} c_1 \{?_2\} \quad \vdash \{?_2\} c_2 \{Q\}}{\vdash \{\ ?_1\ \}\ c_1 ; c_2\ \{Q\}}$$

Hoare Seq!

$\{P\}\, c_1 ; c_2 \{Q\} \equiv$

$\forall \sigma.\ \sigma \models P \rightarrow$

$\quad \forall \sigma'.\ \sigma, c_1 ; c_2 \Downarrow \sigma' \rightarrow \sigma' \models Q$

$$\frac{\vdash \{P\} c_1 \{R\} \quad \vdash \{R\} c_2 \{Q\}}{\vdash \{P\}\ c_1 ; c_2\ \{Q\}} \text{HLSEQ}$$

Hoare Seq!

$$\frac{\vdash \{P\}\, c_1 \,\{R\} \quad \vdash \{R\}\, c_2 \,\{Q\}}{\vdash \{P\}\; c_1 ; c_2 \;\{Q\}}\ \textsc{HLSeq}$$

Hoare If!

$$\frac{\vdash \{P \land b\}\; c_1\; \{Q\} \quad \vdash \{P \land \neg b\}\; c_2\; \{Q\}}{\vdash \{P\}\ \text{if } b \text{ then } c_1 \text{ else } c_2 \text{ end } \{Q\}} \ \text{HLIF}$$

Proof Rules

- What if Assertions don't align?

```
{X=2} X ≔ X + 1 {X = 3}
```

- Have rule for strengthening postconditions and weakening preconditions

$$\frac{\vdash \{P_W\}\ c\ \{Q_S\} \qquad P \rightarrow P_W \qquad Q_S \rightarrow Q}{\vdash \{P\}\ c\ \{Q\}}\ \textsc{HLConseq}$$

$$\frac{\dfrac{}{\vdash \{X+1=3\}\ X \coloneqq X + 1\ \{X = 3\}}\ \textsc{HLAssign} \qquad \dfrac{}{X=2 \rightarrow X+1=3} \qquad \dfrac{}{X=3 \rightarrow X=3}}{\vdash \{X=2\}\ X \coloneqq X + 1\ \{X = 3\}}\ \textsc{HLConseq}$$

Rule Review

$$\frac{}{\vdash \{Q[X{:=}a]\}\, X{:=}a\, \{Q\}}\ \textsc{HLAssign}$$

$$\frac{}{\vdash \{Q\}\ \text{skip}\ \{Q\}}\ \textsc{HLSkip}$$

$$\frac{\vdash \{P\}\ c_1\ \{R\} \qquad \vdash \{R\}\ c_2\ \{Q\}}{\vdash \{P\}\ c_1;c_2\ \{Q\}}\ \textsc{HLSeq}$$

$$\frac{\vdash \{P \wedge b\}\ c_1\ \{Q\} \qquad \vdash \{P \wedge \neg b\}\ c_2\ \{Q\}}{\vdash \{P\}\ \textbf{if}\ b\ \textbf{then}\ c_1\ \textbf{else}\ c_2\ \{Q\}}\ \textsc{HLIf}$$

$$\frac{\vdash \{P_W\}\ c\ \{Q_S\} \quad P \rightarrow P_W \quad Q_S \rightarrow Q}{\vdash \{P\}\ c\ \{Q\}}\ \textsc{HLConseq}$$

Hoare While?

$\vdash \{X < 3\}$ **while** $(X < 3)$ **do** $X := X + 1$ **end** $\{X = 3\}$

$$\frac{\vdash \{?\}\ c\ \{?\}}{\vdash \{?\}\ \text{while}\ b\ \text{do}\ c\ \text{end}\ \{Q\}}$$

Hoare While?

$$\dfrac{\dfrac{\vdash \{X < 4\}\ X := X + 1\ \{X < 4\}}{\vdash \{X < 4\}\ \textbf{while}\ (X < 3)\ \textbf{do}\ \ X := X + 1\ \textbf{end}\ \{X < 4\}}}{\vdash \{X < 3\}\ \textbf{while}\ (X < 3)\ \textbf{do}\ \ X := X + 1\ \textbf{end}\ \{X = 3\}}$$

$$\dfrac{\vdash \{Q \quad\quad\}\ c\ \{Q\}}{\vdash \{Q\}\ \text{while}\ b\ \text{do}\ c\ \text{end}\ \{Q \quad\quad\}}$$

Hoare While?

$$\frac{\dfrac{\vdash \{X < 4 \land X < 3\}\ X := X + 1\ \{X < 4\}}{\vdash \{X < 4\}\ \textbf{while}\ (X < 3)\ \textbf{do}\ \ X := X + 1\ \textbf{end}\ \{X < 4\}}}{\vdash \{X < 3\}\ \textbf{while}\ (X < 3)\ \textbf{do}\ \ X := X + 1\ \textbf{end}\ \{X = 3\}}$$

$$\frac{\vdash \{Q \land b\}\ c\ \{Q\}}{\vdash \{Q\}\ \text{while}\ b\ \text{do}\ c\ \text{end}\ \{Q \quad\ \}}$$

Hoare While?

$$\dfrac{\dfrac{\vdash \{X < 4 \land X < 3\}\ X := X + 1\ \{X < 4\}}{\vdash \{X < 4\}\ \textbf{while}\ (X < 3)\ \textbf{do}\ \ X := X + 1\ \textbf{end}\ \{X < 4 \land \neg X < 3\}}}{\vdash \{X < 3\}\ \textbf{while}\ (X < 3)\ \textbf{do}\ \ X := X + 1\ \textbf{end}\ \{X = 3\}}$$

$$\frac{\vdash \{Q \land b\}\ c\ \{Q\}}{\vdash \{Q\}\ \text{while}\ b\ \text{do}\ c\ \text{end}\ \{Q \land \neg b\}}$$

Hoare While!

I is a *loop invariant*:

- Holds before loop
- Holds after each loop iteration
- Holds when the loop exits

$$\frac{\vdash \{I \wedge b\}\ c\ \{I\}}{\vdash \{I\}\ \text{while } b \text{ do } c \text{ end } \{I \wedge \neg b\}}\ \textsc{HLWhile}$$

Rule Review

$$\frac{}{\vdash \{Q[X:=a]\}\, X:=a\, \{Q\}}\ \textsc{HLAssign}$$

$$\frac{}{\vdash \{Q\}\ \mathrm{skip}\ \{Q\}}\ \textsc{HLSkip}$$

$$\frac{\vdash \{P\}\ c_1\ \{R\} \qquad \vdash \{R\}\ c_2\ \{Q\}}{\vdash \{P\}\ c_1;c_2\ \{Q\}}\ \textsc{HLSeq}$$

$$\frac{\vdash \{P \wedge b\}\ c_1\ \{Q\} \qquad \vdash \{P \wedge \neg b\}\ c_2\ \{Q\}}{\vdash \{P\}\ \mathbf{if}\ b\ \mathbf{then}\ c_1\ \mathbf{else}\ c_2\ \{Q\}}\ \textsc{HLIf}$$

$$\frac{\vdash \{I \wedge b\}\ c\ \{I\}}{\vdash \{I\}\ \mathbf{while}\ b\ \mathbf{do}\ c\ \mathbf{end}\ \{I \wedge \neg b\}}\ \textsc{HLWhile}$$

Hoare in Action

- Want to build proof trees:

$\vdots$

$\vdash \{(z - 1) - (x - 1) = p - m \wedge x = 0 \}\ z := z - 1;\ x := x - 1 \{(z - 1) - (x - 1) = p - m\}$

$\vdash \{(z - 1) - (x - 1) = p - m \}$ **while** x … $z = p - m \wedge (x = 0)\}$

$\vdash \{p=p\}\ z := p\ \{z = p\}$ … $\{z = p - m \wedge (x = 0)\}$

… $\{z = p - m \wedge (x = 0)\}$

$\vdash \{ m = m \}\ x := m\ \{x = m\}$ … $\{z = p - m \wedge (x = 0)\}$

$\vdash \{ m = m \}\ x := m;\ z := p$ … $p - m \wedge (x = 0)\}$

$\vdash \{ True \}\ x := m;\ z := p;$ **while** $x \neq 0$ **do** $z := z - 1;\ x := x - 1$ **end** $\{z = p - m \}$

Proof is *compositional*: it follows structure of program!

Decorated Programs

Idea: include assertions in program

$\{ \text{True} \} \rightarrow \{ m = m \}$
X := m;
$\{ X = m \} \rightarrow \{ X = m \wedge p = p \}$
Z := p;
$\{ X = m \wedge Z = p \} \rightarrow \{ Z - X = p - m \}$
while $X \neq 0$ **do**
$\{ Z - X = p - m \wedge X \neq 0 \} \rightarrow \{ (Z - 1) - (X - 1) = p - m \}$
Z := Z - 1;
$\{ Z - (X - 1) = p - m \}$
X := X - 1
$\{ Z - X = p - m \}$
end;
$\{ Z - X = p - m \wedge \neg (X \neq 0) \} \rightarrow \{ Z = p - m \}$

Decorated Programs

- Idea: include assertions in program
- If each individual command is correct, so is the program

$$
\begin{array}{l}
\{\ X = m \wedge Y = n\ \} \\
\quad X := X + Y \\
\{\ ??\ \} \\
\quad Y := X - Y \\
\{\ ??\ \} \\
\quad X := X - Y \\
\{\ X = n \wedge Y = m\ \}
\end{array}
$$

Decorated Programs

- Idea: include assertions in program
- If each individual command is correct, so is the program

$$
\begin{array}{l}
\{\ X = m \wedge Y = n\ \} \\
\quad X := X + Y \\
\{\ ??\ \} \\
\quad Y := X - Y \\
\{\ X - Y = n \wedge Y = m\ \} \\
\quad X := X - Y \\
\{\ X = n \wedge Y = m\ \}
\end{array}
$$

Decorated Programs

- Idea: include assertions in program
- If each individual command is correct, so is the program

$$
\begin{array}{l}
\{ X = m \wedge Y = n \} \\
\quad X := X + Y \\
\{X - (X - Y) = n \wedge X - Y = m\} \\
\quad Y := X - Y \\
\{ X - Y = n \wedge Y = m \} \\
\quad X := X - Y \\
\{ X = n \wedge Y = m \}
\end{array}
$$

Decorated Programs

- Idea: include assertions in program
- If each individual command is correct, so is the program

$$\{ X = m \wedge Y = n\} \to$$
$$\{(X + Y) - ((X + Y) - Y) = n \wedge (X + Y) - Y = m\}$$
$$X := X + Y$$
$$\{X - (X - Y) = n \wedge X - Y = m\}$$
$$Y := X - Y$$
$$\{ X - Y = n \wedge Y = m \}$$
$$X := X - Y$$
$$\{ X = n \wedge Y = m \}$$

Loop Invariants

- Largely straightforward
- **Except** for loops!

```
{ X = m }
   while X ≠ 0 do
    X ::= X - 1
   end
{ X = 0 }
```

Loop Invariants

- Largely straightf
- **Except** for loo

? needs to

1. be weak enough to be implied by the loop's precondition,
2. be strong enough to imply the loop's postcondition
3. be preserved by one iteration of the loop

{ X = m ∧ Y = n } → { **?** }
while X ≠ 0 do
{ **?** ∧ X ≠ 0 } → { [X:=X-1] [Y:=Y-1] **?**}
Y := Y - 1;
{ [X:=X-1] ? }
X := X - 1
{ **?** }
end
{ **?** ∧ X = 0 } → { Y = n - m }

Loop Invariants

- Largely straightfo
- **Except** for loo

? needs to

1. be weak enough to be implied by the loop's precondition,
2. be strong enough to imply the loop's postcondition
3. be preserved by one iteration of the loop

{ X = m ∧ Y = n } → { **Tru**

  while X ≠ 0 do

  { **True** ∧ X ≠ 0 } → { [X≔X-1] [Y≔Y-1] **True** }

    Y := Y - 1;

  { [X≔X-1] **True** }

    X := X - 1

  { **True** }

  end

{ **True** ∧ X = 0 } → { Y = n - m }

Loop Invariants

- Largely straightf
- **Except** for loo

$\{ X = m \wedge Y = n \} \rightarrow \{ \textbf{True}$

while $X \neq 0$ do

$\{ \textbf{True} \wedge X \neq 0 \} \rightarrow \{ [X{:=}X$ ... $\textbf{True} \}$

Y := Y - 1;

$\{ [X{:=}X-1]\ \textbf{True} \}$

X := X - 1

$\{ \textbf{True} \}$

end

$\{ \textbf{True} \wedge X = 0 \} \rightarrow \{ Y = n - m \}$

? needs to

1. be weak enough to be implied by the loop's precondition,
2. be strong enough to imply the loop's postcondition
3. be preserved by one iteration of the loop

What fails to hold when ? is **True?**

Loop Invariants

- ★ Largely straightforwar
- ★ **Except** for loops!

{ X = m ∧ Y = n } → {**Y-X = n**

while X ≠ 0 do

{**Y-X = n - m** ∧ X ≠ 0}→{[X:= ...][...] **Y-X = n - m**}

Y := Y - 1;

{ **Y - X = n - m** [X:=X-1] }

X := X - 1

{ **Y - X= n - m** }

end

{ **Y - X = n - m** ∧ X = 0 } → { Y = n - m }

? needs to

1. be weak enough to be implied by the loop's precondition,
2. be strong enough to imply the loop's postcondition
3. be preserved by one iteration of the loop

Success!

Recap

- Developed a logic for proving that {P} c {Q} is valid
  - We defined a set of rules (axioms) to build proofs of claims without reasoning directly about states and executions
- Saw how to verify specific programs

$$\boxed{\vdash \{P\}\ c\ \{Q\}}$$