A Bit of network information theory

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Parts of talk are joint work with S. Avestimehr[2], S. Mohajer[1], C. Tian[3], D. Tse[2]

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The holy grail

Information flow over shared networks

- Unicast, multicast, multiple unicast (multicommodity flow).
- Significant progress for graphs (routing, network coding etc).
- Less understood for flows over wireless networks.

Network data compression: Motivation $\longrightarrow$ sensor networks.

- Some successes in side-information coding: Slepian-Wolf, Wyner Ziv etc.
- Many unresolved questions: Distributed source coding, multiple description coding.

Question: How can we make progress to fundamentally characterize flow of information over networks?

Approximate characterizations

Philosophy: Gain insight into central difficulties of problem by identifying underlying deterministic/lossless structures.

Goal: Use the insight of underlying problem to get (provable) approximate characterization for noisy/lossy problem.

- Underlying problem should be characterized exactly to give insight into solution structure for general case.
- **Universal approximation:** Approximation should depend *only* on the problem structure and *not* on parameters (like channel gains, distortions etc.).

Question: Can we identify the appropriate underlying problems and use them to get provable (universal) approximations.

Overall agenda

Central theme: Obtain (universal) approximate characterizations for network flow problems.

Talk outline:

- **Wireless relay networks**
  - Focus on signal interactions $\longrightarrow$ study deterministic networks.
  - Characterize deterministic networks $\longrightarrow$ approximation for noisy case.
- **Multiple description coding**
  - Identify related underlying lossless problem $\longrightarrow$ study multi-level source coding.
  - Use multi-level rate region $\longrightarrow$ approximation for lossy case.

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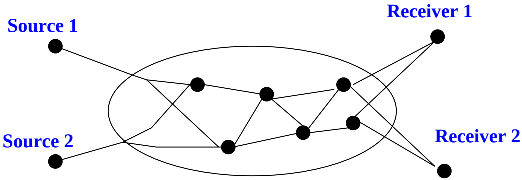
Question: Can we develop cooperative mechanisms to utilize signal interaction?

Signal interaction: Gaussian wireless networks



$$\mathbf{y}_j[t] = \sum_i \mathbf{H}_{ij}\mathbf{x}_i[t] + \mathbf{z}_j[t]$$

- **Broadcast** because transmission $\mathbf{x}_i$ is heard by all receivers.
- **Multiple access** because transmitted signals from all nodes mix linearly at the receiver j.
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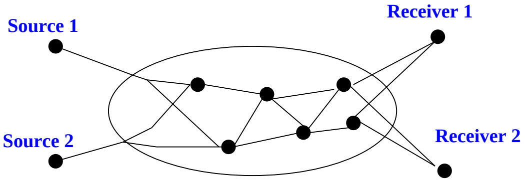

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Question: Can we characterize capacity of such networks?

Gaussian network capacity: state of knowledge

Resolved: Point-to-point channel, multiple access channel, broadcast channel (private messages).

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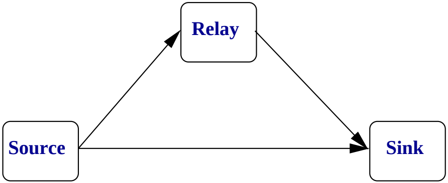

RELAY CHANNEL: Cover, El−Gamal (1979)

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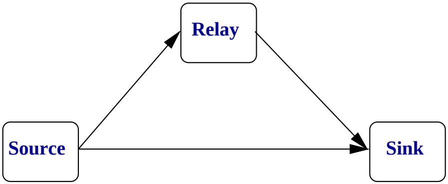

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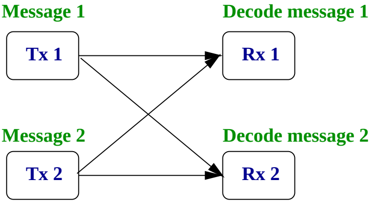

INTERFERENCE CHANNEL: Han−Kobayashi (1981)

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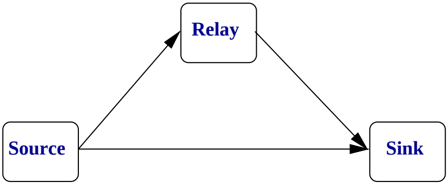

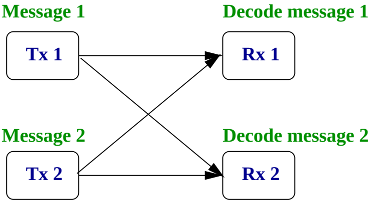

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Question: Thirty years have gone by... How can we make progress from here?

Simplify the model

Focus on signal interaction not noise

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- Many wireless systems are interference rather than noise limited.
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Hope:

- Deterministic models more tractable.
- Use insight to obtain approximate characterizations for noisy (Gaussian) networks.

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Question: Can we develop relevant models and analyze networks with deterministic signal interactions to get the insights?

Agenda: Relay networks

- Introduce deterministic channel model.
- Motivate the utility of deterministic model with examples.
- Develop achievable rates for general deterministic relay networks
- Characterizations for linear finite field deterministic models.

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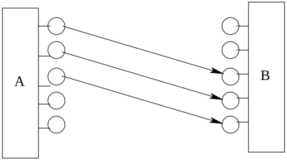
- Motivate the utility of deterministic model with examples.
- Develop achievable rates for general deterministic relay networks
- Characterizations for linear finite field deterministic models.
- **Connection to wireless networks:** Use insights on achievability of deterministic networks to obtain *approximate* characterization of noisy relay networks.

Example 1: Point-to-point link

Deterministic



Gaussian

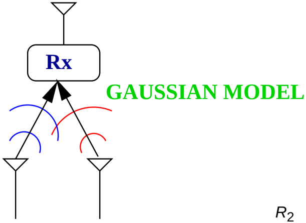
$$y = 2^{\alpha/2}x + z$$

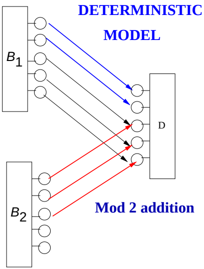
Capacity is $\log(1 + 2^{\alpha}) \approx \alpha \log 2$ assuming unit variance noise.

Receiver observes α most significant bits of transmitted signal.

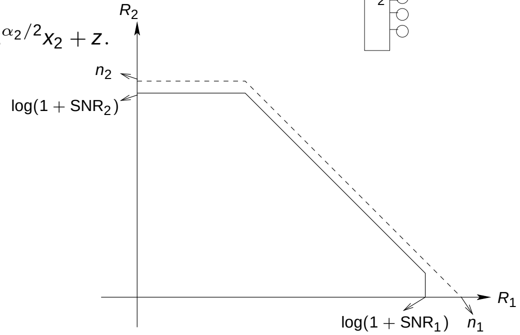
- Number of levels received shows scale of channel strength.
- Scale important when signals interact in broadcast and multiple access.

Example 2: Multiple access channel

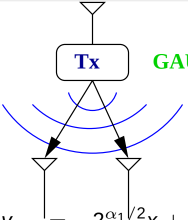




$$y = 2^{\alpha_1/2}x_1 + 2^{\alpha_2/2}x_2 + z.$$

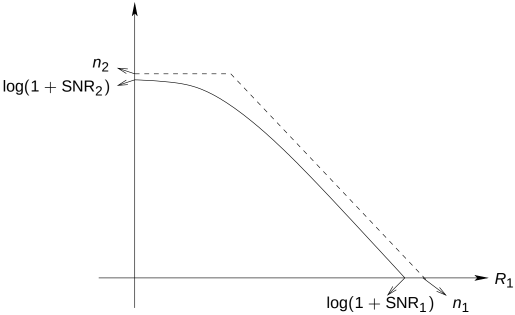


Example 3: Scalar broadcast channel

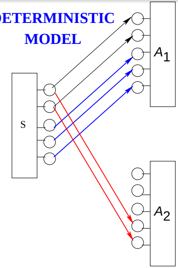


GAUSSIAN MODEL

$$y_1 = 2^{\alpha_1/2}x + z_1, \quad y_2 = 2^{\alpha_2/2}x + z_2$$

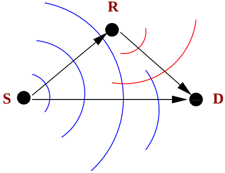


DETERMINISTIC MODEL



Approximation of 1 bit

Relay channel: deterministic approximation



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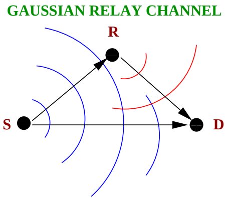

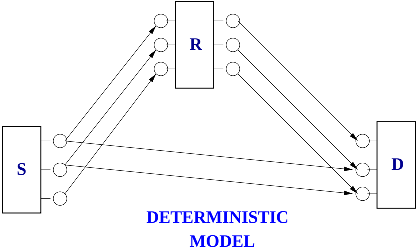

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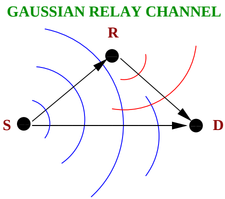

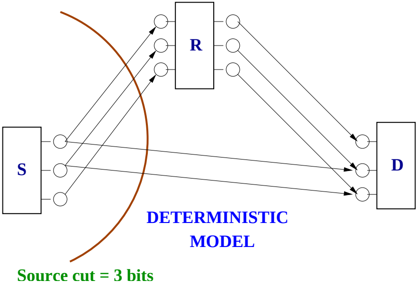

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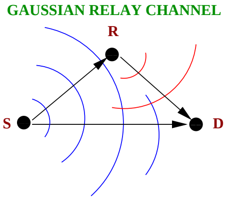

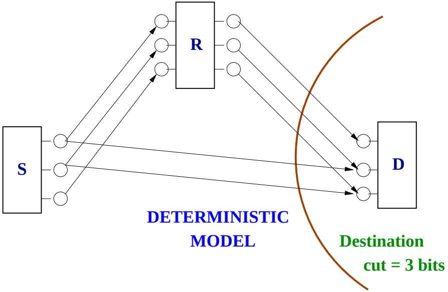

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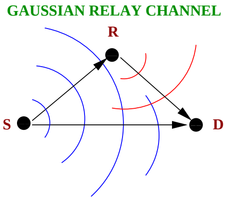

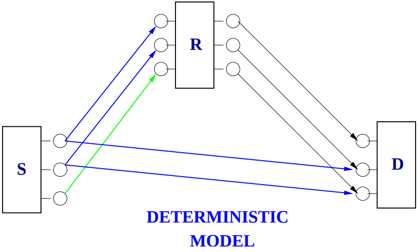

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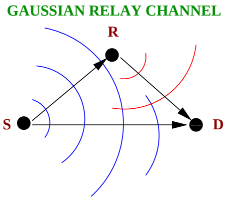

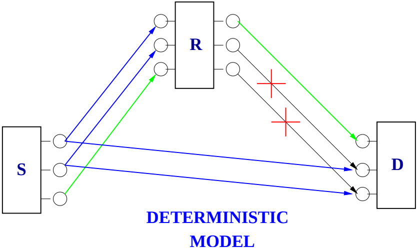

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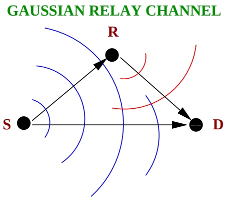

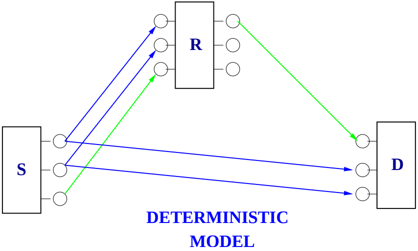

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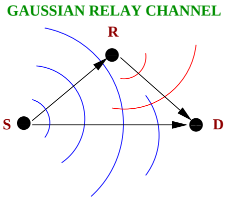
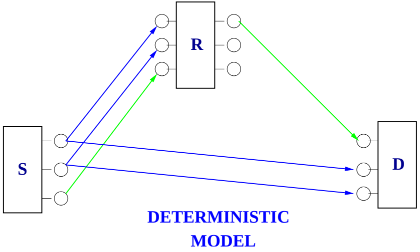


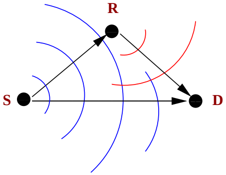


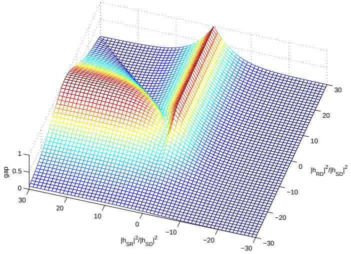
Cut-set bound achievable.

Decode and forward is optimal.

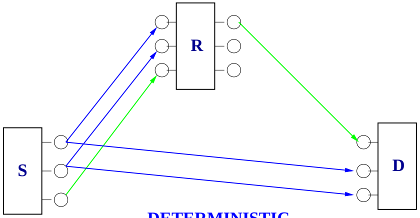
Relay channel: deterministic approximation

GAUSSIAN RELAY CHANNEL





Result: Gap from cut-set less than 1 bit, on average much less.



DETERMINISTIC MODEL

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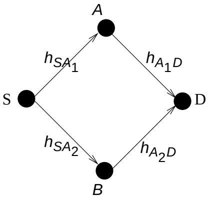
Decode and forward is optimal.

Diamond network

Gaussian



Diamond network

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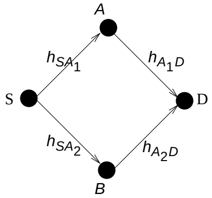
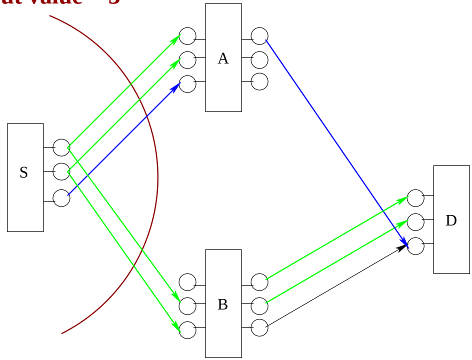


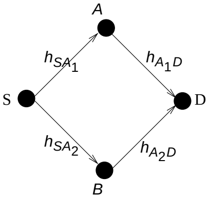
Deterministic

Cut value = 3

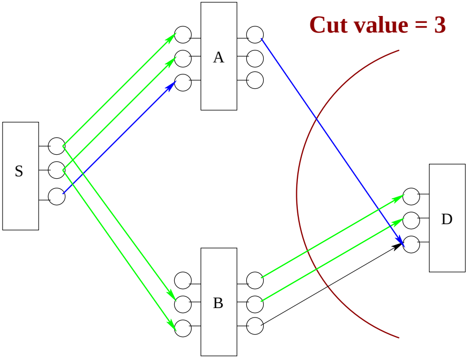


Diamond network

Gaussian

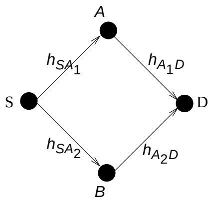


Deterministic

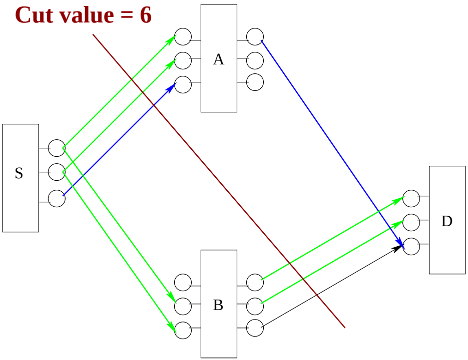


Diamond network

Gaussian



Deterministic



Diamond network

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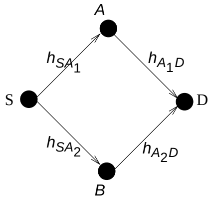
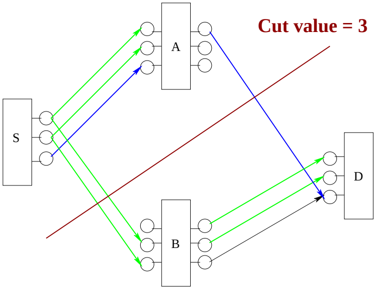
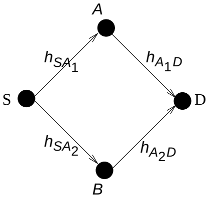


Deterministic



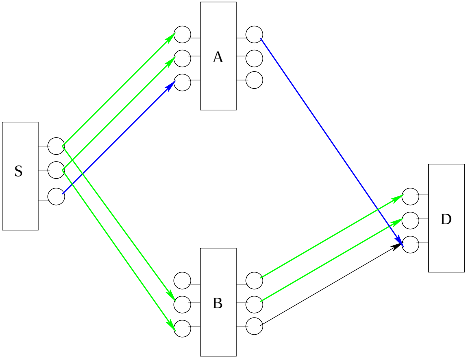
Diamond network

Gaussian



Result: Gap from cut-set less 1 bit.

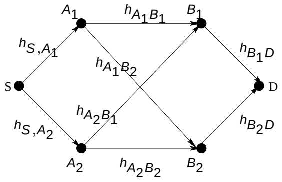
Deterministic



Cut-set bound achievable.
Partial decode-forward is optimal.

Two-layer network

Gaussian



Two-layer network

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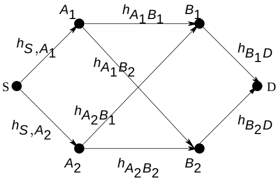
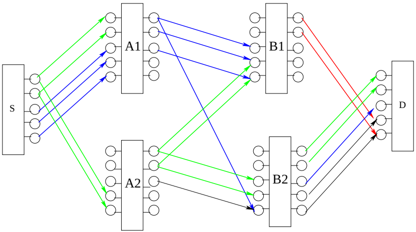


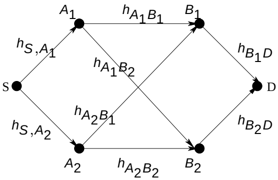
Deterministic



Cut-set bound achievable.
Linear map and forward is optimal.

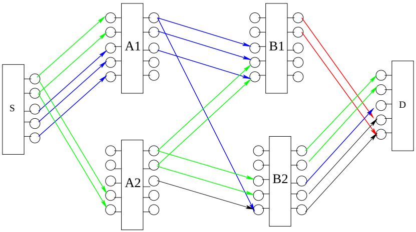
Two-layer network

Gaussian



Result: Gap from cut-set less than constant number of bits.

Deterministic



Cut-set bound achievable.
Linear map and forward is optimal.

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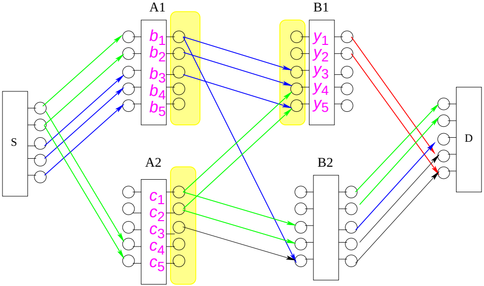
- What is the structure of the optimal strategy?
- Can we use insight from deterministic analysis to get approximately optimal strategy for Gaussian networks?

Algebraic representation



$$\mathbf{S} = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \end{bmatrix}$$

$\mathbf{S}$ is shift matrix of size $q = \max_{i,j} n_{i,j}$.

$$\mathbf{y}_{B_1} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} \oplus \begin{bmatrix} 0 \\ 0 \\ 0 \\ c_1 \\ c_2 \end{bmatrix} = \mathbf{S}^{5-3}\mathbf{x}_{A_1} \oplus \mathbf{S}^{5-2}\mathbf{x}_{A_2} = \mathbf{S}^{5-3}\mathbf{b} \oplus \mathbf{S}^{5-2}\mathbf{c}$$

Generalizations

Linear finite field model

- Channel from i to j is described by channel matrix $\mathbf{G}_{ij}$ operating over $\mathbb{F}_2$.
- Received signal at node j:

$$\mathbf{y}_j[t] = \sum_{i=1}^{N} \mathbf{G}_{ij}\mathbf{x}_i[t]$$

- Special case: our model given in examples

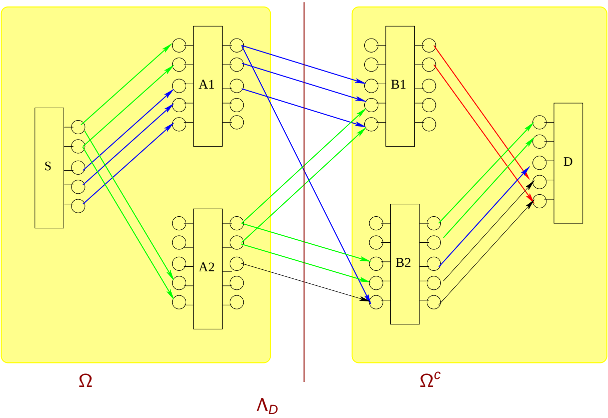
$$\mathbf{G}_{ij} = \mathbf{S}^{q-\alpha_{ij}}$$

General deterministic network:

$$\mathbf{y}[t] = \mathbf{G}(\mathbf{x}_1[t], \ldots, \mathbf{x}_N[t])$$

Observation: Wireline networks are a special case.

Information-theoretic cut-set



Cut: Separates S from D

Cut transfer matrix $\mathbf{G}_{\Omega,\Omega^c}$: Transfer function from nodes in Ω to Ω^c.

Cutset upper bound

General relay network:

$$C_{\text{relay}} \leq \bar{C} = \max_{p(\mathbf{X}_1,\ldots,\mathbf{X}_N)} \min_{\Omega} I(\mathbf{X}_\Omega; \mathbf{Y}_{\Omega^c} | \mathbf{X}_{\Omega^c})$$

Cutset upper bound

General relay network:

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General deterministic relay network:

$$C_{\text{relay}} \leq \bar{C} = \max_{p(\mathbf{X}_1,\ldots,\mathbf{X}_N)} \min_{\Omega} H(\mathbf{Y}_{\Omega^c} | \mathbf{X}_{\Omega^c})$$

Cutset upper bound

General relay network:

$$C_{\text{relay}} \leq \bar{C} = \max_{p(\mathbf{X}_1,\ldots,\mathbf{X}_N)} \min_{\Omega} I(\mathbf{X}_\Omega; \mathbf{Y}_{\Omega^c} | \mathbf{X}_{\Omega^c})$$

General deterministic relay network:

$$C_{\text{relay}} \leq \bar{C} = \max_{p(\mathbf{X}_1,\ldots,\mathbf{X}_N)} \min_{\Omega} H(\mathbf{Y}_{\Omega^c} | \mathbf{X}_{\Omega^c})$$

Linear finite field network: Optimal input distribution $\mathbf{x}_1, \ldots, \mathbf{x}_N$ independent and uniform

$$C_{\text{relay}} \leq \bar{C} = \min_{\Omega} \operatorname{rank}(\mathbf{G}_{\Omega,\Omega^c})$$

where $\mathbf{G}_{\Omega,\Omega^c}$ is the transfer matrix $\mathbf{X}_\Omega \to \mathbf{Y}_{\Omega^c}$.

Main results: Deterministic relay networks

Theorem (Avestimehr, Diggavi and Tse, 2007)

Given a general deterministic relay network (with broadcast and multiple access), we can achieve all rates R upto

$$\max_{\prod_i p(\mathbf{X}_i)} \min_{\Omega} H(\mathbf{Y}_{\Omega^c} | \mathbf{X}_{\Omega^c})$$

Multicast information flow:

Theorem (Avestimehr, Diggavi and Tse, 2007)

Given a general deterministic relay network (with broadcast and multiple access), we can achieve all rates R from S multicasting to all destinations $D \in \mathcal{D}$ up to,

$$\max_{\prod_{i \in \mathcal{V}} p(x_i)} \min_{D \in \mathcal{D}} \min_{\Omega \in \Lambda_D} H(Y_{\Omega^c} | X_{\Omega^c})$$

Application

Linear deterministic models

Corollary (Avestimehr, Diggavi and Tse, 2007)

Given a linear finite-field relay network (with broadcast and multiple access), the capacity C of such a relay network is given by,

$$C = \min_{\Omega \in \Lambda_D} \operatorname{rank}(\mathbf{G}_{\Omega,\Omega^c}).$$

Multicast information flow:

Corollary (Avestimehr, Diggavi and Tse, 2007)

Given a linear finite-field relay network (with broadcast and multiple access), the multicast capacity C of such a relay network is given by,

$$C = \min_{D \in \mathcal{D}} \min_{\Omega \in \Lambda_D} \operatorname{rank}(\mathbf{G}_{\Omega,\Omega^c}).$$

Consequences: Deterministic Relay Networks

General deterministic networks: Cutset upper bound was $C_{\text{relay}} \leq \max_{p(\mathbf{x}_1,\ldots,\mathbf{x}_N)} \min_{\Omega} H(\mathbf{Y}_{\Omega^c}|\mathbf{X}_{\Omega^c}) \Longrightarrow$ achievable if optimum was product distribution.

Linear finite field model: Cutset upper bound was $C_{\text{relay}} \leq \min_{\Omega} \text{rank}(\mathbf{G}_{\Omega,\Omega^c}) \Longrightarrow$ cutset bound achievable

For wireline graph model $\text{rank}(\mathbf{G}_{\Omega,\Omega^c})$ is number of links crossing the cut.

Observation: We have a generalization of Ford-Fulkerson max-flow min-cut theorem to linear finite field relay networks with broadcast and multiple access.

Main results: Gaussian relay networks

Theorem (Avestimehr, Diggavi and Tse, 2007)

Given a Gaussian relay network, $\mathcal{G}$, we can achieve all rates R up to $\overline{C} - \kappa$. Therefore the capacity of this network satisfies

$$\overline{C} - \kappa \leq C \leq \overline{C},$$

where $\overline{C}$ is the cut-set upper bound on the capacity of $\mathcal{G}$, and κ is a constant independent of channel gains.

Theorem (Multicast information flow)

Given a Gaussian relay network, $\mathcal{G}$, we can achieve all multicast rates R up to $\overline{C}_{mcast} - \kappa$, i.e., *for $\overline{C}_{mcast} = \min_{D \in \mathcal{D}} \overline{C}_D$,*

$$\overline{C}_{mcast} - \kappa \leq C \leq \overline{C}_{mcast}$$

Ingredients and insights

Main steps: Gaussian strategy

- **Relay operation:** Quantize received signal at noise-level.
- **Relay function:** Random mapping from received quantized signal to transmitted signal.
- Handle unequal (multiple) paths between nodes like “inter-symbol interference”.

Consequences:

- With probabilistic method we demonstrate min-cut achievability for linear deterministic networks.
- Gaussian networks constant gap independent of SNR operating point.
- Engineering insight of (almost) optimal coding strategies.

Compound relay networks

Compound model: Channel realizations from a set $h_{i,j} \in \mathcal{H}_{i,j}$, unknown to sender.

Observations:

- Relay strategy does not depend on the channel realization.
- Overall network from source to destination behaves like a compound channel.
- Utilize point-to-point compound channel ideas get approximate characterization for compound network.

Theorem

Given a compound Gaussian relay network the capacity C_{cn} satisfies

$$\overline{C}_{cn} - \kappa \leq C_{cn} \leq \overline{C}_{cn},$$

where $\overline{C}_{cn} = \max_{p(\{x_j\}_{j\in\mathcal{V}})} \inf_{h\in\mathcal{H}} \min_{\Omega\in\Lambda_D} I(Y_{\Omega^c}; X_\Omega | X_{\Omega^c})$.

Relay networks: Open questions and extensions

Extensions:

- Outage set behavior for full duplex networks.
- Analysis of half-duplex systems with fixed transmit fractions.
- Ergodic channel variations.

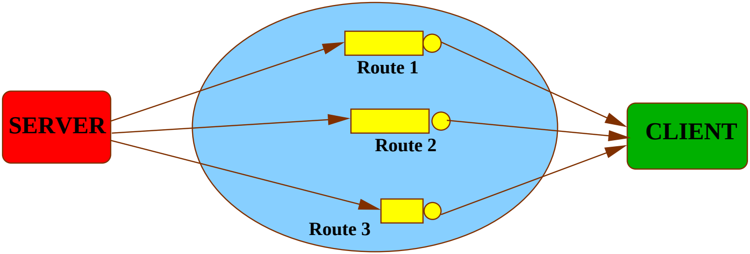
Open questions:

- D-M trade-off for channel dependent half-duplex systems.
- Tightening gap to cut-set bound.
- Use deterministic model directly to get Gaussian result.

Extensions of deterministic approach

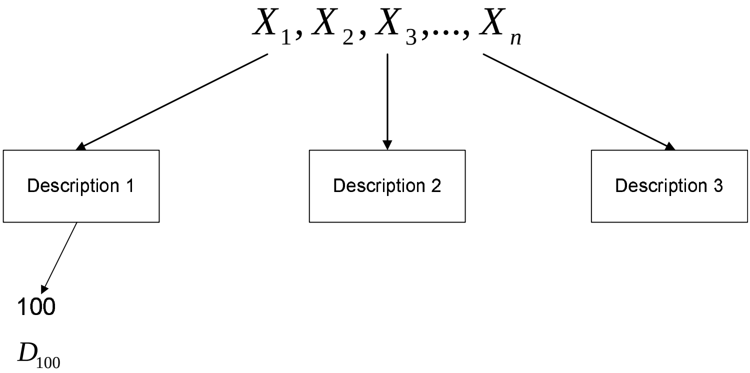
- **Interference channel:** Successfully used to generate approximate characterization (Bresler and Tse, 2007),
- ***K*-user interference channel:** Used to demonstrate new phenomenon of *interference alignment* (Bresler-Tse, 2007, Jafar 2007).
- **Relay-interference networks:** Extension of multiple unicast to wireless networks (Mohajer, Diggavi, Fragouli and Tse, 2008).
- **Wireless network secrecy:** Used to demonstrate secrecy over networks (Diggavi, Perron and Telatar, 2008).
- **Network data compression:** Identify correct multi-terminal lossless structures to get approximations $\longrightarrow$ next topic.

Multiple description coding: route diversity

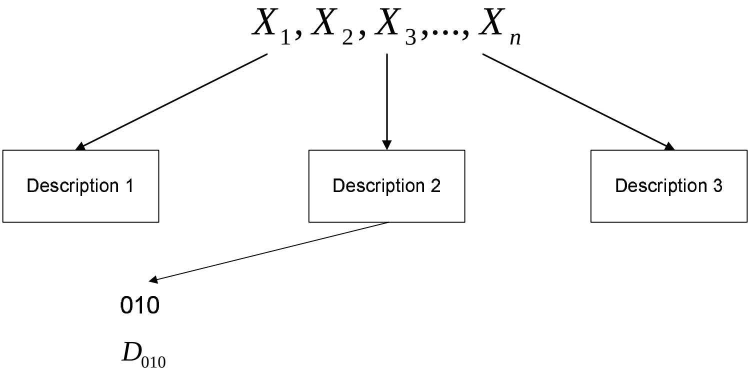


- **Route diversity:** Multiple routes from source to destination.
- **Goal:** Graceful degradation in performance with route failures $\Longrightarrow$ multiple description source coding.
- Generate binary streams with rate constraints with distortion guarantees when only subset of routes succeed.

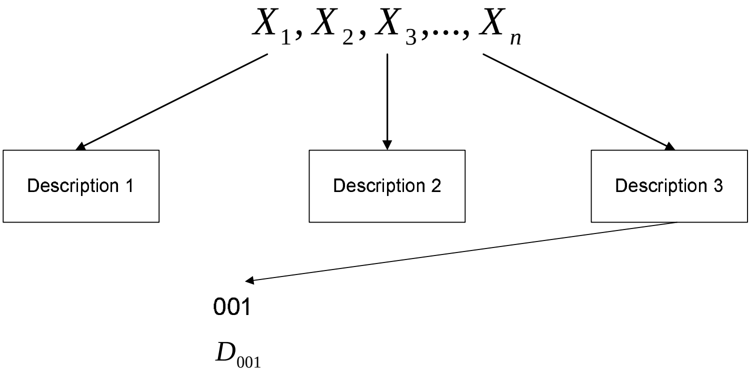
Symmetric multiple description



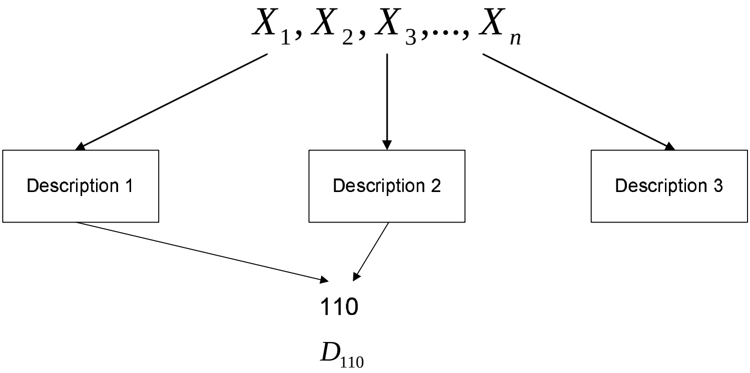
Symmetric multiple description



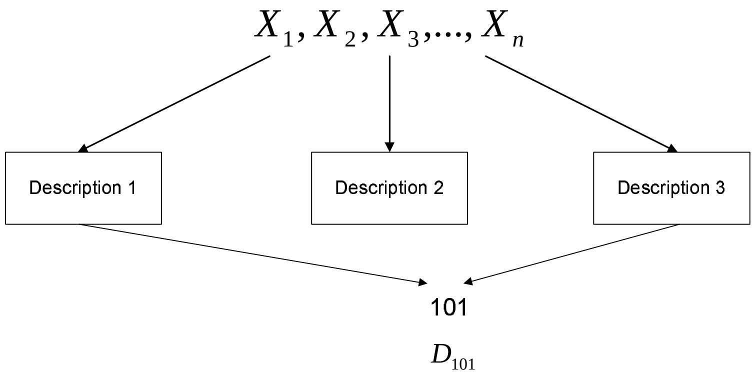
Symmetric multiple description



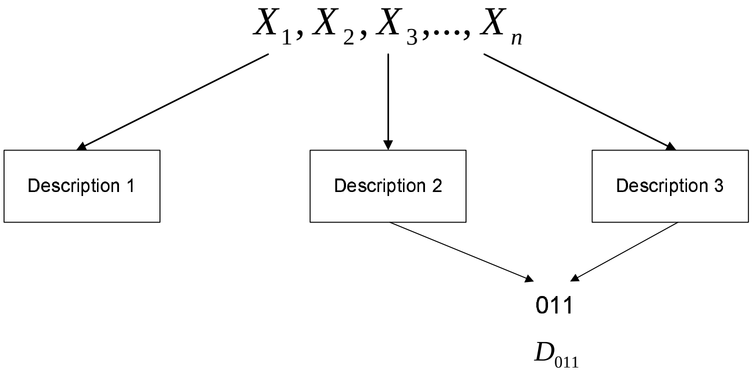
Symmetric multiple description



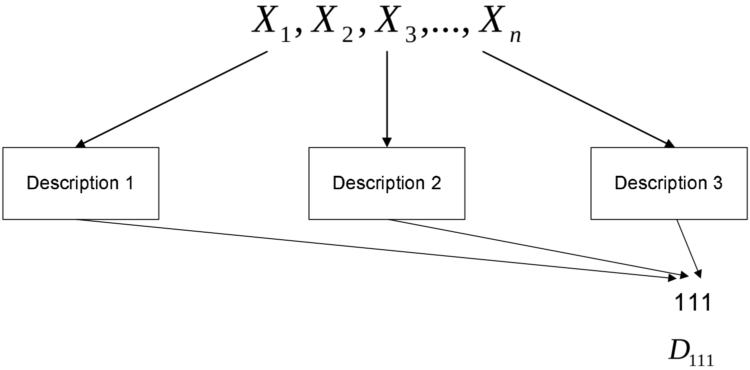
Symmetric multiple description



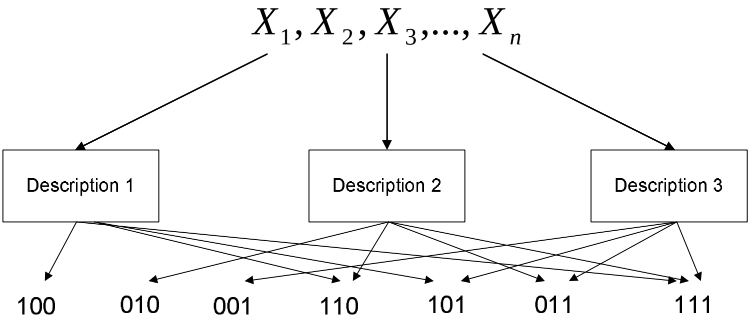
Symmetric multiple description



Symmetric multiple description



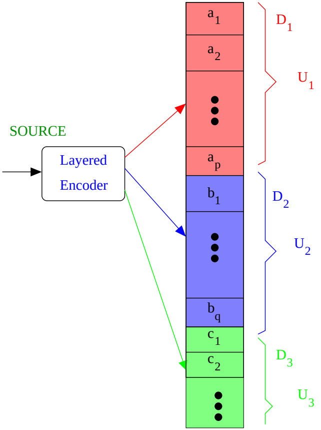
Symmetric multiple description

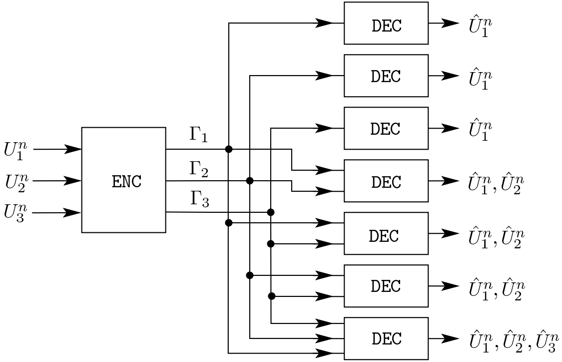


$$D_{100} = D_{010} = D_{001} \geq D_{110} = D_{101} = D_{011} \geq D_{111}$$

Goal: Characterize tuple $(R_1, R_2, R_3, D_1, D_2, D_3)$, for Gaussian quadratic source, where R_i are rates on descriptions and D_i are distortions for i successful descriptions.

Simple architecture: lossless multilevel source codes





Symmetric multi-level coding

Overall strategy

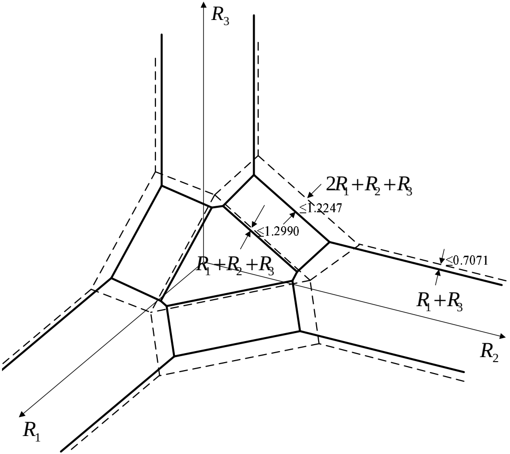
Approach:

- Identify underlying lossless coding problem and solve rate region.
- Use polytopic lossless rate region as “template”.
- Derive outer bound using intuition from the template.

Technical ideas:

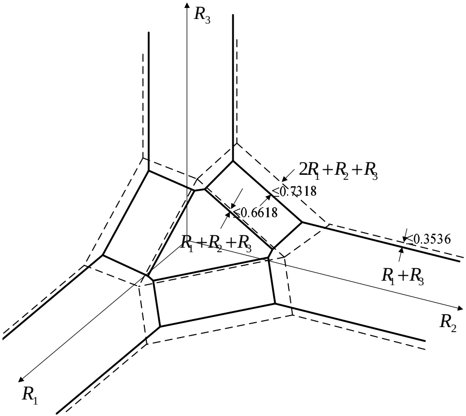
- New lower bounding technique: Expand auxiliary random variable space to $K-1$ for K-descriptions.
- Intuition: Each auxiliary variable captures distortion “level”.
- Use structure for auxiliary variables to obtain lower bound to match inner bound region.

An approximate characterization for SR-MLD



Bottom line: Simple architecture almost optimal!

Improved approximation using binning scheme



Results have been generalized to $K > 3$ using bounding hyperplanes specification.

Extensions

Symmetric MD

- Lower bound extension to arbitrary K description problem (DCC 2008).
- Approximation for $K > 3$ rate-region using the symmetric MLD ($K > 3$) insights (ISIT 2008).
- Extension to non-Gaussian sources.

Asymmetric problem

- Solved underlying 3-description lossless asymmetric multi-level coding rate region (DCC 2008)
- Used insight to approximate asymmetric Gaussian MD rate region (ISIT 2008).

Discussion

Program:

- Focus on underlying deterministic or lossless coding problem
  $\longrightarrow$ this identification is a central challenge.
- Obtain exact characterization of underlying problem.
- Use insight to obtain approximation of rate region of noisy/lossy problem.

Hope:

- The program will yield insight to network flow problems.
- Exposes the central difficulties, solution insights and new schemes?
- Approximations may be sufficient for engineering practice.

Papers/preprints: http://licos.epfl.ch