Information at the Interface between Machine Learning and Game Theory

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Algorithmic complexity and universal prediction

- **Kolmogorov complexity** $K$
  Information content of a binary sequence or integer

- **Universal measure** $M$
  Mixture over all continuous enumerable measures $\mu_1, \mu_2, \ldots$
  \[ M = \sum_{i} 2^{-K(i)} \mu_i \]
The predictive power of $M$

- Any continuous measure $\mu$ corresponds to a **sequential probabilistic predictor** for binary sequences $y = (y_1, y_2, \ldots)$

- How quickly does $M(y_t | y_1, \ldots, y_{t-1}) = M(y_t | y_{t-1})$ converge to 1?

\[
\sum_t (1 - M(y_t | y_{t-1}))^2 \leq -\frac{1}{2} \sum_t \ln M(y_t | y_{t-1}) = -\frac{1}{2} \ln M(y) \approx K(y)
\]

- Solomonoff’s classical result: $M$ can predict a random sequence drawn from any computable measure $\mu_i$ almost as well as $\mu_i$ itself
Theory of repeated games

Play a zero-sum game repeatedly against a possibly suboptimal opponent
Experts are given classes of probabilistic predictors for example, probabilistic FSA with a fixed number of states

Devise a strategy that predicts **any sequence** nearly as well as the best expert for that sequence

Efficiency, nonasymptotic bounds, optimal rates
Example of regret bounds

Fix a class $\mathcal{F} = \{\mu_1, \mu_2, \ldots \}$ of probabilistic predictors (experts) together with some prior $\mu_0$ on $\mathcal{F}$.

There exists a probabilistic prediction strategy whose expected number of mistakes $\hat{L}_T$ on any binary sequence $\mathbf{y} = (y_1, \ldots, y_T)$ satisfies

$$\hat{L}_T - L_T(i) \approx \sqrt{T \ln \frac{1}{\mu_0(i)}} \quad i = 1, 2, \ldots$$

where $L_T(i)$ is expected number of mistakes of $\mu_i$ on $\mathbf{y}$.
**Historical background**

- **Sequential adaptive compression**: predicting as the best finite-state automata under logarithmic loss (Lempel and Ziv, 1976)

- **Gambling and portfolio selection**: linear experts in the simplex (Cover, 1965, 1974 and 1991)

- **FSA prediction**: improved analysis for finite-state automata (Feder, Merhav and Gutman, 1992)

- **Linear classification**: experts as functions in Euclidean, Hilbert and Banach spaces (Novikov, 1962; Freund and Schapire, 1999; Vovk, 2007)
Sequential classification of \((x_1, y_1), (x_2, y_2), \cdots \in \mathbb{R}^d \times \{-1, +1\}\)

Linear classifiers, \(\text{sgn}(w^T x_t)\)
On-line linear classification

- Sequential classification of \((x_1, y_1), (x_2, y_2), \ldots \in \mathbb{R}^d \times \{-1, +1\}\)
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- There are simple reductions from online linear classification to prediction with expert advice
- This yields regret bounds for popular algorithms such as Perceptron (Rosenblatt, 1958) and Winnow (Littlestone, 1989)
- Different approaches for classification in normed spaces
Some issues

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- Extend this program to statistical learning/inference
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- Transfer from game theory to machine learning: concepts (e.g., partial feedback) and analysis (e.g., regret bounds).
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- Extend this program to statistical learning/inference
- Transfer from game theory to machine learning: concepts (e.g., partial feedback) and analysis (e.g., regret bounds)
- Regret analysis in classification/regression: what are the limits?