Variable-to-Variable Codes with Small Redundancy Rates*

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Outline of the Talk

1. Types of Codes
2. Redundancy and Redundancy Rates
3. Main Results
4. Sketch of Proof
5. Algorithm
Types of Codes

Fixed-to-Variable Code:

Fixed length blocks are mapped into variable-length binary code strings. **Example**: Shannon and Huffman codes.

Variable-to-Fixed Code:

- Encoder partitions strings into phrases.
- Phrases belong to dictionary $D$ (complete tree).
- The encoder represents dictionary string by fixed-length code word. **Example**: Tunstall code.

Variable-to-variable (VV) code:

- Encoder consists of a parser and a string encoder.
- The parser works as in the VF code (it partitions a sequence into phrases).
- The string encoder encodes dictionary strings into codeword $C(d)$ of length $|C(d)| = \ell(d)$. 
Redundancy and Redundancy Rates

Let $\mathcal{A} = \{a_1, \ldots, a_m\}$ be the input alphabet of $m \geq 2$ symbols with probabilities $p_1, \ldots, p_m$.

A source $S$ generates a sequence $x$ with the underlying probability $P_S$.

The average and worst case (maximal) redundancy of a fixed-to-variable code are defined, respectively, as

\[
\bar{R} = \sum_{x \in \mathcal{A}^n} P_S[L(x) + \log P_S(x)]
\]

\[
R^* = \max_{x \in \mathcal{A}^n}[L(x) + \log P_S(x)]
\]

where $L(x)$ is the code length assigned to $x \in \mathcal{A}^n$.

Redundancy rates are respectively

\[
\bar{r} = \frac{\bar{R}}{n}
\]

\[
r^* = \frac{R^*}{n}.
\]
**Redundancy Rates for VV Codes**

Let $P := P_D$ be the probability induced by the dictionary $\mathcal{D}$.

Define the average delay $\bar{D}$ as

$$
\bar{D} = \sum_{d \in \mathcal{D}} P_D(d) |d|.
$$

The (asymptotic) average redundancy rate $\bar{r}$ is

$$
\bar{r} = \lim_{n \to \infty} \frac{\sum_{|x|=n} P_S(x) (L(x) + \log P_S(x))}{n}.
$$

Using renewal theory (regeneration theory) we find

$$
\lim_{n \to \infty} \frac{\sum_{|x|=n} P_S(x) L(x)}{n} = \frac{\sum_{d \in \mathcal{D}} P_D(d) \ell(d)}{\bar{D}}.
$$

where $\ell(d)$ is the length of the phrase $d \in \mathcal{D}$. 
A New Definition

Denote
- $H_S$ the binary source entropy,
- $H_D$ the dictionary entropy.

Then

$$\frac{\sum_{d \in D} P_D(d) \ell(d)}{\bar{D}} - H_S = \frac{H_D + (\sum_{d \in D} P_D(d) \ell(d) - H_D)}{\bar{D}} - H_S.$$  

By the Conservation of Entropy Theorem (Savari 99)

$$H_D = H_S \cdot \bar{D},$$

hence

$$\bar{\bar{r}} = \frac{\sum_{d \in D} P_D(d) \ell(d) - H_D}{\bar{D}}$$

which we adopt as our definition of the average redundancy rate.
Maximum Redundancy

Maximum (asymptotic) redundancy rate $r^*$

$$r^* = \lim_{|x| \to \infty} \frac{\max_x [L(x) + \log P_S(x)]}{|x|}.$$ 

Assuming the source is sequence is partitioned into phrases $x^1, \ldots, x^k$, $k \to \infty$ we have

$$r^* = \lim_{k \to \infty} \frac{\max_{x^1, \ldots, x^k} \sum_{i=1}^k [\ell(x^i) + \log P_S(x^i)]}{\sum_{i=1}^k |x^i|}$$

$$= \lim_{k \to \infty} \frac{\sum_{i=1}^k \max_{x^i} [\ell(x^i) + \log P_S(x^i)]}{\sum_{i=1}^k |x^i|}$$

$$= \lim_{k \to \infty} \frac{\sum_{i=1}^k \max_{d \in D} [\ell(d) + \log P(d)]}{\sum_{i=1}^k |x^i|}$$

$$= \lim_{k \to \infty} \frac{k \max_{d \in D} [\ell(d) + \log P(d)]}{\sum_{i=1}^k |x^i|}$$

$$= \frac{\max_{d \in D} [\ell(d) + \log P(d)]}{\bar{D}} \quad (a.s.)$$
Main Result for Average Redundancy

**Theorem 1.** Let $m \geq 2$ and $S$ a memoryless or a Markov source. There exists a variable-to-variable code such that its average redundancy satisfies

$$\bar{r} = O(\bar{D}^{-5/3}).$$

There also exists a variable-to-variable code such that the worst case redundancy satisfies

$$r^* = O(\bar{D}^{-4/3}),$$

however, the maximal code length might be infinite.

The estimate for $\bar{r}$ for the memoryless source is the same as in Khodak’s 1972 paper. However, the method presented in Khodak’s paper is difficult to follow and it is not clear one can construct a VV code.
Main Result for Maximal Redundancy

Theorem 2. Under the same assumption as in Theorem 1. For almost all source parameters \( p_j \) (resp. \( p_{ij} \)):

- There exists a variable-to-variable code such that its average redundancy is bounded by
  \[
  \bar{r} \leq \bar{D}^{-\frac{4}{3} - \frac{m}{3} + \varepsilon},
  \]
  where \( \varepsilon > 0 \) and the maximal length is \( O(\bar{D} \log \bar{D}) \).

- There also exists a variable-to-variable code with worst case redundancy
  \[
  r^* \leq \bar{D}^{-1 - \frac{m}{3} + \varepsilon}
  \]
  for \( \varepsilon > 0 \).
**Theorem 3.** For every variable-to-variable code and almost all parameters \( p_j \), the following lower bound holds

\[
r^* \geq \bar{r} \geq \bar{D}^{-2m-1-\varepsilon}.
\]

for \( \varepsilon > 0 \)

Some comments:

1. **Typically** the best possible average and worst case redundancy are measured in terms of negative powers of \( \bar{D} \) that linearly decrease in term of the alphabet size \( m \).

2. It seems to be very difficult to obtain the optimal exponent (almost surely). Nevertheless the bounds we are obtain are best possible with respect to the methods we use.

3. Note that Theorem 4 of Kodak 1972 paper states a lower bound of for the redundancy of the form \( \bar{r} \geq \bar{D}^{-9}(\log \bar{D})^{-8} \) (for almost all memoryless sources). In view of our Theorem 3 this cannot be true for large \( m \).
Let $\mathcal{A} = \{1, 2, \ldots, m\}$ and $p_1 + \cdots + p_m = 1$. By $k_i$ we denote the number of symbols $i$ in a word $x$.

For the Shannon code the redundancy is

$$
\bar{R} = \sum_x \left( \left\lceil - \log p_1^{k_1} \cdots p_m^{k_m} \right\rceil + \log p_1^{k_1} \cdots p_m^{k_m} \right)
$$

$$
= 1 - \sum_x \langle -k_1 \log p_1 - \cdots - k_m \log p_m \rangle
$$

$$
= \sum_x \langle k_1 \log p_1 + \cdots + k_m \log p_m \rangle
$$

where $\langle a \rangle = a - \lfloor a \rfloor$ is the fractional part of $a$.

**Basic Thrust of our Approach**

In order to minimize the redundancy of a code we must find such $k_1, \ldots, k_m$ that

$$
\langle k_1 \log p_1 + \cdots + k_m \log p_m \rangle
$$

is as close as possible to an integer.
Proof by Picture

- "good" word in $D_j$
- "bad" word in $B_j$

\[
P(D_1) = \frac{c}{N}
\]

\[
P(D_2) = \left(1 - \frac{c}{N}\right) \frac{c}{N}
\]

\[
P(D_3) = \left(1 - \frac{c}{N}\right)^2 \frac{c}{N}
\]

\[
P(D_k) = \left(1 - \frac{c}{N}\right)^{k-1} \frac{c}{N}
\]

\[
K = N \log N
\]

\[
N^2 + O(N)
\]

\[
2N^2 + O(2N)
\]

\[
3N^2 + O(3N)
\]

\[
KN^2 + O(KN)
\]
Algorithm

Input:

- \( m \), an integer \( \geq 2 \),
- positive rational numbers \( p_1, \ldots, p_m \) with \( p_1 + \cdots + p_m = 1 \), \( p_m \) is not a power of 2
- \( \varepsilon < 1 \), a positive real number.

Output:

- A VV-code (given by a dictionary \( \mathcal{D} \), a complete prefix free set on an \( m \)-ary alphabet and by a prefix code \( C : \mathcal{D} \rightarrow \{0, 1\}^* \)) with the average redundancy
  \[
  \overline{r} \leq \frac{\varepsilon}{\overline{D}}
  \]
  where the average dictionary code length \( \overline{D} \) satisfies
  \[
  \overline{D} \geq c(m, p_1, \ldots, p_m)/\varepsilon^3.
  \]
Algorithm

1. Calculate a convergent $\frac{M}{N} = [c_0, c_1, \ldots, c_n]$ of $\log_2 p_m$ for which $N > 4/\varepsilon$.

2. Set $k_j^0 = \lfloor p_j N^2 \rfloor$, $x = \sum_{j=1}^{m} k_j^0 \log_2 p_j$, $n_0 = \sum_{j=1}^{m} k_j^0$.

3. Set $D = \emptyset$, $B = \{\text{empty word}\}$, and $p = 0$.

   while $p < 1 - \varepsilon/4$ do
   
   Take $r \in B$ of minimal length.
   
   $b \leftarrow \log_2 P(r)$
   
   Determine $0 \leq k < N$ that solves
   
   $kM \equiv 1 - \lfloor (x + b)N \rfloor \mod N$
   
   (i.e., $1/N \leq (kM/N + x + b) \leq 2/N$)
   
   $n \leftarrow n_0 + k$
   
   $D' \leftarrow \{d \in A^n : \text{type}(d) = (k_1^0, \ldots, k_m + k)\}$
   
   $D \leftarrow D \cup r \cdot D'$
   
   $B \leftarrow (B \setminus \{r\}) \cup r \cdot (A^n \setminus D')$
   
   $p \leftarrow p + P(r) P(D')$, where
   
   $$P(D') = \frac{n!}{k_1^0! \cdots k_{m-1}^0!(k_m + k)!} p_1^{k_1^0} \cdots p_{m-1}^{k_{m-1}^0} p_m^{k_m + k}.$$  
   
   end while

4. $D \leftarrow D \cup B$

5. Construct a Shannon code $\ell(d) = [-P(d)]$. 

Some Definitions Needed in the Proof

Define **dispersion** of the set $X \subseteq [0, 1)$ as

$$
\delta(X) = \sup_{0 \leq y < 1} \inf_{x \in X} \|y - x\|,
$$

where $\|x\| = \min(\langle x \rangle, \langle 1 - x \rangle)$. That is, for every $y \in [0, 1)$ there exists $x \in X$ with $\|y - x\| \leq \delta(X)$.

**Lemma 1.** Suppose that $\gamma$ is an **irrational** number. The there exists $N$ such that

$$
\delta(\{\langle k\gamma \rangle : 0 \leq k < N\}) \leq \frac{2}{N}.
$$

**Lemma 2.** Let $(\gamma_1, \ldots, \gamma_m) = (\log p_1, \ldots, \log p_m)$ an $m$-vector of real numbers such that at least one of its coordinates is irrational. There exists $N$ such that the dispersion of the set

$$
X = \{\langle k_1\gamma_1 + \cdots + k_m\gamma_m \rangle : 0 \leq k_j < N (1 \leq j \leq m)\}
$$

is bounded by

$$
\delta(X) \leq \frac{2}{N}.
$$
Two Important Lemmas

**Lemma 3.** Let \( D \) be a finite set with probability distribution \( P \) such that for every \( d \in D \) the length \( l_d \) satisfies \( l_d + \log_2 P(d) \leq 1 \). If

\[
\sum_{d \in D} P(d)(l_d + \log_2 P(d)) \geq 2 \sum_{d \in D} P(d)(l_d + \log_2 P(d))^2
\]

then there exists an injective mapping \( C : D \rightarrow \{0, 1\}^* \) such that \( C(D) \) is a prefix-free set and \( |C(d)| = l_d \) for all \( d \in D \).

**Proof:** Kraft’s inequality and Taylor’s expansion.

**Lemma 4.** Let \( D \) be a finite set with probability distribution \( P \). Then

\[
\bar{r} \geq \frac{1}{2D} \sum_{d \in D} P(d)\| \log_2 P(d) \|^2,
\]

for a certain constant \( c > 0 \).
Theorem 4. Suppose that for some $N \geq 1$ and $\eta \geq 1$ the set

$$X = \{ \langle k'_1 \log_2 p_1 + \cdots + k'_m \log_2 p_m \rangle : 0 \leq k'_j < N \}$$

has dispersion

$$\delta(X) \leq \frac{2}{N^\eta}.$$ 

Then there exists a variable-to-variable code (with $\bar{D} = \Theta(N^3)$) such that the average redundancy rate is

$$\bar{r} \leq c'_m \cdot \bar{D}^{-\frac{4+\eta}{3}},$$

There exists also another variable-to-variable code with the worst case redundancy bounded by

$$r^* \leq c''_m \cdot \bar{D}^{-1-\frac{\eta}{3}},$$

where the constants $c'_m, c''_m > 0$ just depend on $m$.

Observe that above theorem and previous lemmas immediately implied our main result after setting $\eta = 1$. 
Sketch of the Proof of Theorem 3

1. Set $k_i^0 := \lfloor p_i N^2 \rfloor$ (1 ≤ $i$ ≤ $m$) and
   
   $x = k_1^0 \log_2 p_1 + \cdots + k_m^0 \log_2 p_m$.

   By Theorem 3 there exist integers 0 ≤ $k_j^1 < N$ such that
   \[
   \left\langle x + k_1^1 \log_2 p_1 + \cdots + k_m^1 \log_2 p_m \right\rangle
   = \left\langle (k_1^0 + k_1^1) \log_2 p_1 + \cdots + (k_m^0 + k_m^1) \log_2 p_m \right\rangle
   < \frac{4}{N^{\eta}}.
   \]

2. Build an $m$-ary tree starting at the root with
   
   $k_1^0 + k_1'$ edges of type 1,
   
   $k_2^0 + k_2'$ edges of type 2, . . . , and
   
   $k_m^0 + k_m'$ edges of type $m$.

   Let $D_1$ denote the set of the corresponding words. Then
   \[
   \frac{c'}{N} \leq P(D_1) = \left( (k_1^0 + k_1') + \cdots + (k_m^0 + k_m') \right) k_1^0 k_1' \cdot \cdots \cdot k_m^0 k_m'
   \leq \frac{c''}{N}
   \]

   for certain positive constants $c'$, $c''$. 
3. By construction, all words $d \in \mathcal{D}_1$ satisfy

$$\langle \log_2 P(d) \rangle < \frac{4}{N^\eta},$$

and have the same length

$$n_1 = (k_1^0 + k_1') + \cdots + (k_m^0 + k_m') = N^2 + O(N).$$

4. Consider words not in $\mathcal{D}_1$, that is, $\mathcal{B}_1 = A^{n_1} \setminus \mathcal{D}_1$ that by above satisfy

$$\frac{c''}{N} \leq P(\mathcal{B}_1) \leq 1 - \frac{c'}{N}.$$
5. Take a word $r \in B_1$ and concatenate it with a word $d_2$ of length $\sim N^2$ such that $\log_2 P(rd_2)$ is close to an integer with high probability.

For every word $r \in B_1$ we set

$$x(r) = \log_2 P(r) + k_1^0 \log_2 p_1 + \cdots + k_m^0 \log_2 p_m.$$  

By condition of Theorem 3 again there exist integers $0 \leq k_j^2(r) < N$ ($1 \leq j \leq m$) such that

$$\langle x(r) + k_1^2(r) \log_2 p_1 + \cdots + k_m^2(r) \log_2 p_m \rangle < \frac{4}{N\eta}.$$
6. We continue extending the tree $\mathcal{T}$ by adding a path starting at $r \in \mathcal{B}_1$ with
$k_1^0 + k_1^1(r)$ edges of type 1,
$k_2^0 + k_2^2(r)$ edges of type 2,
..., and
$k_m^0 + k_m^2(r)$ edges of type $m$.
We observe that
$$P(r) \frac{c'}{N} \leq P(\mathcal{D}_2(r)) \leq P(r) \frac{c''}{N}.$$ Furthermore (by construction) we have
$$\langle \log_2 P(d) \rangle < \frac{4}{N\eta}$$
for all $d \in \mathcal{D}_2(r)$. 
7. This construction is cut after $K = O(N \log N)$ steps so that

$$P(B_K) \leq c'' \left(1 - \frac{c'}{N}\right)^K \leq \frac{1}{N^\beta}$$

for some $\beta > 0$. This also ensures that

$$P(D_1 \cup \cdots \cup D_K) > 1 - \frac{1}{N^\beta}.$$  

8. The complete prefix free set $D$ is

$$D = D_1 \cup \cdots \cup D_K \cup B_K.$$  

By the construction the average delay of $D$ is

$$c_1 N^3 \leq \bar{D} = \sum_{d \in D} P(d) |d| \leq c_2 N^3$$

while the maximal code length satisfies

$$\max_{d \in D} |d| = O N^3 \log N = O \bar{D} \log \bar{D}).$$
For every $d \in \mathcal{D}_1 \cup \cdots \cup \mathcal{D}_K$ we can choose a non-negative integer $\ell_d$ with

$$|\ell_d + \log_2 P(d)| < \frac{2}{N^n}.$$ 

Indeed:

$$0 \leq \ell_d + \log_2 P(d) < \frac{2}{N^n}$$

if $\langle \log_2 P(d) \rangle < 2/N^n$ and

$$-\frac{2}{N^n} < \ell_d + \log_2 P(d) \leq 0$$

if $1 - \langle \log_2 P(d) \rangle < 2/N^n$.

For $d \in \mathcal{B}_K$ we set $\ell_d = \lceil -\log_2 P(d) \rceil$.

The final step is to verify that condition of Lemma 3 is satisfied, which is an easy step.