# Analysis of Some Variable-to-Fixed Codes by Analytic Methods

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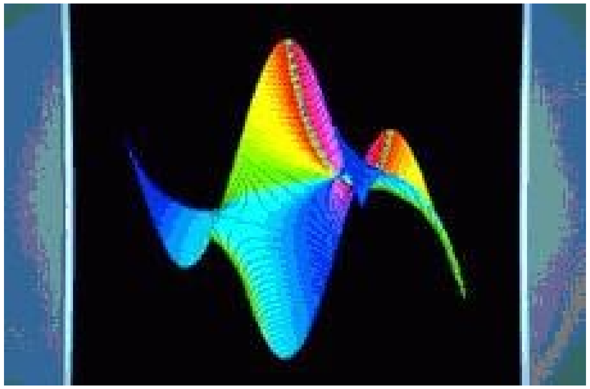

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# Outline of the Talk

1. Tunstall and Khodak VF Codes
2. A Useful Lemma on Trees
3. Some Recurrences
4. Main Results on Tunstall's Code
5. Redundancy of Tunstall's Code
6. Oscillations in Redundancy Formulas

# Variable-to-Fixed Codes

Throughout, we consider an $m$-ary alphabet
$\mathcal{A} = \{1, 2, \ldots, m\}$

**1**. A variable-to-fixed length encoder partitions the source string into a concatenation of variable-length phrases.

**2**. Each phrase belongs to a given dictionary $\mathcal{D}$ of source strings.

**3**. A dictionary can be represented by a complete parsing tree $\mathcal{T}$, i.e., a tree in which every internal node has all $m$ children.
The dictionary entries $d \in \mathcal{D}$ correspond to the leaves of parsing tree.

**4**. The encoder represents each parsed string by the fixed length binary code word.
If the dictionary $\mathcal{D}$ is has $M$ entries, then the code word for each phrase has $\lceil \log_2 M \rceil$ bits.

# Tunstall's Code

Tunstall's construction works as follows (cf. Tunstall, Synthesis of Noiseless Compression Codes, Ph.D. 1968):

**1**. Start with a tree consisting of a root node and $m$ leaves.

**2**. In the $J$'s iteration we select the current leaf corresponding to a string of the **highest probability** and grow $m$ children out it.

**3**. After these $J$ steps, the parsing tree has $J$ non-root internal nodes and

$$M = (m - 1)J + m$$

leaves which correspond to distinct dictionary entries.

# Example

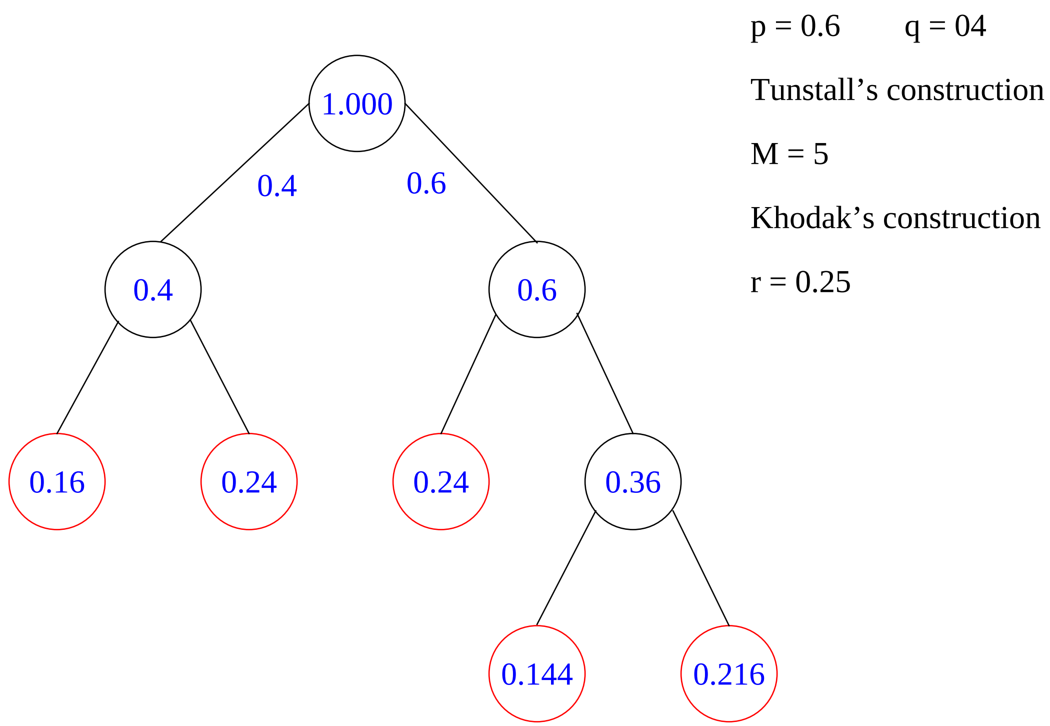



Figure 1: Tunstall's Code for $M = 5$ and Khodak's Construction for $r = 0.25$.

# Khodak's Construction

Khodak's construction of Tunstall's code (cf. Khodak, "Connection Between Redundancy and Average Delay of Fixed-Length Coding", 1969):

**1**. Let $p_i$ be the probability of the $i$th source symbol and let

$$p_{\min} = \min\{p_1, \ldots, p_m\}.$$

**2**. Pick a real number $r \in (0, p_{min})$ and grow a complete parsing tree satisfying

$$p_{\min} r \leq P(d) < r\,, \quad d \in \mathcal{D}.$$

**3**. The resulting parsing tree is exactly the same as a tree constructed by Tunstall's algorithm.

**4**. Observe that if $y$ is a proper prefix of entries of $\mathcal{D} = \mathcal{D}_r$, i.e., $y$ corresponds to an internal node of $\mathcal{T} = \mathcal{T}_r$, then

$$P(y) \geq r.$$

We represent phrases (leaves of $\mathcal{T}$) in terms of the internal nodes of the parsing tree $\mathcal{T}_r$.

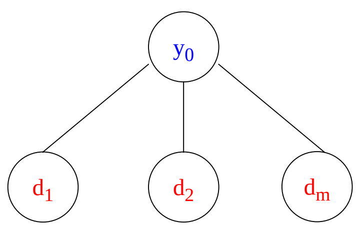
# A Useful Lemma on Trees

**Theorem 1.** *Let $\tilde{\mathcal{D}}$ be a uniquely parsable dictionary (leaves of $\mathcal{T}$) and $\tilde{\mathcal{Y}}$ be the collection of strings which are proper prefixes of one or more dictionary entries (internal nodes of $\mathcal{T}$). Then for all $|z| \leq 1$,*

$$\sum_{d \in \tilde{\mathcal{D}}} P(d) \frac{z^{|d|} - 1}{z - 1} = \sum_{y \in \tilde{\mathcal{Y}}} P(y) z^{|y|}.$$

**Proof** By induction with respect to $\tilde{\mathcal{Y}}$:

**1**. For the inductive step, let $\tilde{\mathcal{Y}}$ have $k+1$ elements.




Choose $y_0 \in \tilde{\mathcal{Y}}$ (of maximum length) so that its single letter extensions correspond to the dictionary entries $d_1, d_2, \ldots, d_m \in \tilde{\mathcal{D}}$.

Observe that $P(y_0) = P(d_1) + P(d_2) + \cdots + P(d_m)$.

**2**. Define an auxiliary dictionary $\tilde{\mathcal{D}}'$ with

$$\tilde{\mathcal{D}}' = \tilde{\mathcal{D}} \cup \{y_0\} \setminus \{d_1, \ldots, d_m\}.$$

Then $\tilde{\mathcal{D}}'$ has a corresponding proper prefix set with $k$ elements

$$\tilde{\mathcal{Y}}' = \tilde{\mathcal{Y}} \setminus \{y_0\}.$$

# Final Step of the Induction

3. By induction we have

$$\begin{aligned}
\sum_{y\in\tilde{\mathcal{Y}}} P(y)z^{|y|} &= \sum_{y\in\tilde{\mathcal{Y}}'} P(y)z^{|y|} + P(y_0)z^{|y_0|} \\
&= \sum_{d\in\tilde{\mathcal{D}}'} P(d)\frac{z^{|d|}-1}{z-1} + P(y_0)z^{|y_0|} \\
&= \sum_{d\in\tilde{\mathcal{D}}'\setminus\{y_0\}} P(d)\frac{z^{|d|}-1}{z-1} \\
&\quad +P(y_0)\left(z^{|y_0|} + \frac{z^{|y_0|}-1}{z-1}\right) \\
&= \sum_{d\in\tilde{\mathcal{D}}'\setminus\{y_0\}} P(d)\frac{z^{|d|}-1}{z-1} \\
&\quad +(P(d_1)+\cdots+P(d_m))\left(\frac{z^{|y_0|+1}-1}{z-1}\right) \\
&= \sum_{d\in\tilde{\mathcal{D}}} P(d)\frac{z^{|d|}-1}{z-1}.
\end{aligned}$$

# Phrase Length

Let $D = |d|$ for $d \in \mathcal{D}$.

Assume now that source strings are generated by a memoryless source over an alphabet $\mathcal{A} = \{1, \ldots, m\}$.

Then we can talk about moments of $D$. We have from previous result

$$\begin{aligned} \mathbf{E}[D] &= \sum_{y \in \tilde{\mathcal{Y}}} P(y), \\ \mathbf{E}[D(D-1)] &= 2 \sum_{y \in \tilde{\mathcal{Y}}} P(y)|y|. \end{aligned}$$

**Moment Generating Functions**: Let

$$D(r, z) := \mathbf{E}[z^D] = \sum_{d \in \mathcal{D}_r} P(d) z^{|d|}.$$

and its corresponding internal nodes generating function

$$S(r, z) = \sum_{y:\ P(y) \geq r} P(y) z^{|y|}.$$

From previous result we also conclude that

$$D(r, z) = 1 + (z-1) S(r, z).$$

# Recurrences

Define for a binary alphabet $\{1,2\}$ with $p_1 < p_2$:
$v = 1/r$, $z$ complex: $\tilde{S}(v,z) = S(v^{-1}, z)$.

Let also

$$A(v) = \sum_{y:P(y)\geq 1/v} 1.$$

denote the number of strings with probability at least $v^{-1}$.

We have

$$A(v) = \begin{cases} 0 & v < 1, \\ 1 + A(vp_1) + A(vp_2) & v \geq 1 \end{cases}$$

and

$$\tilde{S}(v,z) = \begin{cases} 0 & v < 1, \\ 1 + zp_1\tilde{S}(vp_1, z) + zp_2\tilde{S}(vp_2, z) & v \geq 1, \end{cases}$$

since every binary string either is:

- – empty string,
- – string starting with $1$
- – string starting with $2$.

# Some Intermediate Results

$A(v)$ represents the number of internal nodes in Khodak's construction:

$$\begin{aligned} M_r &= A(v) + 1 = |\mathcal{D}_r| \\ \mathbf{E}[D_r] &= \tilde{S}(v, 1) \end{aligned}$$

We shall prove the following.
**Lemma 1.** *If $\ln p_2 / \lg p_1$ is irrational, then*

$$M_r = A(v) + 1 = \frac{v}{H} + o(v),$$

*otherwise (i.e., $\ln p_2 / \lg p_1$ is rational)*

$$M_r = A(v) + 1 = \frac{Q_1(\log v)}{H} v + O(v^{1-\eta})$$

*for some $\eta > 0$, where*

$$Q_1(x) = \frac{L}{1 - e^{-L}} e^{-L \langle \frac{x}{L} \rangle}$$

*$L > 0$ is the largest real number for which*
*$\ln(1/p_1)$ and $\ln(1/p_2)$ are integer multiples of $L$;*
*$H = p_1 \ln(1/p_1) + p_2 \ln(1/p_2)$ is the entropy,*
*$\langle y \rangle = y - \lfloor y \rfloor$ is the fractional part of $y$.*

*Furthermore, if* $\ln p_2/\ln p_1$ *is irrational, then*

$$\mathbf{E}[D_r] = \tilde{S}(v,1) = \frac{\log v}{H} + \frac{H_2}{2H^2} + o(1)$$

*otherwise (i.e.,* $\ln p_2/\lg p_1$ *is rational)*

$$\mathbf{E}[D_r] = \tilde{S}(v,1) = \frac{\log v}{H} + \frac{H_2}{2H^2} + \frac{Q_2(\log v)}{H} + O(v^{-\eta})$$

*for some* $\eta > 0$*, where*

$$Q_2(x) = L \cdot \left(\frac{1}{2} - \left\langle \frac{x}{L} \right\rangle\right)$$

*and* $H_2 = p_1 \ln(1/p_1)^2 + p_2 \ln(1/p_2)^2$.

# Idea of the Proof

The Mellin transform $F^*(s)$ of a function $F(v)$ is

$$F^*(s) = \int_0^\infty F(v) v^{s-1} dv.$$

From the recurrence on $S(v, z)$ we find

$$\tilde{D}^*(s, z) = \frac{1 - z}{s(1 - z p_1^{1-s} - z p_2^{1-s})} - \frac{1}{s}, \qquad \Re(s) < s_0(z),$$

where $s_0(z)$ denotes the real solution of
$zp^{1-s} + zq^{1-s} = 1$.

To find the asymptotics of $\tilde{D}(v, z)$ as $v \to \infty$ we compute the inverse transform of $\tilde{D}^*(s, z)$):

$$\tilde{D}(v, z) = \frac{1}{2\pi i} \lim_{T \to \infty} \int_{\sigma - iT}^{\sigma + iT} \tilde{D}^*(s, z) v^{-s}\, ds,$$

where $\sigma < s_0(z)$.

To determine the polar singularities of the meromorphic continuation of $\tilde{D}^*(s, z)$, we have to analyze the set

$$Z(z) = \{s \in \mathbf{C} : zp^{1-s} + zq^{1-s} = 1\}$$

of all complex roots of $zp^{1-s} + zq^{1-s} = 1$.

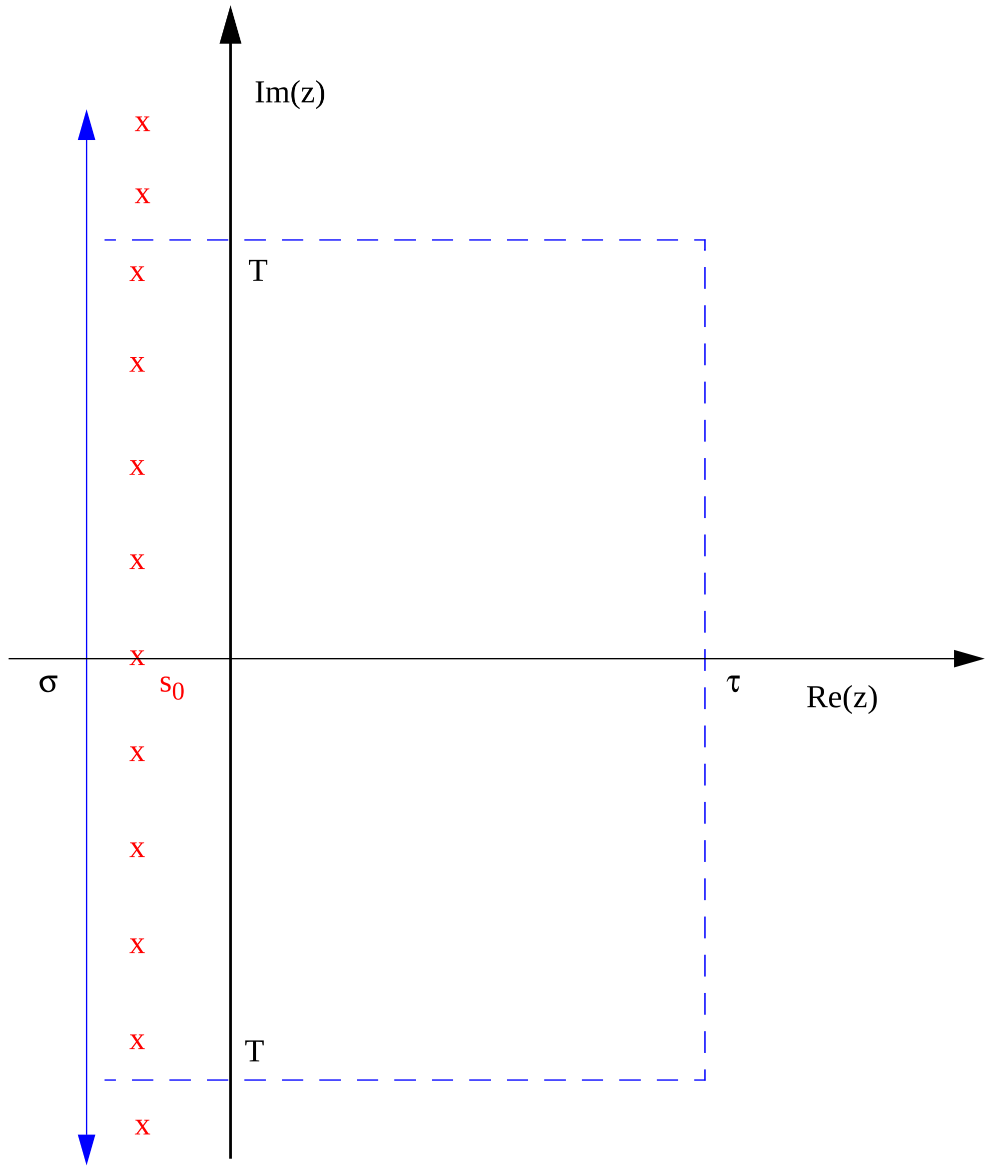



Figure 2: Illustration to the computations of the Mellin transform.

# Difficulties

From Cauchy's residue theorem we obtain
$\tilde{D}(v,z) = \lim_{T\to\infty} F_T(v,z)$ for $\Re(s) < \tau$, where

$$\begin{aligned}
&F_T(v,z)\\
&= - \sum_{s' \in Z(z),\ \Re(s')<\tau, |\Im(s')|>T} \operatorname{Res}(\tilde{D}^*(s,z)\, v^{-s},\ s = s')\\
&+ \frac{1}{2\pi i} \int_{\tau - iT}^{\tau + iT} \left( \frac{1-z}{s(1 - z p_1^{1-s} - z p_2^{1-s})} - \frac{1}{s} \right) v^{-s}\, ds\\
&= - \sum_{s' \in Z(z),\ \Re(s')<\tau, |\Im(s')|>T} \frac{(1-z) v^{-s'}}{z s' p_1^{1-s'} \ln p_1 + z s' p_2^{1-s'} \ln p_2}\\
&+ \frac{1}{2\pi i} \int_{\tau - iT}^{\tau + iT} \left( \frac{1-z}{s(1 - z p_1^{1-s} - z p_2^{1-s})} - \frac{1}{s} \right) v^{-s}\, ds
\end{aligned}$$

provided that the series of residues converges and the limit as $T \to \infty$ of the last integral exists.

The problem is that neither the series nor the integral above are absolutely convergent since the integrand is only of order $1/s$.

## Main Results

**Theorem 2.** *Let $D_r$ and $M_r$ denote the phrase length and the dictionary size in Khodak's construction.*

$$\frac{D_r - \frac{1}{H}\ln M_r}{\sqrt{\left(\frac{H_2}{H^3} - \frac{1}{H}\right)\ln M_r}} \to N(0,1)$$

*where $N(0,1)$ denotes the standard normal distribution. Furthermore,*

$$\mathbf{Var}[D_r] = \left(\frac{H_2}{H^3} - \frac{1}{H}\right)\ln M_r + O(1)$$

*for large $M_r$.*
*In the irrational case*

$$\mathbf{E}[D_r] = \frac{\ln M_r}{H} + \frac{\ln H}{H} + \frac{H_2}{2H^2} + o(1)$$

*and in the rational case (i.e., for $\log p_2/\log p_1 = b/d$)*

$$\mathbf{E}[D_r] = \frac{\ln M_r}{H} + \frac{\ln H}{H} + \frac{H_2}{2H^2} + \frac{Q_2(nv) - \ln Q_1(\ln v)}{H} + O(M_r^{-\eta}).$$

*The above yields (miracle: no oscillations)*

$$Q_2(\ln v) - \log Q_1(\ln v) = -\ln L + \ln(1 - e^{-L}) + \frac{L}{2}.$$

# Redundancy

The average redundancy rate of Tunstall's code is defined as

$$\mathcal{R}_{M_r} = \frac{\ln M_r}{\mathbf{E}[D]} - H$$

As a consequence of our main result we can prove the following.

**Corollary 1.** *Let $\mathcal{D}_r$ denote the dictionary in Khodak's construction of the Tunstall code of size $M_r$. If $\ln p_1 / \ln p_2$ is irrational then*

$$\mathcal{R}_{M_r} = \frac{H}{\ln M_r}\left(-\frac{H_2}{2H} - \ln H\right) + o\left(\frac{1}{\ln M_r}\right).$$

*In the rational case we have*

$$\begin{aligned}\mathcal{R}_{M_r} \quad = \quad & \frac{H}{\ln M_r}\Big( - \frac{H_2}{2H} - \ln H \\ & + \ln L - \ln(e^L - 1) + \frac{L}{2}\Big) + O\left(M_r^{-\eta}\right),\end{aligned}$$

*for some $\eta > 0$, where $L > 0$ is the largest real number for which $\ln(1/p_1)$ and $\ln(1/p_2)$ are integer multiples of $L$ (no oscillation!)*

See Savari and Gallager, Generalized Tunstall codes for sources with memory, IT-43, 1997 (tool: renewal theory).

# Miracles Happen – Oscillations

Oscillations usually occur in redundancy rates.

Lemple-Ziv-78. In this case:

$$r_n = \frac{2h - h\gamma - \frac{1}{2}h_2 + h\beta - h\delta_0(n)}{\log n} + O\left(\frac{\log\log n}{\log^2 n}\right) ,$$

where

$$\begin{aligned} h &= -p\log p - q\log q \\ h_2 &= p\log^2 p + q\log^2 q , \end{aligned}$$

$\gamma = 0.577\ldots$ is the Euler constant,
and $\delta_0(x)$ is a fluctuating functions with small amplitude for $\log p/\log q$ rational, and

$$\lim_{x\to\infty} \delta_0(x) = 0$$

otherwise. Finally, the constant $\beta$ is defined as:

$$\beta = -\sum_{k=1}^{\infty} \frac{p^{k+1}\log p + q^{k+1}\log q}{1 - p^{k+1} - q^{k+1}} .$$

# Shannon and Huffman Codes

**Shannon's Code**. In this case:

$$\bar{R}_n^{SF} = \begin{cases} \frac{1}{2} + o(1) & \alpha \quad \text{irrational} \\ \frac{1}{2} - \frac{1}{M}\left(\langle Mn\beta\rangle - \frac{1}{2}\right) + O(\rho^n) & \alpha = \frac{N}{M}, \end{cases}$$

where $\rho < 1$ and $N, M$ are integers such that $\gcd(N, M) = 1$, and

$$\alpha = \log_2\left(\frac{1-p}{p}\right), \quad \beta = \log_2\left(\frac{1}{1-p}\right).$$

**Huffman's Code**. In this case:

$$\bar{R}_n^{H} = \begin{cases} \frac{3}{2} - \frac{1}{\log 2} + o(1) \approx 0.057304, & \alpha \quad \text{irrational} \\ \frac{3}{2} - \frac{1}{M}\left(\langle \beta M n\rangle - \frac{1}{2}\right) - \frac{1}{M(1-2^{-1/M})}2^{-\langle n\beta M\rangle/M} + O(\rho^n) & \alpha = \frac{N}{M} \end{cases}$$

where $\rho < 1$ and $N, M$ are integers such that $\gcd(N, M) = 1$.