A Master Theorem for Discrete Divide and Conquer Recurrences*

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Outline

1. Divide and Conquer
2. Example: Boncelet's Algorithm
3. Continuous Relaxation of the Recurrence
4. Master Theorem
5. Examples
6. Boncelet's Algorithm Revisited
7. Sketch of Proof.

Divide and Conquer

Divide and Conquer:
A divide and conquer algorithm splits the input into several smaller subproblems, solving each subproblem separately, and then knitting together to solve the original problem.

Complexity:
A problem of size n is divided into $m \geq 2$ subproblems of size $\lfloor p_j n + \delta_j \rfloor$ and $\lceil p_j n + \delta'_j \rceil$ and each subproblem contributes b_j, b'_j fraction to the final solution; there is a cost a_n associated with combining subproblems.

Total Cost:
The total cost $T(n)$ satisfies the **discrete divide and conquer** recurrence:

$$T(n) = a_n + \sum_{j=1}^{m} b_j T\left(\lfloor p_j n + \delta_j \rfloor\right) + \sum_{j=1}^{m} b'_j T\left(\left\lceil p_j n + \delta'_j \right\rceil\right) \qquad (n \geq 2)$$

where $0 \leq p_j < 1$ (e.g., $\sum_{i=1}^{m} p_i = 1$).

(**Flajolet** & Golin, *Acta Informatica*, 1994, simpler version for $p_1 = p_2 = 1/2$.)

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Example: Boncelet's Algorithm

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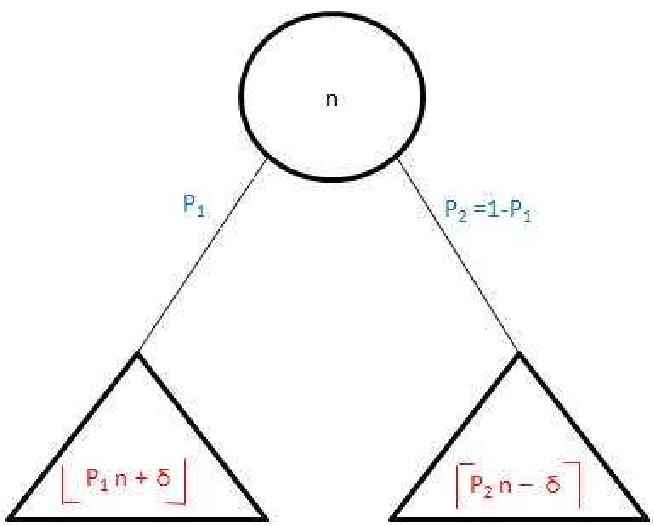
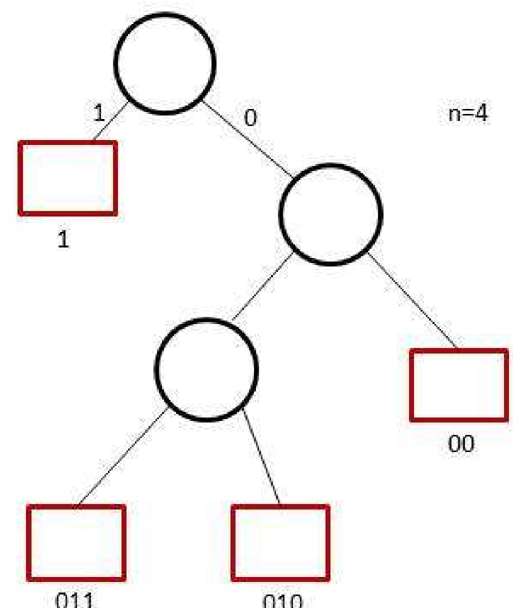
Boncelet's algorithm is a variable-to-fixed block arithmetic data compression coder with low complexity.

1. A variable-to-fixed length encoder partitions a source string over an m-ary alphabet into variable-length phrases.
2. Each phrase belongs to a given dictionary.
3. A dictionary is represented by a complete parsing tree.
4. The dictionary entries correspond to the leaves of the parsing tree.





Note: Tunstall variable-to-fixed scheme requires searching a codebook, so is more complex.

Example: Boncelet's Algorithm Recurrences

Let a sequence X be generated by a memoryless source over alphabet $\mathcal{A}$ of size m with symbol probabilities p_i, $i \in \mathcal{A}$.

Using the Boncelet's parsing tree, we parse X into phrases $\{v_1, \ldots v_n\}$ of length $\ell(v_1), \ldots, \ell(v_n)$ with phrase probabilities $P(v_1), \ldots, P(v_n)$.

Phrase Length and its Probability Generating Function:
Let D_n be the phrase length while its probability generating function is $C(n, y) = \mathbf{E}[y^{D_n}]$. It satisfies the following divide & conquer recurrence:

$$C(n, y) = y \sum_{i=1}^{m} p_i C([p_i n + \delta_i], y)$$

where $[x]$ is the quantized value of x.
The average redundancy R_n of the Boncelet code is (H is the entropy):

$$R_n = \frac{\log n}{\mathbf{E}[D_n]} - H = \frac{\log n}{d(n)} - H.$$

The expected phrase length $d(n) = \mathbf{E}[D_n] = C'(n, 1)$ satisfies the following recurrence with $d(0) = \cdots = d(m-1) = 0$

$$d(n) = 1 + \sum_{i=1}^{m} p_i d([p_i n + \delta_i])$$

These are **discrete divide & conquer recurrences**.

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Continuous Relaxation

We relax the discrete nature of the recurrence and consider a continuous version:

$$T(x) = a(x) + \sum_{j=1}^{m} b_j T(p_j x)), \quad x > 1, \quad b_j' = 0.$$

Akra and Bazzi (1998) proved that

$$T(x) = \Theta\left(x^{s_0}\left(1 + \int_1^x \frac{a(u)}{u^{s_0+1}} du\right)\right)$$

where s_0 is a unique real root of $\sum_j b_j {p_j}^{s_0} = 1$.

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where s_0 is a unique real root of $\sum_j b_j p_j^{s_0} = 1$.

Indeed, by taking Mellin transform of the relaxed recurrence:

$$t(s) = \int_0^\infty T(x) x^{s-1} dx$$

we find (for some $a(s)$ and $g(s)$)

$$t(s) = \frac{a(s) + g(s)}{1 - \sum_{j=1}^m b_j p_i^{-s}}.$$

An application of the Wiener-Ikehara theorem leads to

$$T(x) \sim C x^{s_0} \quad \text{with} \quad C = \frac{a(-s_0) + g(-s_0)}{\sum_j b_j p_j^{s_0} \log(1/p_j)}.$$

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Discrete Divide & Conquer Recurrence by Dirichlet Series

For a sequence $c(n)$ define the Dirichlet series as

$$C(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}$$

provided it exists for $\Re(s) > \sigma_c$ for some $\sigma_c \geq -\infty$.
Theorem 1 (Perron-Mellin Formula). *For all $\sigma > \sigma_c$ and all $x > 0$*

$$\sum_{n<x} c(n) + \frac{c(\lfloor x \rfloor)}{2} [\![x \in \mathbb{Z}]\!] = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{\sigma - iT}^{\sigma + iT} C(s) \frac{x^s}{s} \, ds.$$

where $[\![P]\!]$ is 1 if P is a true proposition and 0 otherwise.

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Theorem 1 (Perron-Mellin Formula). *For all* $\sigma > \sigma_c$ *and all* $x > 0$

$$\sum_{n<x} c(n) + \frac{c(\lfloor x \rfloor)}{2} [\![x \in \mathbb{Z}]\!] = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{\sigma-iT}^{\sigma+iT} C(s) \frac{x^s}{s}\, ds.$$

where $[\![P]\!]$ *is* 1 *if* P *is a true proposition and* 0 *otherwise.*

Example: Define $c(n) = T(n+2) - T(n+1)$. Then

$$T(n) = T(2) + \lim_{T\to\infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \widetilde{T}(s) \frac{(n-\frac{3}{2})^s}{s}\, ds$$

for some $c > \sigma_{\widetilde{T}}$ with

$$\widetilde{T}(s) = \sum_{n=1}^{\infty} \frac{T(n+2) - T(n+1)}{n^s}.$$

where $\Re(s) > \sigma_{\widetilde{T}}$.

Assumptions

Let a_n be a nondecreasing sequence. Define

$$\widetilde{A}(s) = \sum_{n=1}^{\infty} \frac{a_{n+2} - a_{n+1}}{n^s}$$

which is postulated to exists for $\Re(s) > \sigma_a$.

Example. Define $a_n = n^\sigma (\log n)^\alpha$. Then

$$\widetilde{A}(s) = \sigma \frac{\Gamma(\alpha+1)}{(s-\sigma)^{\alpha+1}} + \frac{\Gamma(\alpha+1)}{(s-\sigma)^{\alpha}} + \tilde{F}(s),$$

where $\tilde{F}(s)$ is analytic for $\Re(s) > \sigma - 1$ and $\Gamma(s)$ is the gamma function.

Define s_0 to be the **unique real** root of

$$\sum_{j=1}^{m} (b_j + b_j') \, p_j{}^s = 1.$$

Other zeros depend on the relation among $\log(1/p_1), \ldots, \log(1/p_m)$.

Rationally and Irrationally Related Numbers

Definition 1. (i) $\log(1/p_1), \ldots, \log(1/p_m)$ *are rationally related if* $\log(1/p_1), \ldots, \log(1/p_m)$ *are integer multiples of* L*, that is,* $\log(1/p_j) = n_j L$*,* $n_j \in \mathbb{Z}$*,* $(1 \leq j \leq m)$*.*
(ii) *Otherwise* $\log(1/p_1), \ldots, \log(1/p_m)$ *are irrationally related.*

Example. If $m = 1$, then we are always in the rationally related case.
For $m = 2$, if $\log(1/p_1)/\log(1/p_2) = m/n$, $(m, n$ integers), then rationally related.

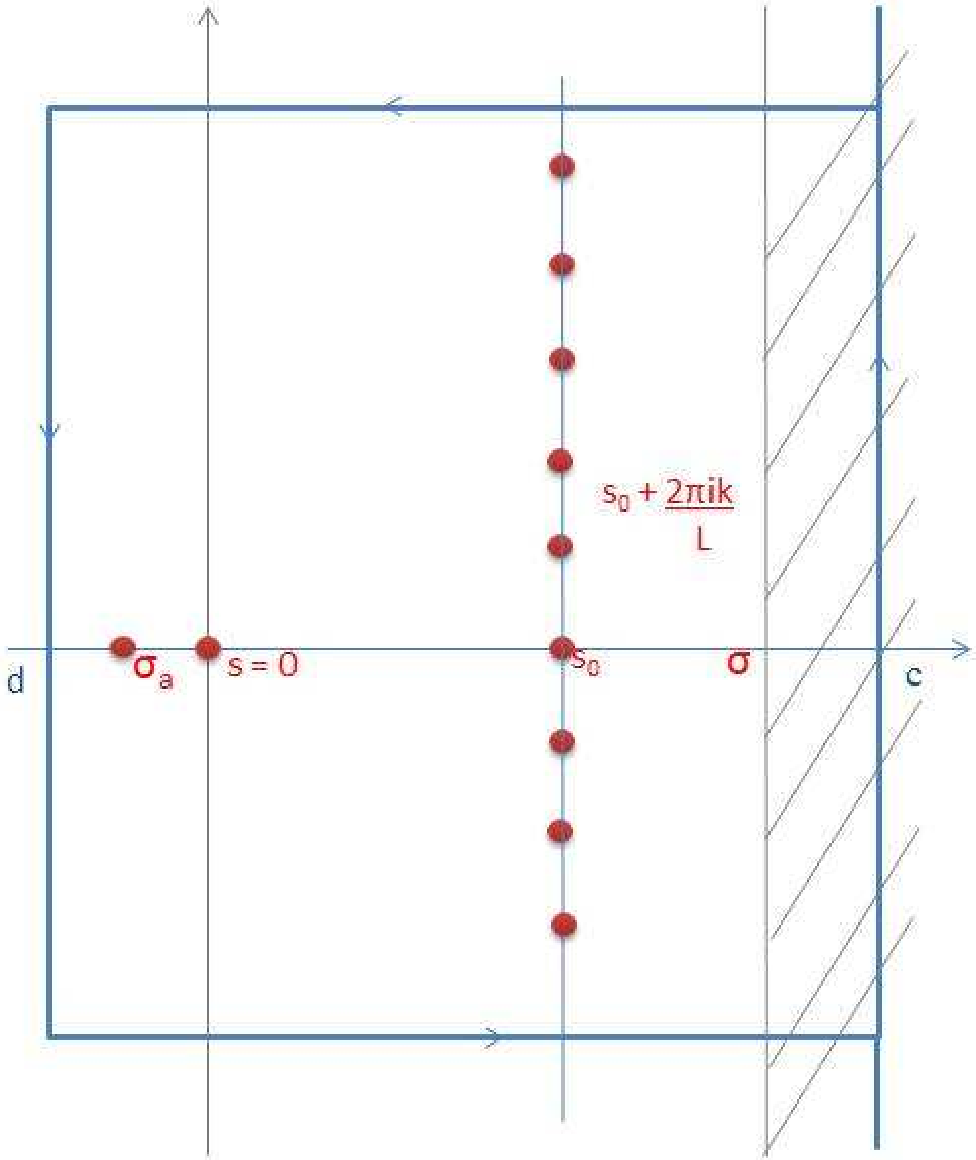
Lemma 1. (i) *If* $\log(1/p_1), \ldots, \log(1/p_m)$ *are irrationally related, then* s_0 *is the only solution on* $\Re(s) = s_0$*.*

(ii) *If* $\log(1/p_1), \ldots, \log(1/p_m)$ *are rationally related, then there are infinitely many solutions*

$$s_k = s_0 + \frac{2\pi i k}{L} \qquad (k \in \mathbb{Z})$$

where $\log(1/p_j)$ *are all integer multiples of* L*.*

Evaluation of $T(n)$: A Bird View



$$\widetilde{T}(s) = \frac{\widetilde{A}(s)+B(s)}{1-\sum_{j=1}^{m}(b_j+b_j')\,p_j^s},$$

$$T(n) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \widetilde{T}(s)\frac{(n-\frac{3}{2})^s}{s}\,ds$$

Main Master Theorem

Theorem 2 (Discrete Master Theorem). *Let* $a_n = Cn^{\sigma_a}(\log n)^\alpha$ *with* $\min\{\sigma, \alpha\} \geq 0$.

(i) If $\log(1/p_1), \dots, \log(1/p_m)$ *are irrationally related, then*

$$T(n) = \begin{cases} C_1 + o(1) & \text{if } \sigma_a \leq 0 \text{ and } s_0 < 0, \\ C_2 \log n + C_2' + o(1) & \text{if } \sigma_a < s_0 = 0, \\ C_3(\log n)^{\alpha+1}(1 + +o(1)) & \text{if } \sigma_a = s_0 = 0 \\ C_4\, n^{s_0} \cdot (1 + o(1)) & \text{if } \sigma_a < s_0 \text{ and } s_0 > 0, \\ C_5 n^{s_0}(\log n)^{\alpha+1} \cdot (1 + o(1)) & \text{if } \sigma_a = s_0 > 0 \text{ and } \alpha \neq -1, \\ C_5 n^{s_0} \log\log n \cdot (1 + o(1)) & \text{if } \sigma_a = s_0 > 0 \text{ and } \alpha = -1, \\ C_6(\log n)^\alpha(1 + o(1)) & \text{if } \sigma_a = 0 \text{ and } s_0 < 0, \\ C_7 n^{\sigma_a}(\log n)^\alpha \cdot (1 + o(1)) & \text{if } \sigma_a > s_0 \text{ and } \sigma_a > 0. \end{cases}$$

(ii) *If* $\log(1/p_1), \dots, \log(1/p_m)$ *are rationally related, then* $T(n)$ *behaves as in the irrationally related case with the following two exceptions:*

$$T(n) = \begin{cases} C_2 \log n + \Psi_2(\log n) + o(1) & \text{if } \sigma_a < s_0 = 0, \\ \Psi_4(\log n)\, n^{s_0} \cdot (1 + o(1)) & \text{if } \sigma_a < s_0 \text{ and } s_0 > 0, \end{cases}$$

where C_2 *is positive and* $\Psi_2(t)$, $\Psi_4(t)$ *are periodic functions with period* L *(with usually countably many discontinuities).*

Extensions and Remarks

1. We can handle any a_n sequence with Dirichlet series $\widetilde{A}(s)$:

$$\widetilde{A}(s) = g_0(s)\frac{\left(\log\frac{1}{s-\sigma_a}\right)^{\beta_0}}{(s-\sigma_a)^{\alpha_0}} + \sum_{j=1}^{J} g_j(s)\frac{\left(\log\frac{1}{s-\sigma_a}\right)^{\beta_j}}{(s-\sigma_a)^{\alpha_j}} + \tilde{F}(s),$$

$\tilde{F}(s)$ is analytic, $g_0(\sigma_a) \neq 0$, β_j non-negative integers, and α_0 real.
Then (under some additional conditions on the Fourier series of $\tilde{A}(s)$):

$$T(n) \sim C\, n^{\sigma'} (\log n)^{\alpha'} (\log\log n)^{\beta'} \quad or \quad T(n) \sim \Psi(\log n)\, n^{s_0}$$

$\sigma' = \max\{\sigma, s_0\}$), depending whether $\log p_1, \ldots \log p_m$ are irrationally or rationally related.

2. The periodic function $\Psi(t)$ has the following building blocks

$$\lambda^{-t} \sum_{n\geq 1} B_n \frac{\lambda^{\left\lfloor t-\frac{\log n}{L}\right\rfloor+1}}{\lambda - 1}$$

where $\lambda > 1$ and B_n is such that $\sum_{n\geq 1} B_n \lambda^{-(\log n)/L}$ converges absolutely.
This function is discontinuous at

$$t = \{\log n/L\},$$

where $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of a real number x.

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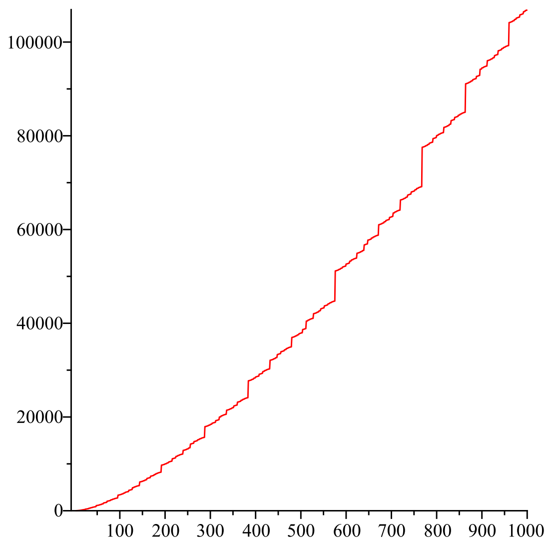
Examples

Example 1. **Irrationally Related; Case 4**:



$$T(n) = 2\,T(\lfloor n/2 \rfloor) + 3\,T(\lfloor n/6 \rfloor) + n \log n$$

Here $\sigma_a = 1$ since $a_n = n \log n$.
The equation

$$2 \cdot 2^{-s} + 3 \cdot 6^{-s} = 1$$

has the (real) solution $s_0 = 1.402\ldots > 1$, and finally $\log(1/2)/\log(1/6)$ are irrationally related. Thus by our **Master Theorem** Case 4

$$T(n) \sim Cn^{s_0}$$

for some constant $C > 0$

Examples

Example 2. **Irrationally Related; Case 6**:
Consider the recurrence

$$T(n) = 2\,T(\lfloor n/2 \rfloor) + \frac{8}{9}T(\lfloor 3n/4 \rfloor) + \frac{n^2}{\log n}.$$

Here $\sigma_a = s_0 = 2$, and we deal with irrationally related case. Furthermore,

$$\widetilde{A}(s) = s \log \frac{1}{s-2} + G(s)$$

for $G(s)$ analytic for $\Re(s) > 1$. By Master Theorem Case 6

$$T(n) \sim C n^2 \log\log n.$$

Example 3. **Rationally Related ($m = 1$); Case 3**:
Next consider

$$T(n) = T(\lfloor n/2 \rfloor) + \log n.$$

Here $\sigma_a = s_0 = 0$, and we have rational case ($m = 1$). Since

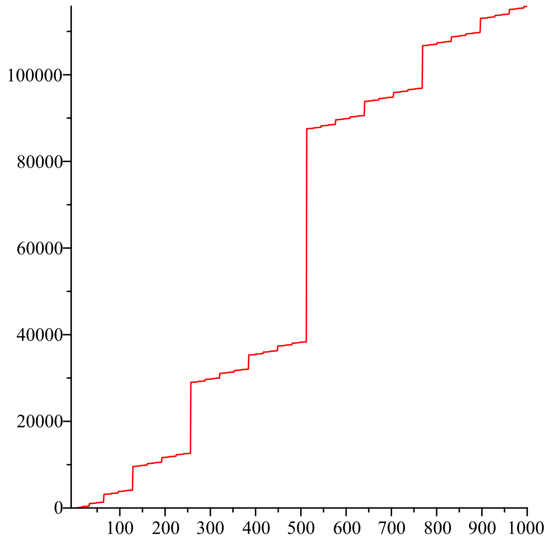
$$\widetilde{A}(s) = \frac{1}{s} + G(s)$$

we conclude

$$T(n) \sim C(\log n)^2.$$

Examples

Example 4: Karatsuba algorithm: Rationally Related $(m = 1)$:



$$T(n) = 3T(\lceil n/2 \rceil) + n$$

Here, $s_0 = (\log 3)/(\log 2) = 1.5849\ldots$ and $s_0 > \sigma_a = 1$. Thus

$$T(n) = \Psi(\log n)\, n^{\frac{\log 3}{\log 2}} \cdot (1 + o(1))$$

for some periodic function $\Psi(t)$.

Examples

Example 5. **Rationally Related** $(m = 1)$. The recurrence

$$T(n) = \frac{1}{2} T(\lfloor n/2 \rfloor) + \frac{1}{n}$$

is not covered by our Master Theorem but our methodology still works. Here $\sigma_a = s_0 = -1 < 0$. It follows that

$$T(n) = C \frac{\log n}{n} + \frac{\Psi(\log n)}{n} + o\left(\frac{1}{n}\right)$$

for a periodic function $\Psi(t)$.

Example 6: Mergesort. **Rationally Related**.
The mergesort recurrences are

$$T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + n - 1,$$

$$Y(n) = Y(\lfloor n/2 \rfloor) + Y(\lceil n/2 \rceil) + \lfloor n/2 \rfloor.$$

Here $\sigma_a = s_0 = 1$ and we deal with the rationally related case. By our Master Theorem (cf. Flajolet & Golin, 1994)

$$T(n) = \frac{1}{\log 2} n \log n + n \Psi(\log n) + o(n),$$

$$Y(n) = \frac{1}{2 \log 2} n \log n + n \Psi(\log n) + o(n).$$

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Using the Boncelet's parsing tree, we parse X into phrases $\{v_1, \ldots v_n\}$ of length $\ell(v_1), \ldots, \ell(v_n)$ with phrase probabilities $P(v_1), \ldots, P(v_n)$.

Phrase Length and its Probability Generating Function:
Let D_n denote the phrase length and define the probability generating function as

$$C(n, y) = \mathbf{E}[y^{D_n}]$$

It satisfies the following discrete divide and conquer recurrence:

$$C(n, y) = y \sum_{i=1}^{m} p_i C([p_i n + \delta_i], y)$$

The expected phrase length $d(n) = \mathbf{E}[D_n] = C'(n, 1)$ satisfies the following discrete divide and conquer recurrence:

$$d(n) = 1 + \sum_{i=1}^{m} p_i d([p_i n + \delta_i])$$

with $d(0) = \cdots = d(m-1) = 0$.

Main Results for Boncelet's Algorithm

Theorem 3. *Consider an m-ary memoryless source with probabilities $p_i > 0$ and the entropy rate $H = \sum_{i=1}^{m} p_i \log(1/p_i)$.*

(i) *If $\log(1/p_1), \ldots \log(1/p_m)$ are irrationally related, then*

$$d(n) = \frac{1}{H} \log n - \frac{\alpha}{H} + o(1),$$

where

$$\alpha = E'(0) - H - \frac{H_2}{2H},$$

$H_2 = \sum_{i=1}^{m} p_i \log^2 p_i$, and $E'(0)$ is the derivative at $s = 0$ of a Dirichlet series $E(s)$ arises from the **discrete** *nature of the recurrence.*

(ii) *If $\log(1/p_1), \ldots \log(1/p_m)$ are rationally related, then*

$$d(n) = \frac{1}{H} \log n - \frac{\alpha + \Psi(\log n)}{H} + O(n^{-\eta})$$

for some $\eta > 0$, where $\Psi(t)$ is a periodic function of bounded variation that has usually an infinite number of discontinuities.

Redundancy of the Boncelet's Algorithm

Corollary 1. *Let R_n denote the redundancy of the Boncelet code:*

$$R_n = \frac{\log n}{\mathbf{E}[D_n]} - H = \frac{\log n}{d(n)} - H.$$

(i) *If* $\log(1/p_1), \ldots \log(1/p_m)$ *are irrationally related, then*

$$R_n = \frac{H\alpha}{\log n} + o\left(\frac{1}{\log n}\right).$$

(ii) *If* $\log(1/p_1), \ldots \log(1/p_m)$ *are rationally related, then*

$$R_n = \frac{H\alpha + \Psi(\log n)}{\log n} + o\left(\frac{1}{\log n}\right).$$

Tunstall Code Redundancy:

$$R_n^T = \frac{H}{\log n}\left(-\log H - \frac{H_2}{2H}\right) + o\left(\frac{1}{\log n}\right)$$

for irrational case; in the rational case there is aperiodic function.

Example. Consider $p = 1/3$ and $q = 2/3$. Then one computes $\alpha = E'(0) - H - \frac{H_2}{2H} \approx 0.322$ while for the Tunstall code $-\log H - \frac{H_2}{2H} \approx 0.0496.$

Limiting Distribution for the Phrase length

Theorem 4. *Consider a memoryless source generating a sequence of length n parsed by the Boncelet algorithm. If $(p_1, \ldots, p_m)$ is not the uniform distribution, then the phrase length D_n satisfies the* **central limit law**, *that is,*

$$\frac{D_n - \frac{1}{H}\log n}{\sqrt{\left(\frac{H_2}{H^3} - \frac{1}{H}\right)\log n}} \to N(0,1),$$

where $N(0,1)$ denotes the standard normal distribution, $H_2 = \sum_{i=1}^{m} p_i \log^2 p_i$, and and

$$\mathbf{E}[D_n] = \frac{\log n}{H} + O(1),$$

$$\mathbb{V}\text{ar}\, D_n \sim \left(\frac{H_2}{H^3} - \frac{1}{H}\right)\log n$$

for $n \to \infty$.

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Sketch of Proof

1. From the recurrence we have

$$\widetilde{T}(s) = \widetilde{A}(s) + \sum_{j=1}^{m} b_j \sum_{n=1}^{\infty} \frac{T\left(\lfloor p_j(n+2)+\delta_j \rfloor\right) - T\left(\lfloor p_j(n+1)+\delta_j \rfloor\right)}{n^s}.$$

But defining

$$n = \left\lfloor \frac{k+2-\delta_j}{p_j} \right\rfloor - 2$$

for some integer k for which we have
$\lfloor p_j(n+1)+\delta_j \rfloor = k+1$ and $\lfloor p_j(n+2)+\delta_j \rfloor = k+2$. Then

$$\sum_{n=1}^{\infty} \frac{T\left(\lfloor p_j(n+2)+\delta_j \rfloor\right) - T\left(\lfloor p_j(n+1)+\delta_j \rfloor\right)}{n^s} = G_j(s) + \sum_{k=1}^{\infty} \frac{T(k+2) - T(k+1)}{\left(\left\lfloor \frac{k+2-\delta_j}{p_j} \right\rfloor - 2\right)^s}.$$

for an explicit (and simple) analytic function $G_j(s)$, namely

$$G_j(s) = \sum_{3p_j+\delta_j-2 \le k \le 0} \frac{T(k+2) - T(k+1)}{\left(\left\lfloor \frac{k+2-\delta_j}{p_j} \right\rfloor - 2\right)^s}.$$

Sketch of Proof – Continuation

2. We now compare the last sum to $p_j^s \widetilde{T}(s)$ and obtain

$$\sum_{k=1}^{\infty} \frac{T(k+2)-T(k+1)}{\left(\left\lfloor \frac{k+2-\delta_j}{p_j} \right\rfloor - 2\right)^s} = \sum_{k=1}^{\infty} \frac{T(k+2)-T(k+1)}{(k/p_j)^s} - E_j(s) = p_j^s \widetilde{T}(s) - E_j(s),$$

where

$$E_j(s) = \sum_{k=1}^{\infty} (T(k+2)-T(k+1)) \left(\frac{1}{(k/p_j)^s} - \frac{1}{\left(\left\lfloor \frac{k+2-\delta_j}{p_j} \right\rfloor - 2\right)^s} \right).$$

3. Defining

$$E(s) = \sum_{j=1}^{m} b_j E_j(s) \quad \text{and} \quad G(s) = \sum_{j=1}^{m} b_j G_j(s)$$

we finally obtain our final formula

$$\widetilde{T}(s) = \frac{\widetilde{A}(s) + G(s) - E(s)}{1 - \sum_{j=1}^{m} b_j\, p_j^s}.$$

Asymptotics – Tauberian Theorem

For any sequence $c(n)$ with Dirichlet series $C(s)$ define

$$\overline{c}(v) = \sum_{n \le v} c(n).$$

Notice that the Mellin-Stieltjes transform of $C(s)$ becomes

$$C(s) = \sum_{n \ge 1} c(n) n^{-s} = \int_{1-}^{\infty} v^{-s} d\overline{c}(v) = s \int_{1}^{\infty} \overline{c}(v) v^{-s-1}\, dv.$$

Theorem 5 (Wiener-Ikehara). *Suppose that for some constant $A_0 > 0$, the analytic function*

$$F(s) = C(s) - \frac{A_0}{s-1} \qquad (\Re(s) > 1)$$

has a continuous extension to the closed half-plane $\Re(s) \ge 1$. Then

$$\overline{c}(v) \sim A_0 v, \qquad v \to \infty.$$

More general version by Delange that covers singularities of algebraic-logarithmic type.

Asymptotics – Perron-Mellin Formula

Inn order to provide error term and second order terms, one needs to use the Perron-Mellin formula:

$$T(n) = T(2) + \lim_{T\to\infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \widetilde{T}(s) \frac{(n-\frac{3}{2})^s}{s}\, ds$$

Unfortunately, the integrals and series (of residues) are not absolutely convergent because of the terms $1/s$.

To remedy it we consider the auxiliary function (for any sequence $(c(n))$)

$$\overline{c}_1(v) = \int_0^v \left(\sum_{n \le w} c(n) \right) dw$$

which is also given by

$$\overline{c}_1(v) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} C(s) \frac{v^{s+1}}{s(s+1)}\, ds.$$

But to recover $\overline{c}(v)$, and then $T(n)$, we need a Wiener-Ikehara Tauberian result.

Asymptotics – Rationally Related Case

Previous methods generally cannot handle infinitely many poles on the line $\Re(s) = s_0$! That is, it is not true that

$$\overline{c}_1(v) = \int_0^v \overline{c}(w)\, dw \sim \Psi_1(\log v) \cdot v^{s_0+1}$$

implies $\overline{c}(v) \sim \Psi(\log v) \cdot v^{s_0}$.

Suppose that $\log p_j = -n_j L$ for some real $L > 0$. In our case, we replace the denominator $1 - \sum_{j=1}^m b_j p_j^s$ with a single real root $z_0 = e^{-Ls_0}$ by

$$1 - \sum_{j=1}^{m} b_j z^{n_j} = (1 - e^{Ls_0} z) P(z), \quad P(z) \text{ polynomial.}$$

Then we prove the following

$$\frac{1}{2\pi i} \lim_{T\to\infty} \int_{c-iT}^{c+iT} \frac{1}{1 - e^{-Ls}\lambda} \frac{x^s}{s}\, ds = \frac{\lambda^{\left\lfloor \frac{\log x}{L} \right\rfloor + 1} - 1}{\lambda - 1} - \frac{1}{2} \lambda^{\left\lfloor \frac{\log x}{L} \right\rfloor} [\![\log x / L \in \mathbb{Z}]\!].$$

where $\left\lfloor \frac{\log x}{L} \right\rfloor$ lead to fluctuations.

Sketch of Proof – Binary Boncelet's Algorithm

1. Define

$$C(s,y)=\sum_{n=1}^{\infty}\frac{C(n+2,y)-C(n+1,y)}{n^s}.$$

which from the basic recurrence becomes

$$C(s,y)=\frac{(y-1)-E(s,y)}{1-y(p^{s+1}+q^{s+1})},$$

where $E(s,y)$ converges (in the right half a plane) and satisfies $E(0,y)=0$ and $E(s,1)=0$.

2. Let $s_0(y)$ be the real zero of

$$y(p^{s+1}+q^{s+1})=1, \quad q=1-p.$$

3. By Mellin-Perron formula and residue theorem we can prove that

$$C(n,y)=(1+O(y-1))n^{s_0(y)}(1+o(1))$$

where

$$s_0(y)=\frac{y-1}{H}+\left(\frac{H_2}{2H^3}-\frac{1}{H}\right)(y-1)^2+O((y-1)^3).$$

Continuation

4. By setting $y = e^{t/(\log n)^{1/2}}$ we obtain

$$
\begin{aligned}
n^{s_0(y)} &= \exp\left(\log n \left(\frac{y-1}{H} - \left(\frac{1}{H} - \frac{H_2}{2H^3}\right)(y-1)^2 + O(|y-1|^3)\right)\right) \\
&= \exp\left(\frac{1}{H} t\sqrt{\log n} + \frac{1}{H}\frac{t^2}{2} - \left(\frac{1}{H} - \frac{H_2}{2H^3}\right) t^2 + O(t^3/\sqrt{\log n})\right) \\
&= \exp\left(\frac{1}{H} t\sqrt{\log n} + \left(\frac{H_2}{H^3} - \frac{1}{H}\right)\frac{t^2}{2} + O(t^3/\sqrt{\log n})\right)
\end{aligned}
$$

and consequently

$$
\mathbb{E}\left[e^{D_n t/\sqrt{\log n}}\right] = C\left(n, e^{t/\sqrt{\log n}}\right) = \exp\left(\frac{1}{H} t\sqrt{\log n} + \left(\frac{H_2}{H^3} - \frac{1}{H}\right)\frac{t^2}{2}\right)(1+o(1)).
$$

arriving at

$$
\begin{aligned}
\mathbb{E}\left[e^{t(D_n - \frac{1}{H}\log n)/\sqrt{\log n}}\right] &= e^{-(t/H)\sqrt{\log n}}\,\mathbb{E}\left[e^{D_n t/\sqrt{\log n}}\right] \\
&= e^{\frac{t^2}{2}\left(\frac{H_2}{H^3} - \frac{1}{H}\right)} + o(1).
\end{aligned}
$$

By Goncharev's theorem, this completes the proof.

That's It

THANK YOU