Analysis of Some Variable-to-Fixed Codes by Analytic Methods

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Outline of the Talk

1. Tunstall and Khodak VF Codes
2. A Useful Lemma on Trees
3. Some Recurrences
4. Main Results on Tunstall’s Code
5. Redundancy of Tunstall’s Code
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Variable-to-Fixed Codes

Throughout, we consider an \( m \)-ary alphabet \( \mathcal{A} = \{1, 2, \ldots, m\} \).

1. A variable-to-fixed length encoder partitions the source string into a concatenation of variable-length phrases.

2. Each phrase belongs to a given dictionary \( \mathcal{D} \) of source strings.

3. A dictionary can be represented by a complete parsing tree \( T \), i.e., a tree in which every internal node has all \( m \) children. The dictionary entries \( d \in \mathcal{D} \) correspond to the leaves of parsing tree.

4. The encoder represents each parsed string by the fixed length binary code word. If the dictionary \( \mathcal{D} \) is has \( M \) entries, then the code word for each phrase has \( \lceil \log_2 M \rceil \) bits.
Tunstall’s construction works as follows (cf. Tunstall, Synthesis of Noiseless Compression Codes, Ph.D. 1968):

1. Start with a tree consisting of a root node and \( m \) leaves.

2. In the \( J \)'s iteration we select the current leaf corresponding to a string of the highest probability and grow \( m \) children out it.

3. After these \( J \) steps, the parsing tree has \( J \) non-root internal nodes and

\[
M = (m - 1)J + m
\]

leaves which correspond to distinct dictionary entries.
Figure 1: Tunstall’s Code for $M = 5$ and Khodak’s Construction for $r = 0.25$. 

Tunstall’s construction
$M = 5$
Khodak’s construction
$r = 0.25$
Khodak’s construction of Tunstall’s code (cf. Khodak, “Connection Between Redundancy and Average Delay of Fixed-Length Coding”, 1969):

1. Let $p_i$ be the probability of the $i$th source symbol and let

$$p_{\text{min}} = \min\{p_1, \ldots, p_m\}.$$

2. Pick a real number $r \in (0, p_{\text{min}})$ and grow a complete parsing tree satisfying

$$p_{\text{min}} r \leq P(d) < r, \quad d \in D.$$

3. The resulting parsing tree is exactly the same as a tree constructed by Tunstall’s algorithm.

4. Observe that if $y$ is a proper prefix of entries of $D = D_r$, i.e., $y$ corresponds to an internal node of $T = T_r$, then

$$P(y) \geq r.$$

We represent phrases (leaves of $T$) in terms of the internal nodes of the parsing tree $T_r$. 
A Useful Lemma on Trees

**Theorem 1.** Let $\tilde{D}$ be a uniquely parsable dictionary (leaves of $T$) and $\tilde{Y}$ be the collection of strings which are proper prefixes of one or more dictionary entries (internal nodes of $T$). Then for all $|z| \leq 1$,

$$\sum_{d \in \tilde{D}} P(d) \frac{z^{|d|} - 1}{z - 1} = \sum_{y \in \tilde{Y}} P(y) z^{|y|}.$$ 

**Proof** By induction with respect to $\tilde{Y}$:

1. For the inductive step, let $\tilde{Y}$ have $k + 1$ elements. Choose $y_0 \in \tilde{Y}$ (of maximum length) so that its single letter extensions correspond to the dictionary entries $d_1, d_2, \ldots, d_m \in \tilde{D}$. Observe that $P(y_0) = P(d_1) + P(d_2) + \cdots + P(d_m)$.

2. Define an auxiliary dictionary $\tilde{D}'$ with

$$\tilde{D}' = \tilde{D} \cup \{y_0\} \setminus \{d_1, \ldots, d_m\}.$$ 

Then $\tilde{D}'$ has a corresponding proper prefix set with $k$ elements

$$\tilde{Y}' = \tilde{Y} \setminus \{y_0\}.$$
3. By induction we have

\[
\sum_{y \in \tilde{Y}} P(y) z^{\left|y\right|} = \sum_{y \in \tilde{Y}'} P(y) z^{\left|y\right|} + P(y_0) z^{\left|y_0\right|}
\]

\[
= \sum_{d \in \tilde{D}'} P(d) \frac{z^{\left|d\right|} - 1}{z - 1} + P(y_0) z^{\left|y_0\right|}
\]

\[
= \sum_{d \in \tilde{D}' \setminus \{y_0\}} P(d) \frac{z^{\left|d\right|} - 1}{z - 1}
\]

\[
+ P(y_0) \left( z^{\left|y_0\right|} + \frac{z^{\left|y_0\right|} - 1}{z - 1} \right)
\]

\[
= \sum_{d \in \tilde{D}' \setminus \{y_0\}} P(d) \frac{z^{\left|d\right|} - 1}{z - 1}
\]

\[
+ (P(d_1) + \cdots + P(d_m)) \left( \frac{z^{\left|y_0\right|} + 1 - 1}{z - 1} \right)
\]

\[
= \sum_{d \in \tilde{D}} P(d) \frac{z^{\left|d\right|} - 1}{z - 1}.
\]
Let $D = |d|$ for $d \in D$.

Assume now that source strings are generated by a memoryless source over an alphabet $A = \{1, \ldots, m\}$.

Then we can talk about moments of $D$. We have from previous result

$$
E[D] = \sum_{y \in \hat{Y}} P(y),
$$
$$
E[D(D - 1)] = 2 \sum_{y \in \hat{Y}} P(y)|y|.
$$

**Moment Generating Functions:** Let

$$
D(r, z) := E[z^D] = \sum_{d \in D_r} P(d)z^{|d|}.
$$

and its corresponding internal nodes generating function

$$
S(r, z) = \sum_{y: P(y) \geq r} P(y)z^{|y|}.
$$

From previous result we also conclude that

$$
D(r, z) = 1 + (z - 1)S(r, z).
$$
Define for a binary alphabet \( \{1, 2\} \) with \( p_1 < p_2 \):
\[
v = 1/r, \ z \text{ complex: } \tilde{S}(v, z) = S(v^{-1}, z).
\]

Let also
\[
A(v) = \sum_{y: P(y) \geq 1/v} 1.
\]
denote the number of strings with probability at least \( v^{-1} \).

We have
\[
A(v) = \begin{cases} 
0 & v < 1, \\
1 + A(vp_1) + A(vp_2) & v \geq 1
\end{cases}
\]
and
\[
\tilde{S}(v, z) = \begin{cases} 
0 & v < 1, \\
1 + zp_1 \tilde{S}(vp_1, z) + zp_2 \tilde{S}(vp_2, z) & v \geq 1
\end{cases}
\]
since every binary string either is:
- empty string,
- string starting with 1
- string starting with 2.
Some Intermediate Results

$A(v)$ represents the number of internal nodes in Khodak’s construction:

$$M_r = A(v) + 1 = |D_r|$$

$$E[D_r] = \tilde{S}(v, 1)$$

We shall prove the following. 

**Lemma 1.** If $\ln p_2 / \lg p_1$ is irrational, then

$$M_r = A(v) + 1 = \frac{v}{H} + o(v),$$

otherwise (i.e., $\ln p_2 / \lg p_1$ is rational)

$$M_r = A(v) + 1 = \frac{Q_1(\log v)}{H}v + O(v^{1-\eta})$$

for some $\eta > 0$, where

$$Q_1(x) = \frac{L}{1 - e^{-L}} e^{-L\langle x \rangle}$$

$L > 0$ is the largest real number for which $\ln(1/p_1)$ and $\ln(1/p_2)$ are integer multiples of $L$;

$H = p_1 \ln(1/p_1) + p_2 \ln(1/p_2)$ is the entropy,

$\langle y \rangle = y - \lfloor y \rfloor$ is the fractional part of $y$. 
Furthermore, if $\ln p_2 / \ln p_1$ is irrational, then

$$E[D_r] = \tilde{S}(v, 1) = \frac{\log v}{H} + \frac{H_2}{2H^2} + o(1)$$

otherwise (i.e., $\ln p_2 / \lg p_1$ is rational)

$$E[D_r] = \tilde{S}(v, 1) = \frac{\log v}{H} + \frac{H_2}{2H^2} + \frac{Q_2(\log v)}{H} + O(v^{-\eta})$$

for some $\eta > 0$, where

$$Q_2(x) = L \cdot \left( \frac{1}{2} - \left\langle \frac{x}{L} \right\rangle \right)$$

and $H_2 = p_1 \ln(1/p_1)^2 + p_2 \ln(1/p_2)^2$. 
Idea of the Proof

The Mellin transform $F^*(s)$ of a function $F(v)$ is

$$F^*(s) = \int_0^\infty F(v)v^{s-1}dv.$$  

From the recurrence on $S(v, z)$ we find

$$\tilde{D}^*(s, z) = \frac{1 - z}{s(1 - zp_1^{1-s} - zp_2^{1-s})} - \frac{1}{s}, \quad \Re(s) < s_0(z),$$

where $s_0(z)$ denotes the real solution of $zp_1^{1-s} + zq_1^{1-s} = 1$.

To find the asymptotics of $\tilde{D}(v, z)$ as $v \to \infty$ we compute the inverse transform of $\tilde{D}^*(s, z)$:

$$\tilde{D}(v, z) = \frac{1}{2\pi i} \lim_{T \to \infty} \int_{\sigma - iT}^{\sigma + iT} \tilde{D}^*(s, z)v^{-s}ds,$$

where $\sigma < s_0(z)$.

To determine the polar singularities of the meromorphic continuation of $\tilde{D}^*(s, z)$, we have to analyze the set

$$Z(z) = \{ s \in \mathbb{C} : zp_1^{1-s} + zq_1^{1-s} = 1 \}$$

of all complex roots of $zp_1^{1-s} + zq_1^{1-s} = 1$. 
Computing the Inverse Mellin Transform

Figure 2: Illustration to the computations of the Mellin transform.
From Cauchy’s residue theorem we obtain
\[ \tilde{D}(v, z) = \lim_{T \to \infty} F_T(v, z) \] for \( \Re(s) < \tau \), where

\[ F_T(v, z) = - \sum_{s' \in Z(z), \, \Re(s') < \tau, \, |\Im(s')| > T} \text{Res}(\tilde{D}^*(s, z) v^{-s}, \, s = s') \]

\[ + \frac{1}{2\pi i} \int_{\tau-iT}^{\tau+iT} \left( \frac{1 - z}{s(1 - z p_1^{1-s} - z p_2^{1-s})} - \frac{1}{s} \right) v^{-s} \, ds \]

\[ = - \sum_{s' \in Z(z), \, \Re(s') < \tau, \, |\Im(s')| > T} \frac{(1 - z)v^{-s'}}{z s' p_1^{1-s'} \ln p_1 + z s' p_2^{1-s'} \ln p_2} \]

\[ + \frac{1}{2\pi i} \int_{\tau-iT}^{\tau+iT} \left( \frac{1 - z}{s(1 - z p_1^{1-s} - z p_2^{1-s})} - \frac{1}{s} \right) v^{-s} \, ds \]

provided that the series of residues converges and the limit as \( T \to \infty \) of the last integral exists.

The problem is that neither the series nor the integral above are absolutely convergent since the integrand is only of order \( 1/s \).
Main Results

**Theorem 2.** Let $D_r$ and $M_r$ denote the phrase length and the dictionary size in Khodak's construction.

\[ \frac{D_r - \frac{1}{H} \ln M_r}{\sqrt{\left(\frac{H_2}{H^3} - \frac{1}{H}\right) \ln M_r}} \to N(0, 1) \]

where $N(0, 1)$ denotes the standard normal distribution. Furthermore,

\[ \text{Var}[D_r] = \left(\frac{H_2}{H^3} - \frac{1}{H}\right) \ln M_r + O(1) \]

for large $M_r$.

In the **irrational case**

\[ E[D_r] = \frac{\ln M_r}{H} + \frac{\ln H}{H} + \frac{H_2}{2H^2} + o(1) \]

and in the **rational case** (i.e., for $\log p_2 / \log p_1 = b/d$)

\[ E[D_r] = \frac{\ln M_r}{H} + \frac{\ln H}{H} + \frac{H_2}{2H^2} + \frac{Q_2(nv) - \ln Q_1(ln v)}{H} + O(M_r^{-\eta}). \]

The above yields (miracle: no oscillations)

\[ Q_2(ln v) - \log Q_1(ln v) = -\ln L + \ln(1 - e^{-L}) + \frac{L}{2}. \]
Redundancy

The average redundancy rate of Tunstall’s code is defined as

\[ R_{M_r} = \frac{\ln M_r}{E[D]} - H \]

As a consequence of our main result we can prove the following.

**Corollary 1.** Let \( D_r \) denote the dictionary in Khodak’s construction of the Tunstall code of size \( M_r \). If \( \ln p_1 / \ln p_2 \) is irrational then

\[ R_{M_r} = \frac{H}{\ln M_r} \left( -\frac{H_2}{2H} - \ln H \right) + o \left( \frac{1}{\ln M_r} \right). \]

In the rational case we have

\[ R_{M_r} = \frac{H}{\ln M_r} \left( -\frac{H_2}{2H} - \ln H + \ln L - \ln(e^L - 1) + \frac{L}{2} \right) + O \left( M_r^{-\eta} \right), \]

for some \( \eta > 0 \), where \( L > 0 \) is the largest real number for which \( \ln(1/p_1) \) and \( \ln(1/p_2) \) are integer multiples of \( L \) (no oscillation!)

Oscillations usually occur in redundancy rates.

**Lempel-Ziv-78.** In this case:

\[ r_n = 2h - h\gamma - \frac{1}{2}h_2 + h\beta - h\delta_0(n) \frac{\log \log n}{\log n} + O \left( \frac{\log \log n}{\log^2 n} \right), \]

where

\[ h = -p \log p - q \log q \]
\[ h_2 = p \log^2 p + q \log^2 q, \]

\[ \gamma = 0.577 \ldots \] is the Euler constant,
and \( \delta_0(x) \) is a fluctuating functions with small amplitude for \( \log p/\log q \) rational, and

\[ \lim_{x \to \infty} \delta_0(x) = 0 \]

otherwise. Finally, the constant \( \beta \) is defined as:

\[ \beta = -\sum_{k=1}^{\infty} \frac{p^{k+1} \log p + q^{k+1} \log q}{1 - p^{k+1} - q^{k+1}}. \]
Shannon and Huffman Codes

Shannon’s Code. In this case:

$$\bar{R}_n^{SF} = \begin{cases} 
\frac{1}{2} + o(1) & \alpha \text{ irrational} \\
\frac{1}{2} - \frac{1}{M} (\langle Mn\beta \rangle - \frac{1}{2}) + O(\rho^n) & \alpha = \frac{N}{M},
\end{cases}$$

where $\rho < 1$ and $N, M$ are integers such that $\gcd(N, M) = 1$, and

$$\alpha = \log_2 \left( \frac{1 - p}{p} \right), \quad \beta = \log_2 \left( \frac{1}{1 - p} \right).$$

Huffman’s Code. In this case:

$$\bar{R}_n^H = \begin{cases} 
\frac{3}{2} - \frac{1}{\log 2} + o(1) \approx 0.057304, & \alpha \text{ irrational} \\
\frac{3}{2} - \frac{1}{M} (\langle \beta Mn \rangle - \frac{1}{2}) - \frac{1}{M(1-2^{-1/M})} 2^{-\langle n\beta M \rangle/M} + O(\rho^n) & \alpha = \frac{N}{M}
\end{cases}$$

where $\rho < 1$ and $N, M$ are integers such that $\gcd(N, M) = 1$. 