A Master Theorem for Discrete Divide and Conquer Recurrences*

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January 20, 2011

NSF CSoI

SODA, 2011

*Research supported by NSF Science & Technology Center, and Humboldt Foundation.
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Divide and Conquer

Divide and Conquer:
A divide and conquer algorithm splits the input into several smaller subproblems, solving each subproblem separately, and then knitting together to solve the original problem.

Complexity:
A problem of size $n$ is divided into $m \geq 2$ subproblems of size $\lfloor p_j n + \delta_j \rfloor$ and $\lceil p_j n + \delta'_j \rceil$ and each subproblem contributes $b_j, b'_j$ fraction to the final solution; there is a cost $a_n$ associated with combining subproblems.

Total Cost:
The total cost $T(n)$ satisfies the discrete divide and conquer recurrence:

$$T(n) = a_n + \sum_{j=1}^{m} b_j T(\lfloor p_j n + \delta_j \rfloor) + \sum_{j=1}^{m} b'_j T(\lceil p_j n + \delta'_j \rceil) \quad (n \geq 2)$$

where $0 \leq p_j < 1$ (e.g., $\sum_{i=1}^{m} p_i = 1$).
Boncelet’s algorithm is a variable-to-fixed data compression scheme:

1. A variable-to-fixed length encoder partitions a source string over an $m$-ary alphabet into variable-length phrases.
2. Each phrase belongs to a given dictionary.
3. A dictionary is represented by a complete parsing tree.
4. The dictionary entries correspond to the leaves of the parsing tree.
Example: Boncelet’s Algorithm

Boncelet’s algorithm is a variable-to-fixed data compression scheme:

1. A variable-to-fixed length encoder partitions a source string over an $m$-ary alphabet into variable-length phrases.
2. Each phrase belongs to a given dictionary.
3. A dictionary is represented by a complete parsing tree.
4. The dictionary entries correspond to the leaves of the parsing tree.

Example: A binary string ($m = 2$) with symbol probabilities $p_1$ and $p_2$.
The expected phrase length $d(n)$ satisfies:

$$d(n) = 1 + p_1 d(\lceil p_1 n + \delta \rceil) + p_2 d(\lfloor p_2 n - \delta \rfloor)$$
Continuous Relaxation

We relax the discrete nature of the recurrence and consider a continuous version:

\[ T(x) = a(x) + \sum_{j=1}^{m} b_j T(p_j x)), \quad x > 1, \quad b'_j = 0. \]

Akra and Bazzi (1998) proved that

\[ T(x) = \Theta \left( x^{s_0} \left( 1 + \int_{1}^{x} \frac{a(u)}{u^{s_0+1}} du \right) \right) \]

where \( s_0 \) is a unique real root of \( \sum_j b_j p_j^{s_0} = 1 \).
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where \( s_0 \) is a unique real root of \( \sum_j b_j p_j^{s_0} = 1 \).

Indeed, by taking Mellin transform of the relaxed recurrence:

\[ t(s) = \int_{0}^{\infty} T(x) x^{s-1} dx \]

we find (for some \( a(s) \) and \( g(s) \))

\[ t(s) = \frac{a(s) + g(s)}{1 - \sum_{j=1}^{m} b_j p_j^{-s}}. \]

An application of the Wiener-Ikehara theorem leads to

\[ T(x) \sim C x^{s_0} \quad \text{with} \quad C = \frac{a(-s_0) + g(-s_0)}{\sum_j b_j p_j^{s_0} \log(1/p_j)}. \]
For a sequence $c(n)$ define the Dirichlet series as

$$C(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}$$

provided it exists for $\Re(s) > \sigma_c$ for some $\sigma_c \geq -\infty$.

**Theorem 1** (Perron-Mellin Formula). For all $\sigma > \sigma_c$ and all $x > 0$

$$\sum_{n<x} c(n) + \frac{c([x])}{2}[x \in \mathbb{Z}] = \lim_{T \to \infty} \frac{1}{2\pi i} \int_{\sigma-iT}^{\sigma+iT} C(s) \frac{x^s}{s} \, ds.$$

where $[P]$ is 1 if $P$ is a true proposition and 0 otherwise.
Discrete Divide & Conquer Recurrence by Dirichlet Series

For a sequence \( c(n) \) define the Dirichlet series as

\[
C(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}
\]

provided it exists for \( \Re(s) > \sigma_c \) for some \( \sigma_c \geq -\infty \).

**Theorem 1 (Perron-Mellin Formula).** For all \( \sigma > \sigma_c \) and all \( x > 0 \)

\[
\sum_{n<x} c(n) + \frac{c([x])}{2} \mathbb{[}x \in \mathbb{Z}] = \lim_{T \to \infty} \frac{1}{2\pi i} \int_{\sigma-iT}^{\sigma+iT} C(s) \frac{x^s}{s} \, ds.
\]

where \( \mathbb{[}P\] \) is 1 if \( P \) is a true proposition and 0 otherwise.

**Example:** Define \( c(n) = T(n + 2) - T(n + 1) \). Then

\[
T(n) = T(2) + \lim_{T \to \infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \tilde{T}(s) \frac{(n - \frac{3}{2})^s}{s} \, ds
\]

for some \( c > \sigma_{\tilde{T}} \) with

\[
\tilde{T}(s) = \sum_{n=1}^{\infty} \frac{T(n + 2) - T(n + 1)}{n^s}
\]

where \( \Re(s) > \sigma_{\tilde{T}} \).
Let \( a_n \) be a nondecreasing sequence. Define

\[
\tilde{A}(s) = \sum_{n=1}^{\infty} \frac{a_{n+2} - a_{n+1}}{n^s}
\]

which is postulated to exists for \( \Re(s) > \sigma_a \).

**Example.** Define \( a_n = n^\sigma (\log n)^\alpha \). Then

\[
\tilde{A}(s) = \sigma \frac{\Gamma(\alpha + 1)}{(s - \sigma)^{\alpha + 1}} + \frac{\Gamma(\alpha + 1)}{(s - \sigma)^\alpha} + \tilde{F}(s),
\]

where \( \tilde{F}(s) \) is analytic for \( \Re(s) > \sigma - 1 \) and \( \Gamma(s) \) is the gamma function.

Define \( s_0 \) to be the unique real root of

\[
\sum_{j=1}^{m} (b_j + b'_j) p_j^s = 1.
\]

Other zeros depend on the relation among \( \log(1/p_1), \ldots, \log(1/p_m) \).
Definition 1. (i) \( \log(1/p_1), \ldots, \log(1/p_m) \) are rationally related if
\( \log(1/p_1), \ldots, \log(1/p_m) \) are integer multiples of \( L \), that is, \( \log(1/p_j) = n_j L, n_j \in \mathbb{Z}, (1 \leq j \leq m) \).
(ii) Otherwise \( \log(1/p_1), \ldots, \log(1/p_m) \) are irrationally related.

Example. If \( m = 1 \), then we are always in the rationally related case.
For \( m = 2 \), if \( \log(1/p_1)/\log(1/p_2) = m/n, (m, n \text{ integers}) \), then rationally related.

Lemma 1. (i) If \( \log(1/p_1), \ldots, \log(1/p_m) \) are irrationally related, then \( s_0 \) is the only solution on \( \Re(s) = s_0 \).
(ii) If \( \log(1/p_1), \ldots, \log(1/p_m) \) are rationally related, then there are infinitely many solutions
\[
s_k = s_0 + \frac{2\pi ik}{L} \quad (k \in \mathbb{Z})
\]
where \( \log(1/p_j) \) are all integer multiples of \( L \).
Evaluation of $T(n)$: A Bird View

We estimate $T(n)$ by the Cauchy residue theorem.

$$\tilde{T}(s) = \frac{\tilde{A}(s) + B(s)}{1 - \sum_{j=1}^{\infty} (b_j + b'_j) p_j^s},$$

$$T(n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{T}(s) \frac{(n-\frac{3}{2})^s}{s} \, ds$$
Main Master Theorem

**Theorem 2 (Discrete Master Theorem).** Let \(a_n = C n^{\sigma_a} (\log n)^\alpha\) with \(\min\{\sigma, \alpha\} \geq 0\).

(i) If \(\log(1/p_1), \ldots, \log(1/p_m)\) are irrationally related, then

\[
T(n) = \begin{cases} 
C_1 + o(1) & \text{if } \sigma_a \leq 0 \text{ and } s_0 < 0, \\
C_2 \log n + C'_2 + o(1) & \text{if } \sigma_a < s_0 = 0, \\
C_3 (\log n)^{\alpha+1} (1 + o(1)) & \text{if } \sigma_a = s_0 = 0, \\
C_4 n^{s_0} (1 + o(1)) & \text{if } \sigma_a < s_0 \text{ and } s_0 > 0, \\
C_5 n^{s_0} (\log n)^{\alpha+1} (1 + o(1)) & \text{if } \sigma_a = s_0 > 0 \text{ and } \alpha \neq -1, \\
C_5 n^{s_0} \log \log n (1 + o(1)) & \text{if } \sigma_a = s_0 > 0 \text{ and } \alpha = -1, \\
C_6 (\log n)^{\alpha} (1 + o(1)) & \text{if } \sigma_a = 0 \text{ and } s_0 < 0, \\
C_7 n^{\sigma_a} (\log n)^{\alpha} (1 + o(1)) & \text{if } \sigma_a > s_0 \text{ and } \sigma_a > 0.
\end{cases}
\]

(ii) If \(\log(1/p_1), \ldots, \log(1/p_m)\) are rationally related, then \(T(n)\) behaves as in the irrationally related case with the following two exceptions:

\[
T(n) = \begin{cases} 
C_2 \log n + \Psi_2 (\log n) + o(1) & \text{if } \sigma_a < s_0 = 0, \\
\Psi_4 (\log n) n^{s_0} (1 + o(1)) & \text{if } \sigma_a < s_0 \text{ and } s_0 > 0,
\end{cases}
\]

where \(C_2\) is positive and \(\Psi_2(t), \Psi_4(t)\) are periodic functions with period \(L\) (with usually countably many discontinuities).
Extensions and Remarks

1. We can handle any $a_n$ sequence with Dirichlet series $\tilde{A}(s)$:

$$
\tilde{A}(s) = g_0(s) \left( \frac{\log \frac{1}{s-\sigma_a}}{s - \sigma_a} \right)^{\beta_0} + \sum_{j=1}^{J} g_j(s) \left( \frac{\log \frac{1}{s-\sigma_a}}{s - \sigma_a} \right)^{\beta_j} + \tilde{F}(s),
$$

$\tilde{F}(s)$ is analytic, $g_0(\sigma_a) \neq 0$, $\beta_j$ non-negative integers, and $\alpha_0$ real. Then the solution $T(n)$ of the divide and conquer recurrence has the form:

$$
T(n) \sim C n^{\sigma'} (\log n)^{\alpha'} (\log \log n)^{\beta'} \quad \text{or} \quad T(n) \sim \Psi(\log n) n^{s_0}
$$

$\sigma' = \max\{\sigma, s_0\}$, depending whether $\log p_1, \ldots \log p_m$ are irrationally or rationally related.

2. The periodic function $\Psi(t)$ has the following building blocks

$$
\lambda^{-t} \sum_{n \geq 1} B_n \frac{\lambda^{\left[t - \log n \right]/L} + 1}{\lambda - 1}
$$

where $\lambda > 1$ and $B_n$ is such that $\sum_{n \geq 1} B_n \lambda^{-(\log n)/L}$ converges absolutely. This function is discontinuous at

$$
t = \{\log n/L\},
$$

where $\{x\} = x - [x]$ denotes the fractional part of a real number $x$. 
Example 1. Irrationally Related; Case 4:

\[ T(n) = 2T(\lfloor n/2 \rfloor) + 3T(\lfloor n/6 \rfloor) + n \log n \]

Here \( \sigma_a = 1 \) since \( a_n = n \log n \).

The equation

\[ 2 \cdot 2^{-s} + 3 \cdot 6^{-s} = 1 \]

has the (real) solution \( s_0 = 1.402 \ldots > 1 \), and finally \( \log(1/2)/\log(1/6) \) are irrationally related. Thus by our **Master Theorem** Case 4

\[ T(n) \sim Cn^{s_0} \]

for some constant \( C > 0 \).
Examples

Example 2. Irrationally Related; Case 6:
Consider the recurrence

\[ T(n) = 2T(\lfloor n/2 \rfloor) + \frac{8}{9}T(\lfloor 3n/4 \rfloor) + \frac{n^2}{\log n}. \]

Here \( \sigma_a = s_0 = 2 \), and we deal with irrationally related case. Furthermore,

\[ \tilde{A}(s) = s \log \frac{1}{s - 2} + G(s) \]

for \( G(s) \) analytic for \( \Re(s) > 1 \). By Master Theorem Case 6

\[ T(n) \sim Cn^2 \log \log n. \]

Example 3. Rationally Related \((m = 1)\); Case 3:
Next consider

\[ T(n) = T(\lfloor n/2 \rfloor) + \log n. \]

Here \( \sigma_a = s_0 = 0 \), and we have rational case \((m = 1)\). Since

\[ \tilde{A}(s) = \frac{1}{s} + G(s) \]

we conclude

\[ T(n) \sim C(\log n)^2. \]
Example 4: **Karatsuba algorithm**: Rationally Related \((m = 1)\):

\[
T(n) = 3T(\lceil n/2 \rceil) + n
\]

Here, \(s_0 = (\log 3)/(\log 2) = 1.5849 \ldots\) and \(s_0 > \sigma_a = 1\). Thus

\[
T(n) = \Psi(\log n) n^{\frac{\log 3}{\log 2}} \cdot (1 + o(1))
\]

for some periodic function \(\Psi(t)\).
Examples

Example 5. Rationally Related \((m = 1)\). The recurrence

\[
T(n) = \frac{1}{2} T\left(\lfloor n/2 \rfloor \right) + \frac{1}{n}
\]

is not covered by our Master Theorem but our methodology still works. Here \(\sigma_a = s_0 = -1 < 0\). It follows that

\[
T(n) = \frac{C \log n}{n} + \frac{\Psi(\log n)}{n} + o\left(\frac{1}{n}\right)
\]

for a periodic function \(\Psi(t)\).

Example 6: Mergesort. Rationally Related.
The mergesort recurrences are

\[
T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + n - 1,
\]

\[
Y(n) = Y(\lfloor n/2 \rfloor) + Y(\lceil n/2 \rceil) + \lfloor n/2 \rfloor.
\]

Here \(\sigma_a = s_0 = 1\) and we deal with the rationally related case. By our Master Theorem (cf. Flajolet & Golin, 1994)

\[
T(n) = \frac{1}{\log 2} n \log n + n \Psi(\log n) + o(n),
\]

\[
Y(n) = \frac{1}{2 \log 2} n \log n + n \Psi(\log n) + o(n).
\]
Boncellet’s Algorithm Revisited

Let a sequence \( X \) be generated by a memoryless source over alphabet \( \mathcal{A} \) of size \( m \) with symbol probabilities \( p_i, i \in \mathcal{A} \).

Using the Boncellet’s parsing tree, we parse \( X \) into phrases \( \{v_1, \ldots, v_n\} \) of length \( \ell(v_1), \ldots, \ell(v_n) \) with phrase probabilities \( P(v_1), \ldots, P(v_n) \).

**Phrase Length and its Probability Generating Function:**

Let \( D_n \) denote the phrase length and define the probability generating function as

\[
C(n, y) = \mathbb{E}[y^{D_n}]
\]

It satisfies the following discrete divide and conquer recurrence:

\[
C(n, y) = y \sum_{i=1}^{m} p_i C([p_i n + \delta_i], y)
\]

The expected phrase length \( d(n) = \mathbb{E}[D_n] = C'(n, 1) \) satisfies the following discrete divide and conquer recurrence:

\[
d(n) = 1 + \sum_{i=1}^{m} p_i d([p_i n + \delta_i])
\]

with \( d(0) = \cdots = d(m - 1) = 0 \).
Main Results for Boncelet’s Algorithm

Theorem 3. Consider an \( m \)-ary memoryless source with probabilities \( p_i > 0 \) and the entropy rate \( H = \sum_{i=1}^{m} p_i \log(1/p_i) \).

(i) If \( \log(1/p_1), \ldots, \log(1/p_m) \) are irrationally related, then

\[
d(n) = \frac{1}{H} \log n - \frac{\alpha}{H} + o(1),
\]

where

\[
\alpha = E'(0) - H - \frac{H_2}{2H},
\]

\( H_2 = \sum_{i=1}^{m} p_i \log^2 p_i, \) and \( E'(0) \) is the derivative at \( s = 0 \) of a Dirichlet series \( E(s) \) arises from the discrete nature of the recurrence.

(ii) If \( \log(1/p_1), \ldots, \log(1/p_m) \) are rationally related, then

\[
d(n) = \frac{1}{H} \log n - \alpha + \frac{\Psi(\log n)}{H} + O(n^{-\eta})
\]

for some \( \eta > 0 \), where \( \Psi(t) \) is a periodic function of bounded variation that has usually an infinite number of discontinuities.
Redundancy of the Boncelet's Algorithm

Corollary 1. Let $R_n$ denote the redundancy of the Boncelet code:

$$R_n = \frac{\log n}{\mathbb{E}[D_n]} - H = \frac{\log n}{d(n)} - H.$$

(i) If $\log(1/p_1), \ldots, \log(1/p_m)$ are irrationally related, then

$$R_n = \frac{H \alpha}{\log n} + o \left( \frac{1}{\log n} \right).$$

(ii) If $\log(1/p_1), \ldots, \log(1/p_m)$ are rationally related, then

$$R_n = \frac{H \alpha + \Psi(\log n)}{\log n} + o \left( \frac{1}{\log n} \right).$$

Tunstall Code Redundancy:

$$R_n^T = \frac{H}{\log n} \left( -\log H - \frac{H_2}{2H} \right) + o \left( \frac{1}{\log n} \right)$$

for irrational case; in the rational case there is aperiodic function.

Example. Consider $p = 1/3$ and $q = 2/3$. Then one computes $\alpha = E'(0) - H - \frac{H_2}{2H} \approx 0.322$ while for the Tunstall code $-\log H - \frac{H_2}{2H} \approx 0.0496$. 

Theorem 4. Consider a memoryless source generating a sequence of length $n$ parsed by the Boncellet algorithm. If $(p_1, \ldots, p_m)$ is not the uniform distribution, then the phrase length $D_n$ satisfies the central limit law, that is,

$$
\frac{D_n - \frac{1}{H} \log n}{\sqrt{\left( \frac{H_2}{H^3} - \frac{1}{H} \right) \log n}} \to N(0, 1),
$$

where $N(0, 1)$ denotes the standard normal distribution, and

$$\mathbb{E}[D_n] = \frac{\log n}{H} + O(1),$$

$$\text{Var } D_n \sim \left( \frac{H_2}{H^3} - \frac{1}{H} \right) \log n$$

for $n \to \infty$. 

1. From the recurrence we have

\[ \tilde{T}(s) = \tilde{A}(s) + \sum_{j=1}^{m} b_j \sum_{n=1}^{\infty} \frac{T(\lfloor p_j(n + 2) + \delta_j \rfloor) - T(\lfloor p_j(n + 1) + \delta_j \rfloor)}{n^s}. \]

But defining

\[ n = \left\lfloor \frac{k + 2 - \delta_j}{p_j} \right\rfloor - 2 \]

for some integer \( k \), we arrive at

\[ \sum_{n=1}^{\infty} \frac{T(\lfloor p_j(n + 2) + \delta_j \rfloor) - T(\lfloor p_j(n + 1) + \delta_j \rfloor)}{n^s} = G_j(s) + \sum_{k=1}^{\infty} \frac{T(k + 2) - T(k + 1)}{\left(\left\lfloor \frac{k+2-\delta_j}{p_j} \right\rfloor - 2\right)^s}. \]

for an explicit (and simple) analytic function \( G_j(s) \).
Sketch of Proof – Continuation

2. We now compare the last sum to $p_j^s \tilde{T}(s)$ and obtain

$$
\sum_{k=1}^{\infty} \frac{T(k + 2) - T(k + 1)}{\left(\left\lfloor \frac{k+2-\delta_j}{p_j} \right\rfloor - 2 \right)^s} = \sum_{k=1}^{\infty} \frac{T(k + 2) - T(k + 1)}{(k/p_j)^s} = p_j^s \tilde{T}(s) - E_j(s),
$$

where

$$
E_j(s) = \sum_{k=1}^{\infty} (T(k + 2) - T(k + 1)) \left( \frac{1}{(k/p_j)^s} - \frac{1}{\left(\left\lfloor \frac{k+2-\delta_j}{p_j} \right\rfloor - 2 \right)^s} \right).
$$

3. Defining

$$
E(s) = \sum_{j=1}^{m} b_j E_j(s) \quad \text{and} \quad G(s) = \sum_{j=1}^{m} b_j G_j(s)
$$

we finally obtain our final formula

$$
\tilde{T}(s) = \frac{\tilde{A}(s) + G(s) - E(s)}{1 - \sum_{j=1}^{m} b_j p_j^s}.
$$
Sketch of Proof – Binary Boncellet’s Algorithm

1. Let $s_0(y)$ be the real zero of

$$y(p^{s+1} + q^{s+1}) = 1, \quad q = 1 - p.$$ 

2. Define

$$C(s, y) = \sum_{n=1}^{\infty} \frac{C(n + 2, y) - C(n + 1, y)}{n^s}.$$ 

which from the basic recurrence becomes

$$C(s, y) = \frac{(y - 1) - E(s, y)}{1 - y(p^{s+1} + q^{s+1})},$$ 

where $E(s, y)$ converges for $\Re(s) > s_0(y) - 1$ and satisfies $E(0, y) = 0$ and $E(s, 1) = 0$.

3. By Mellin-Perron formula and residue theorem we can prove that

$$C(n, y) = (1 + O(y - 1))n^{s_0(y)}(1 + o(1))$$ 

where

$$s_0(y) = \frac{y - 1}{H} + \left(\frac{H_2}{2H^3} - \frac{1}{H}\right)(y - 1)^2 + O((y - 1)^3).$$
That's It

THANK YOU