Is the Internet Fractal? The Multicast Power Law Revisited*

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Outline

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Multicasting over the Internet

In **multicast communication** senders transmit to logical address while receivers join a logical group. Multicast routing ensures that only a **single copy of a packet destined to multiple destinations traverses each link**, so that the overall traffic load is reduced significantly.

The main question we want to address is:

By how much the overall traffic is reduced in the multicast communication.

NOTE. By **global multicast tree** we mean a tree composed of the shortest paths from the source to all the routing nodes of the network G.

Formulation of the Problem

Define:

- The **number of routing nodes** is denoted as N.
- The **number of multicast destination hosts** is equal to n. The number of destination hosts n can be bigger than the number of routing nodes N since multiple destinations hosts may be connected to a routing node (e.g., a LAN network). Define

$$a = \frac{n}{N}.$$

- Let $L(n)$ be the **average** number of links in the multicast tree needed to reach n randomly selected destination hosts.
- The **average** number of links in a unicast path is denoted as U. Hence nU is the total number of links used to deliver packets to **all** n destination hosts.
- The **reduction ratio** is defined as

$$R(n) = \frac{L(n)}{U}.$$

Chuang-Sirbu Power Law

In "Pricing Multicast Communications: A Cost-based Approach" INET'98, Chunag and Sirbu (CMU) observed **experimentally** that

$$R(n) = Cn^{0.8}, \quad a \ll 1 \ (i.e., n \ll N),$$

where C is a constant that does not change significantly with the size of the network.

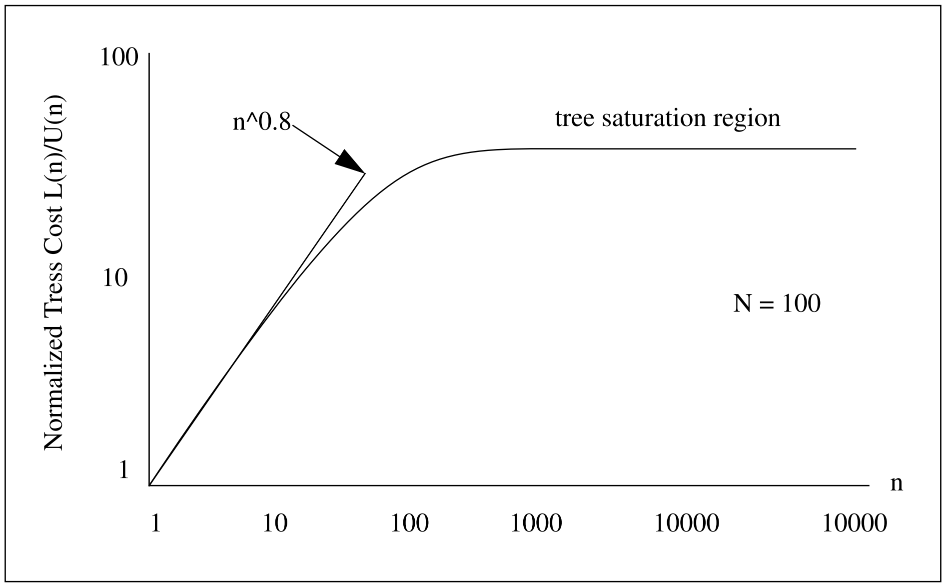

Figure 1: This is Figure 7 from Chunag and Sirbu showing the phase transition of the ratio of the number of links traverse in multicast and the average path length in unicast versus the number of destinations n.

Regular Complete Multicast Tree

Let the multicast tree be a **complete** V-ary tree (all nodes but the leaves have degree V) of depth D i.e., all leaves are located on level D. This model was proposed by: Phillips, Shenker, Tangmunarunkit, SIGCOMM,99.

We assume that all routing nodes are located at leaf nodes. Clearly there the number of leaves N satisfies

$$N = V^D.$$

As before, the multicast group consist of n hosts and $a = n/N$.

Average Number of Links

We estimate now the average number, $L(n)$, of links that connect n randomly selected hosts (leaves).

Observe that at level k of the tree, $1 \leq k \leq D$, there are V^k links; the probability that a particular link is in the multicast tree after n destination hosts have been selected is

$$1 - (1 - 1/V^k)^n.$$

Thus the average number of links in the multicast tree is

$$L(n) = \sum_{k=1}^{D=\log_V N} V^k \left(1 - \left(1 - V^{-k}\right)^n\right).$$

Our goal is to estimate $L(n)$ asymptotically as $n \to \infty$ for fixed a, as well as for $a \to 0$, $a \to 1$ and $a \to \infty$.

How to Estimate $L(n)$?

A standard way of estimating sums like this is to apply the Mellin transform to the following approximation of $L(n)$:

$$\sum_{k=1}^{D=\log_V N} V^k \left(1 - \left(1 - V^{-k}\right)^n\right) \approx \sum_{k=1}^{\infty} V^k \left(1 - e^{-nV^{-k}}\right).$$

But it does not work in this case ...

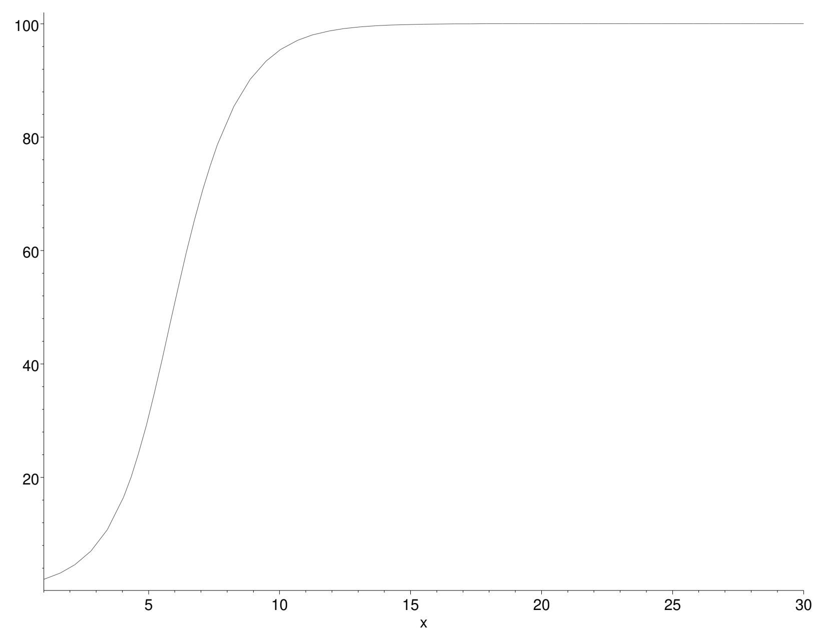

Figure 2: Plot of $2^x(1 - (1 - (1 - 2^{-x})^{100})$.

Estimation of $L(n)$ Revisited

We proceed as follows (for a fixed):

$$
\begin{aligned}
L(n) \quad &= \quad \sum_{k=1}^{D} V^k \left(1 - \left(1 - V^{-k}\right)^n\right) \\
&\overset{l=D-k}{=} \quad V^D \sum_{l=0}^{D-1} V^{-l} \left(1 - \left(1 - \frac{V^l}{V^D}\right)^n\right) \\
&= \quad V^D \left(\frac{1 - V^{-D}}{1 - V^{-1}}\right) - V^D \sum_{l=0}^{D-1} V^{-l} \left(1 - \frac{V^l}{V^D}\right)^n \\
&= \quad N \left(\frac{1 - N^{-\bar{\theta}}}{1 - V^{-\bar{\theta}}}\right) - \sum_{l=1}^{\infty} V^{-l} e^{-aV^l} + O(1/\log n).
\end{aligned}
$$

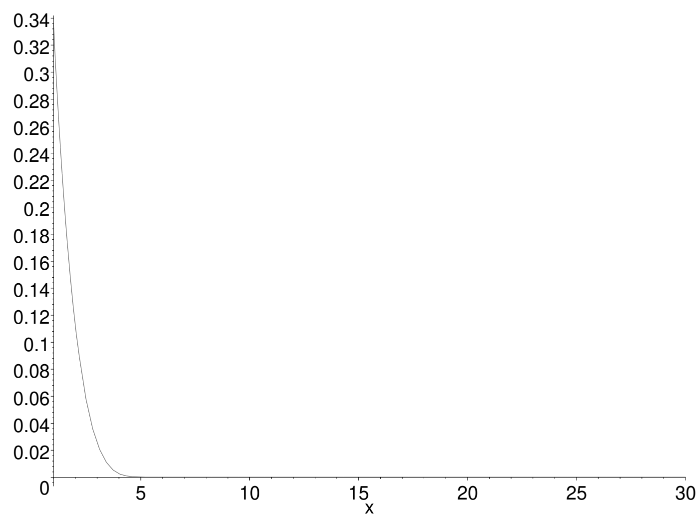

Figure 3: Plot of $2^{-x}e^{-0.1\cdot 2^x}$.

Asymptotics of $L(n)$

Theorem 1. *For fixed a, $0 < a = n/N < \infty$, and large n (hence large N) the average number of traversed links $L(n)$ attains the following asymptotic expansion*

$$L(n) = N\left(\frac{V}{V-1} - c_1(a)\right) - \frac{V}{V-1} - \frac{1}{2}c_2(a) + O\left(\frac{1}{\log n}\right),$$

where

$$\begin{aligned} c_1(a) &= \sum_{l=0}^{\infty} V^{-l} \exp\left(-aV^l\right) \\ c_2(a) &= \sum_{l=0}^{\infty} aV^l \exp\left(-aV^l\right). \end{aligned}$$

Comparison to PST Results

Table 1: Comparison of the exact $L(n)$ with the asymptotic expansion $L^{\mathrm{asym}}(n)$ obtained in Theorem 1 and the approximation $L^{\mathrm{PST}}(n)$ proposed by Phillips, Shenker, and Tangmunarunkit for $a = 0.5$, that is, $N = 2n$.

N	$L(n)$	$L^{\mathrm{asym}}(n)$	$L^{\mathrm{PST}}(n)$
2	3.250	3.202	4.885
4	7.919	7.895	9.771
8	17.295	17.283	19.541
16	36.083	36.057	39.083
32	73.609	73.605	78.166
64	148.704	148.702	156.332
128	298.896	298.896	312.665
256	599.283	599.282	625.330
512	1200.056	1200.056	1250.650
1024	2401.602	2401.602	2501.320

Asymptotics for $a \to 0$

Corollary 1. *Under the same conditions as in Theorem 1, we find:*

(i) *For* $\mathbf{a} \to \mathbf{0}$ *the quantity* $L(n)$ *attains the following asymptotics*

$$\begin{aligned} L(n) &= n\left(D + \frac{1}{\ln V} - \frac{\ln n}{\ln V} + \left(\frac{1}{2} - \frac{\gamma}{\ln V}\right) + \psi_1(\ln a)\right) \\ &- \frac{V}{V-1} - \frac{1}{2\ln V} + \frac{1}{2}\psi_2(\ln a) + O\left(\frac{1}{\log n}\right) \end{aligned}$$

where $\gamma = 0.571\ldots$ *is the Euler constant, and* $\psi_1(x)$*,* $\psi_2(x)$*, are oscillation periodic functions of small amplitude for small* V *that can be expressed as*

$$\begin{aligned} \psi_1(x) &= \sum_{k\neq 0} \frac{\Gamma(-1-(2\pi ik)/\ln V)}{\ln V} \exp\left((2\pi ik)\frac{x}{\ln V}\right), \\ \psi_2(x) &= 2\pi i \sum_{k\neq 0} \frac{k\Gamma(-(2\pi ik)/\ln V)}{\ln^2 V} \exp\left((2\pi ik)\frac{x}{\ln V}\right). \end{aligned}$$

In fact, $|\psi_1(x)| < 0.0000001725,\ 0.00041227,\ 0.0085, 0.068,\ 0.153$ *for* $V = 2, 3, 5, 100, 1000$*, respectively.*

Asymptotics for $a \to 1$ and $a \to \infty$

(ii) *For* $\mathbf{a} \to \mathbf{1}$ *we have*

$$L(n) = N\left(\frac{V}{V-1} - C_1 - C_2\,(a-1) + C_3\,(a-1)^2\right) - \frac{V}{V-1} - \frac{C_3}{2}a + O\left(\frac{1}{\log n}\right),$$

where

$$C_1 = \sum_{l=0}^{\infty} V^{-l} e^{-V^l}, \quad C_2 = \sum_{l=0}^{\infty} e^{-V^l}, \quad C_3 = \frac{\sum_{l=0}^{\infty} V^l e^{-V^l}}{2}.$$

(iii) For $\mathbf{a} \to \infty$ *we arrive at*

$$L(n) = N\left(\frac{V}{V-1} - e^{-a}\right) - \frac{V}{V-1} - \frac{1}{2}\left(ae^{-a} + aVe^{-aV}\right) + O\left(\frac{1}{\log n}\right).$$

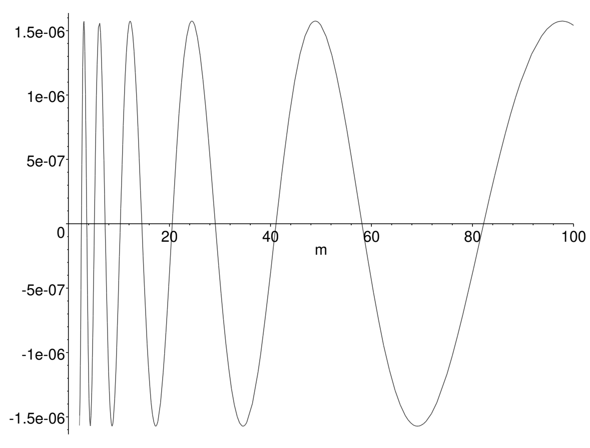

Figure 4: The periodic function $\Psi(x)$.

Conclusion

We can now compare the theoretical results of Corollary 1 to the experimental results of Chuang and Sirbu presented in Figure 1. In particular, we see that the ratio $R(n)$ for $a \to 0$ can be approximated by

$$R_D(n) \approx n \left(1 + \frac{1-\gamma}{\ln N} + \frac{\ln V}{2 \ln N} - \frac{\ln n}{\ln N} + \frac{\ln V}{\ln N} \psi_1(\ln a) \right)$$

which **most decidedly** is not of the power law form.

However, one can argue that for large $a \to \infty$ and fixed N Corollary 1(ii) and (iii) could explain respectively the transition and saturation region of Figure 1.

Self-Similar Tree

Consider again a $V-$ary tree but now we assume that the link connecting a node at level k and a node at level $k-1$ consists of a concatenation of a random number of links. Let ℓ_k be the average number of these links. We postulate that ℓ_k is a fraction of ℓ_{k-1}, that is, for some A we have $\ell_1 = A$ and $\ell_k = \phi\ell_{k-1}, \quad 0 \leq \phi \leq 1$. Therefore, $\ell_k = \phi^{k-1}A$. Setting $\phi = V^{-\theta}$ we have,

$$\ell_k = V^{(D-k)\theta}, \qquad \theta > 0.$$

Observe that there are $K = V^{D-k}$ leaves hanging from a node at level k; thus we reproduce such a node K^θ times.

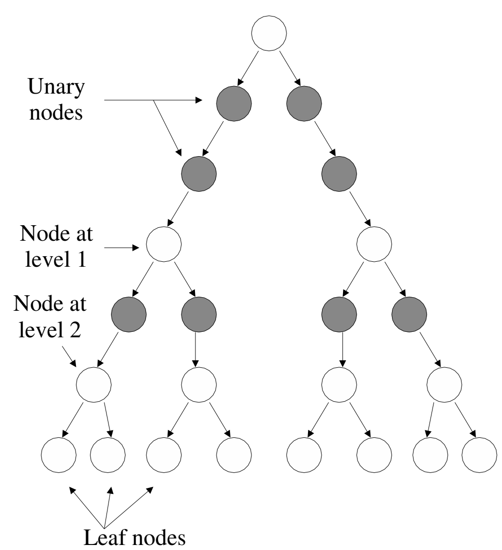

Figure 5: A Self-Similar Tree with $D = 3$ and $\theta = 1$.

How $L(n)$ and $U(n)$ look like?

For self-similar trees we have:

$$L_\theta(n) = \sum_{k=1}^{D} V^{(D-k)\theta} V^k \left(1 - \left(1 - V^{-k}\right)^n\right),$$

and for the average path length in a unicast connection we find

$$U_\theta = \sum_{k=1}^{D} V^{(D-k)\theta} = \frac{N^\theta - 1}{V^\theta - 1}.$$

Theorem 2. *For fixed a, $0 < a = n/N < \infty$, and large n the average number of links $L_\theta(n)$ in the self-similar tree attains the following asymptotic expansion*

$$L_\theta(n) = N\left(\frac{V^{\bar\theta}}{V^{\bar\theta} - 1} - c_1(a,\theta)\right) - \frac{N^\theta V^{\bar\theta}}{V^{\bar\theta} - 1} - \frac{1}{2} c_2(a,\theta) + O\left(\frac{1}{\log n}\right),$$

where $\overline{\theta} = 1 - \theta$ and

$$\begin{aligned} c_1(a,\theta) &= \sum_{l=0}^{\infty} V^{-\bar\theta l} \exp\left(-aV^l\right), \\ c_2(a,\theta) &= \sum_{l=0}^{\infty} aV^{(1+\theta)l} \exp\left(-aV^l\right). \end{aligned}$$

Asymptotics of $L(n)$ for Self-Similar Trees

Corollary 2. *Under the same conditions as in Theorem 2, we find:*

(i) *For* $\mathbf{a} \to \mathbf{0}$ *the quantity* $L(n)$ *attains the following asymptotics*

$$L_\theta(n) = N^\theta \left(n^{\bar{\theta}} \left(\frac{\Gamma\left(\theta\right)}{\bar{\theta} \ln V} - \psi_3(\ln a) \right) - \frac{V^{\bar{\theta}}}{V^{\bar{\theta}} - 1} - \frac{1}{2} \frac{\theta \Gamma\left(\theta\right)}{n^\theta \ln V} - \frac{1}{n^\theta} \psi_4(\ln a) \right) + O\left(\frac{1}{\log n} \right),$$

where $\gamma = 0.571\ldots$ *is the Euler constant,* $\Gamma(\theta)$ *is the Gamma function,* $\psi_3(a)$ *and* $\psi_4(a)$ *are oscillating periodic functions of small amplitude for small* V *that can be expressed as*

$$\begin{aligned}
\psi_3(x) &= \sum_{k \neq 0} \frac{\Gamma(-1 + \theta - (2\pi i k)/\ln V)}{\ln V} \exp\left(2\pi i k \frac{x}{\ln V}\right), \\
\psi_4(x) &= \sum_{k \neq 0} \frac{(\theta - i2\pi k/\ln V)\Gamma(\theta - 2\pi i k/\ln V)}{\ln V} \exp\left(2\pi i k \frac{x}{\ln V}\right).
\end{aligned}$$

Asymptotics for $a \to 1$ and $a \to \infty$

(ii) *For* $\mathbf{a} \to \mathbf{1}$ *we have*

$$\begin{aligned} L(n) &= N\left(\frac{V^{\bar{\theta}}}{V^{\bar{\theta}}-1} - C_1(\theta) - C_2(\theta)\,(a-1) + C_3(\theta)\,(a-1)^2\right) \\ &- \frac{V}{V-1} - \frac{C_3(\theta)}{2}a + O\left(\frac{1}{\log n}\right) \end{aligned}$$

where

$$C_1(\theta) = \sum_{l=0}^{\infty} V^{-\theta l} e^{-V^l}, \quad C_2(\theta) = \sum_{l=0}^{\infty} V^{\theta l} e^{-V^l}, \quad C_3(\theta) = \frac{\sum_{l=0}^{\infty} V^{(1+\theta)l} e^{-V^l}}{2}.$$

(iii) *For* $\mathbf{a} \to \infty$ *we arrive at*

$$L_\theta(n) = N\left(\frac{V^{\bar{\theta}}}{V^{\bar{\theta}}-1} - e^{-a}\right) - \frac{V^{\bar{\theta}}}{V^{\bar{\theta}}-1} - \frac{1}{2}\left(ae^{-a} + aV^{1+\theta}e^{-aV}\right) + O\left(\frac{1}{\log n}\right)$$

Chunag-Sirbu Law Explained!

Now we are in a position to explain Figure 1 and to justify the power law of Chuang and Sirbu. Observe that for small a Corollary 2 suggest the following approximation.

$$\begin{aligned} R(n,\theta) &= \frac{L(n,\theta)}{U_\theta} \\ &\approx n^{1-\theta}\left(V^\theta - 1\right)\left(\frac{\Gamma(\theta)}{(1-\theta)\ln V} - \psi_3(\ln a)\right) \\ &- \frac{V - V^{1-\theta}}{V^{1-\theta}-1}. \end{aligned}$$

We obtain the power law with exponent of n equal to $1-\theta$.

We also observe that:

• the constant that multiplies $n^{1-\theta}$ is independent of the tree size;

• Corollary 2(iii) predicts that for $a \to \infty$ with N fixed the ratio $R(n)$ tends to the constant equal to $N^{1-\theta}V^{1-\theta}(V^\theta - 1)/(V^{1-\theta}-1)$;

• Around $a = 1$ we have a transitive behavior predicted by Corollary 2(ii).

Figure 1 **is explained** under the assumption of self-similar multicast trees.

Experiments

We conducted a set of experiments on a real network in order to provide further evidence of the power law. We used data that was graciously made available by Bill Cheswick and Hal Burch on their site Internet Mapping Project: They run frequent traceroute-style path probe on registered Internet networks: these probes give a good approximation of the paths used to route packets to many Internet networks, and all together give an approximation of the tree starting from one host in the Internet to all other networks. We will use this data to construct a tree topology and a multicast tree.

Here is a brief description of our experiment:

- The probes were taken on May 2000.
- The total number of networks probed is 103625.
- The total number of successful probes is $N = 28587$.
- The destination hosts were all drawn randomly.
- After each draw, the size of the multicast tree is computed and updated along with the size of the unicast tree.

Experimental Results

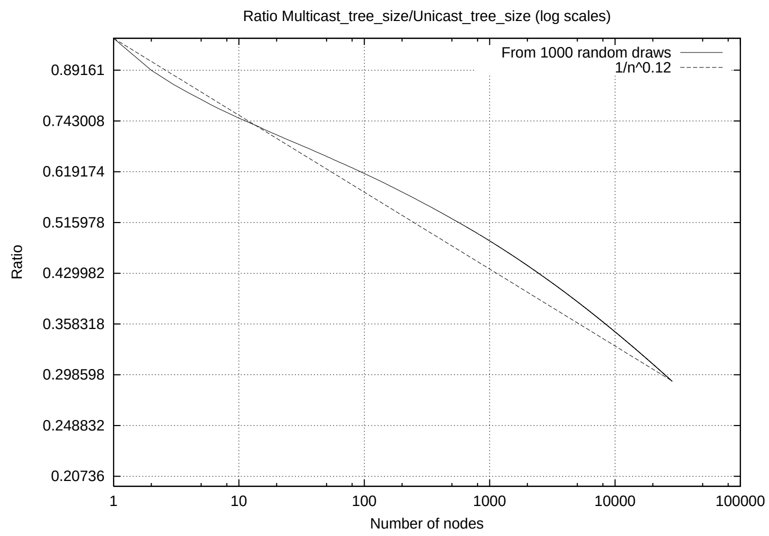

Figure 6: Experimental Ratio $R(n)/n$.

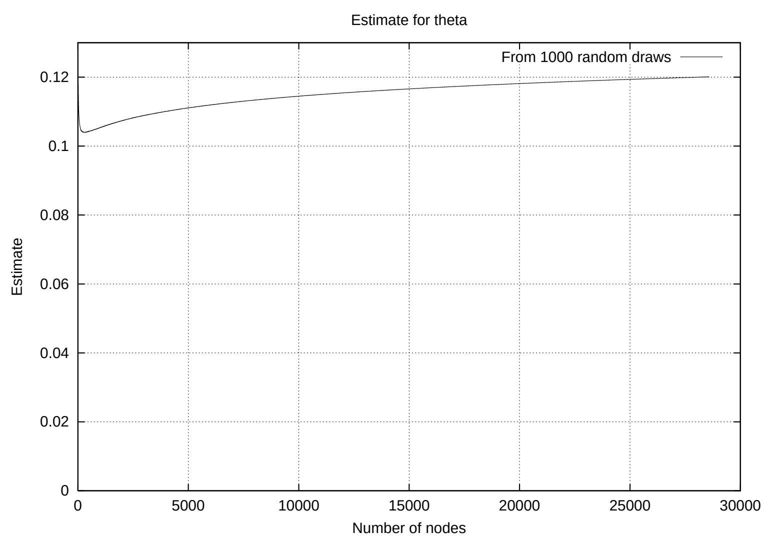

Figure 7: Experimental Curve for Estimating θ versus n.

Is the Multicast Tree a Self-Similar Tree?

The next question one may ask is whether the multicast tree resembles a self-similar tree. This is not easy to verify so we tried to design indirect experiments to verify it. In particular, we worked with **reachability**.

For a node A on the global multicast tree, define by $r(A)$ the number of destination tree nodes that can be reached by A, using the multicast tree links. We call $r(A)$ the "reachability degree of A".

Let $Q(k)$ the number of nodes with reachability degree k. Observe that if destination nodes are at the leaves of the global multicast tree, then

- For **regular** V**-ary trees**, we have

  $$Q(k) = V^D k^{-1} \quad \text{if} \quad k = V^l$$

  for $0 \leq l \leq D$, and $Q(k) = 0$ otherwise.
- For **self-similar** trees, there are $V^{\theta(D-l)} - 1$ unary nodes between levels $l-1$ and l, hence

  $$Q(k) = k^{-1+\theta} V^D \quad \text{if} \quad k = V^l,$$

  for $1 \leq l \leq D$ and $Q(k) = 0$ when k is not a power of V.

Reachability for Self-Similar Trees

Now, let us define by $F(k)$ the number of tree nodes with reachability degree at least k. That is,

$$F(k) = \sum_{l=k}^{N} Q(k).$$

For a self-similar tree of depth D, we have for $1 \leq k \leq N$.

$$\begin{aligned} F(k) &= \sum_{l=\lceil \log_V k \rceil}^{D} Q(k) = N \sum_{l=\lceil \log_V k \rceil}^{D} V^{l(-1+\theta)} \\ &= NV^{\lceil \log_V k \rceil (-1+\theta)} \frac{1 - V^{(D - \lceil \log_V k \rceil + 1)(-1+\theta)}}{1 - V^{-1+\theta}}. \end{aligned}$$

Hence,

$$NV^{-1+\theta} k^{-1+\theta} \leq F(k) \leq \frac{N}{1 - V^{-1+\theta}} k^{-1+\theta}.$$

Experimental Results for Reachability

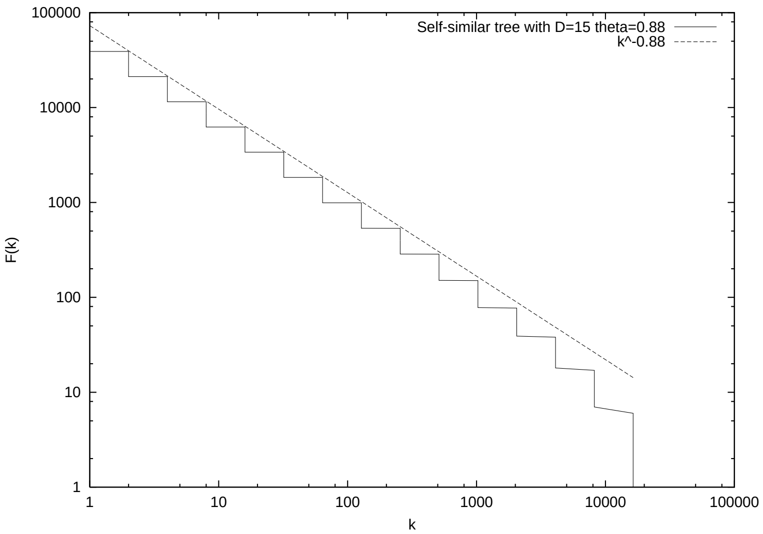

Figure 8: Log-log plots of $F(k)$ versus k for self-similar trees.

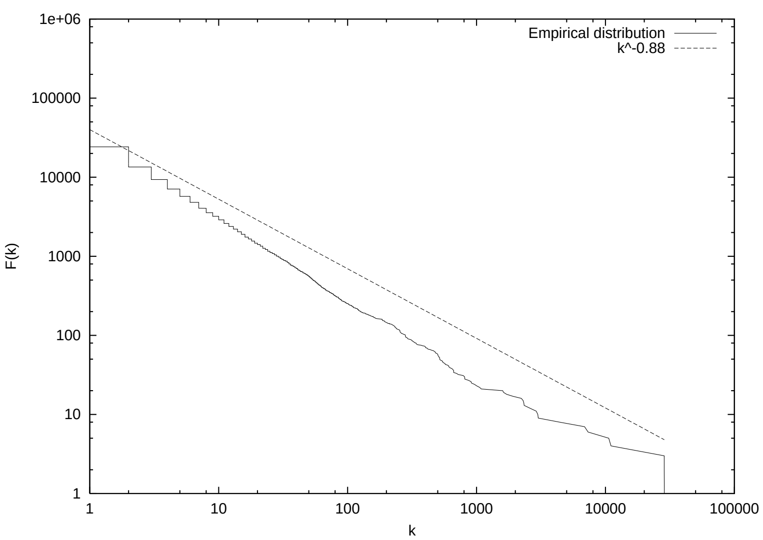

Figure 9: Log-log plots of $F(k)$ versus k for actual internet trees.

Generalization

From the previous discussion we conclude that for self-similar trees, $F(k)$ is decreasing according to $k^{-1+\theta}$. The experimental results shown below confirm that this latter relation holds in real networks as well. Hence it seems natural to ask whether networks satisfying relations of this type give rise to the multicast power law. It turns out that indeed this is the case as shown in the next theorem.

Theorem 3. *If for the global multicast tree and* $0 < \theta < 1$ *it holds for large* N*,*

$$a_N k^{-1+\theta} \leq F(k) \leq A_N k^{-1+\theta}$$

then we have for $n \leq N$*.*

$$\frac{a_N}{A_N}\Gamma(\theta)\theta n^{1-\theta}\left(1 + O\left(\max\{n^{-1}, n/N\}\right)\right)$$

$$\leq R(n) \leq$$

$$\frac{A_N}{a_N}\Gamma(\theta)\theta n^{1-\theta}\left(1 + O\left(\max\{n^{-1}, n/N\}\right)\right)$$

Derivations of Corollary 1 for $a \to 0$

We shall study

$$c_1(a, \theta) = \sum_{l=0}^{\infty} V^{-\bar{\theta} l} \exp\left(-aV^l\right)$$

for $a \to 0$.

Observe that the sum above falls under the paradigm of **harmonic sum** discussed in property (M3) of the Mellin transform. Thus the Mellin transform $c_1^*(s)$ of $c_1(a, \theta)$ is

$$c_1^*(s) = \frac{\Gamma(s)}{1 - V^{-(1+s-\theta)}}.$$

where $\Gamma(s)$ is the Euler gamma function. The inverse Mellin is

$$c_1(a) = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\Gamma(s)}{1 - V^{-(1+s-\theta)}} x^{-s} ds.$$

Finishing up ...

We now evaluate the residues. We note that the function $c_1^*(s)$ has poles at

$$s_k = -1 + \theta - (2\pi i k)/\ln V,$$

$k = \pm 1, \pm 2, \ldots$. All these poles are single. The pole at 0 has residue

$$c_{0,0} = \frac{V^{\bar{\theta}}}{V^{\bar{\theta}} - 1}$$

The poles at s_k, $k \neq 1$ have residues

$$c_{0,s_k} = \frac{\Gamma\left(-1 + \theta - (2\pi i k)/\ln V\right)}{\ln V}.$$

Using now the Reverse Mapping Theorem (M4) and the property $\Gamma(-1+\theta) = \Gamma(\theta)/(-1+\theta)$, we see that for $a \to 0$ we have the expansion for any $M > 0$

$$c_1(a,\theta) = \frac{v^{\bar{\theta}}}{v^{\bar{\theta}} - 1} - \frac{\Gamma\left(\theta\right)}{\bar{\theta}\ln v} a^{\bar{\theta}} + a^{\bar{\theta}}\psi_3(\ln a) + O(a^M)$$

where

$$\psi_3(x) = \sum_{k \neq 0} \frac{\Gamma(-1 + \theta - (2\pi i k)/\ln V)}{\ln V} \exp\left((2\pi i k)\frac{x}{\ln V}\right).$$

Mellin Transform Properties

(M1) Direct and Inverse Mellin Transforms. Let c belong to the *fundamental strip* defined below.

$$f^*(s) := \mathcal{M}(f(x); s) = \int_0^\infty f(x)x^{s-1}dx$$

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f^*(s)x^{-s}ds.$$

(M2) Fundamental Strip. The Mellin transform of $f(x)$ exists in the *fundamental strip* $\Re(s) \in (-\alpha, -\beta)$, where

$$f(x) = O(x^\alpha) \qquad (x \to 0), \qquad\qquad f(x) = O(x^\beta) \qquad (x \to \infty)$$

for $\beta < \alpha$.

(M3) Harmonic Sum Property. By linearity and the scale rule $\mathcal{M}(f(ax); s) = a^{-s}\mathcal{M}(f(x); s)$,

$$f(x) = \sum_{k\geq 0} \lambda_k g(\mu_k x) \qquad \Longleftrightarrow \qquad f^*(s) = g^*(s)\sum_{k\geq 0} \lambda_k \mu_k^- s.$$

(M4) <u>MAPPING PROPERTIES (Asymptotic expansion of $f(x)$</u> and singularities of $f^*(s)$).

$$f(x) = \sum_{(\xi,k)\in A} c_\xi, k x^\xi (\log x)^k + O(x^M) \qquad (*)$$

and

$$f^*(s) \asymp \sum_{(\xi,k)\in A} c_\xi, k \frac{(-1)^k k!}{(s+\xi)^k + 1}. \qquad (**)$$

— (i) *Direct Mapping.* Assume that $f(x)$ admits as $x \to 0^+$ the asymptotic expansion (*) for some $-M < -\alpha$ and $k > 0$. Then for $\Re(s) \in (-M, -\beta)$, the transform $f^*(s)$ satisfies the singular expansion (**).

— (ii) *Reverse Mapping.* Assume that $f^*(s) = O(|s|^{-r})$ with $r > 1$, as $|s| \to \infty$ and that $f^*(s)$ admits the singular expansion (**) for $\Re(s) \in (-M, -\beta)$. Then $f(x)$ satisfies the asymptotic expansion (*) at $x = 0^+$.

More Experiments – Really Self-Similar Tree?

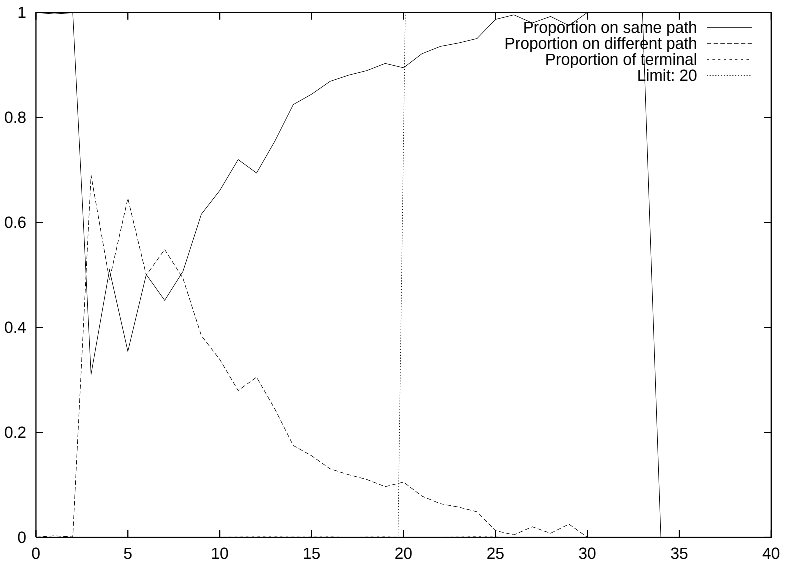

Figure 10: Proportion of paths going through the same link at depth k.