On the Entropy of a Hidden Markov Process

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Outline of the Talk

1. Hidden Markov Model and Its Applications
2. Product of Random Matrices
3. Entropy Rate as a Lyapunov Exponent
4. Asymptotic Expansion
5. Experimental Verifications
6. Sketch of the Proof
7. Rényi’s Entropy
Problem Formulation

1. Let $X = \{X_k\}_{k \geq 1}$ be a first order stationary Markov process over a binary alphabet, with transition matrix $P = \{\pi_{ab}\}_{a,b \in \{0,1\}}$;

   \[ \pi_{ab} = P(X_k = b | X_{k-1} = a) \]

2. Let $E = \{E_k\}_{k \geq 1}$ be a Bernoulli (binary i.i.d.) noise process independent of $X$, such that

   \[ P(E_i = 1) = \varepsilon \]

3. Define $Z = \{Z_k\}_{k \geq 1}$ such that

   \[ Z_k = X_k \oplus E_k, \quad k \geq 1, \]

   where $\oplus$ denotes addition modulo 2 (exclusive-or).
Hidden Markov Process

The process $Z$ is, in a sense, one of the simplest examples of a hidden Markov process (HMP).

**Basic question:** what is the entropy rate of such a process?

In general, a HMP is a process resulting from observing any discrete-time, finite state homogeneous Markov chain through a discrete-time memoryless channel.

In particular, $Z = f(X, E)$ for a Markov process $X$, i.i.d. $E$, and a function $f$.

**Applications:**
- data compression
- automatic character recognition
- speech recognition
- statistics
- communications and information theory
- DNA sequencing
- denoising
- performance of digital trees.
Figure 1: A trie and its parameters.

*Hidden Markov Source:* Binary sequences generated by a Markov source with an i.i.d. error sequence.
Some Previous Works

Blackwell derived in 1957 an expression for the entropy of HMP in terms of a measure $Q$, which solves an integral equation. The measure is hard to extract from the equation in any explicit way.

Ordentlich and Weissman in 2003 obtained explicit formulas for the entropy rate when $\pi_{ab} \to 0$.

In contrast, our study focuses on the regime where the channel parameter (noise) $\varepsilon \to 0$ is small.
Joint Distribution of $P(Z_1^n)$

For any sequence $\{Y_k\}_{k \geq 1}$, let

$$Y_i^j = Y_iY_{i+1} \ldots Y_j.$$ 

Also $\bar{Y} = 1 \oplus Y$.

In particular, $Z_i = X_i$ if $E_i = 0$ and $Z_i = \bar{X}_i$ if $E_i = 1$.

We have

$$P(Z_1^n, E_n) = P(Z_1^n, E_{n-1} = 0, E_n) + P(Z_1^n, E_{n-1} = 1, E_n) =$$

$$= P(Z_1^{n-1}, Z_n, E_{n-1} = 0, E_n) + P(Z_1^{n-1}, Z_n, E_{n-1} = 1, E_n)$$

$$= P(Z_n, E_n|Z_1^{n-1}, E_{n-1} = 0)P(Z_1^{n-1}, E_{n-1} = 0) +$$

$$P(Z_n, E_n|Z_1^{n-1}, E_{n-1} = 1)P(Z_1^{n-1}, E_{n-1} = 1)$$

$$= P(E_n)P_X(Z_n \oplus E_n|Z_{n-1})P(Z_1^{n-1}, E_{n-1} = 0)$$

$$+ P(E_n)P_X(Z_n \oplus E_n|\bar{Z}_{n-1})P(Z_1^{n-1}, E_{n-1} = 1)$$
Let
\[ p_n = [P(Z_1^n, E_n = 0), P(Z_1^n, E_n = 1)] \]
and
\[ M(Z_{n-1}, Z_n) = \begin{bmatrix}
(1-\varepsilon)P_X(Z_n|Z_{n-1}) & \varepsilon P_X(Z_n|\tilde{Z}_{n-1}) \\
(1-\varepsilon)P_X(Z_n|\tilde{Z}_{n-1}) & \varepsilon P_X(Z_n|\tilde{Z}_{n-1})
\end{bmatrix} \]
where the expressions \( P_X(Z_i|Z_{i-1}) \) are the Markov transition probabilities computed on the components of the HMP \( Z \).

From the previous slide we conclude that
\[ p_n = p_{n-1}M(Z_{n-1}, Z_n). \]

Since \( P(Z_1^n) = p_n1^t \) \((1^t = (1, \ldots, 1))\) we finally obtain
\[ P(Z_1^n) = p_1M(Z_1, Z_2) \cdots M(Z_{n-1}, Z_n)1^t, \]
that is, product of random matrices since \( P_X(Z_i|Z_{i-1}) \) are random variables.
Theorem 1 (Furstenberg and Kesten, 1960). Let $M_1, \ldots, M_n$ form a stationary ergodic sequence and $E[\log^+ ||M_1||] < \infty$ Then

$$\lim_{n \to \infty} \frac{1}{n} E[\log ||M_1 \cdots M_n||] = \lim_{n \to \infty} \frac{1}{n} \log ||M_1 \cdots M_n|| = \mu \text{ a.s.}$$

where $\mu$ is called top Lyapunov exponent.

**Corollary 1.** Consider the HMP $Z$ as defined above. The entropy rate $^1$

$$h(Z) = \lim_{n \to \infty} E\left[-\frac{1}{n} \log P(Z_1^n)\right]$$

$$= \lim_{n \to \infty} \frac{1}{n} E[-\log \left(p_1 M(Z_1, Z_2) \cdots M(Z_{n-1}, Z_n) 1^t\right)]$$

is a top Lyapunov exponent of $M(Z_1, Z_2) \cdots M(Z_{n-1}, Z_n)$.

Unfortunately, it is notoriously difficult to compute top Lyapunov exponents as proved in Tsitsiklis and Blondel. Therefore, in next we derive an explicit asymptotic expansion of the entropy rate $h(Z)$.

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^1When no base is specified, logarithms are to base 2; $\ln x$ will denote the natural logarithm of $x$. 
We now assume that \( P(E_i = 1) = \varepsilon \to 0 \) is small.

**Theorem 2.** The entropy rate of the process \( Z \) is

\[
h(Z) = \lim_{n \to \infty} \frac{1}{n} H_n(Z^n) = h(X) + f_1(\pi_{01}, \pi_{10}) \varepsilon + O(\varepsilon^2),
\]

with

\[
f_1(\pi_{01}, \pi_{10}) = \mathbb{D}(P_X(z_1z_2z_3) || P_X(z_1\bar{z}_2z_3))
\]

\[
= \sum_{z_1z_2z_3} P_X(z_1z_2z_3) \log \frac{P_X(z_1z_2z_3)}{P_X(z_1\bar{z}_2z_3)},
\]

where \( h(X) \) is the entropy rate of the Markov process \( X \), \( \mathbb{D} \) denotes the Kullback-Liebler divergence, and the summation is over all binary triplets.
Consider a Markov process with symmetric transition probabilities \( \pi_{01} = \pi_{10} = \pi, \pi_{00} = \pi_{11} = 1 - \pi \). This process has stationary probabilities \( P_X(0) = P_X(1) = \frac{1}{2} \).

The probabilities \( P_X(z_1^3) \) of binary triplets are readily computed as

\[
\begin{align*}
P_X(000) &= P_X(111) = \frac{1}{2}(1 - \pi)^2, \\
P_X(001) &= P_X(011) = P_X(100) = P_X(110) = \frac{1}{2}\pi(1 - \pi), \\
P_X(010) &= P_X(101) = \pi^2.
\end{align*}
\]

Thus we obtain from our Theorem

\[
f_1(\pi, \pi) = 2(1 - 2\pi) \log \frac{1 - \pi}{\pi},
\]

and

\[
h(Z) = -\pi \log \pi - (1 - \pi) \log(1 - \pi) + \epsilon 2(1 - 2\pi) \log \frac{1 - \pi}{\pi} + O(\epsilon^2).
\]
HMPs for various values of the parameters $\varepsilon$ and $\pi_{01} = \pi_{10} = \pi$ were simulated, generating pseudo-random HMP sequences of lengths between $n = 10^8$ and $n = 4 \cdot 10^9$. For each generated sequence $z_1^n$, the probability $P_Z(z_1^n)$ assigned by the hidden Markov model of the given parameters was computed, and $-\frac{1}{n} \log P_Z(z_1^n)$ was taken as an estimate for the entropy rate.

<table>
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<th>Parameters</th>
<th>Calculated</th>
<th>Empirical</th>
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<td>$n$</td>
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<td>$1 \cdot 10^8$</td>
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Table 1: First order approximation of $h(Z)$ according to our Theorem and empirical estimation.
Figure 2: Values of $f_1$ and empirical estimation of $\partial h / \partial \varepsilon |_{\varepsilon=0}$ as a function of $\pi$. 
1. Instead of computing entropy $H(Z^n_1)$ we evaluate the following sum

$$R(s, \varepsilon) = \sum_{z^n_1} P^s_Z(z^n_1),$$

where $s$ is a complex variable. Observe that

$$H(Z^n_1) = \mathbb{E} \left[ -\log P(Z^n_1) \right] = -(\ln 2) \left. \frac{\partial}{\partial s} R(s, \varepsilon) \right|_{s=1}.$$

The entropy of the underlying Markov sequence is

$$H(X^n_1) = (-\ln 2) \left. \frac{\partial}{\partial s} R(s, 0) \right|_{s=1}$$

and

$$R(s, 0) = \sum_{z^n_1} P^s_X(z^n_1) = \pi(s) P^{n-1}(s) 1^t.$$
2. By Taylor expansion

\[ R(s, \varepsilon) = R(s, 0) + \varepsilon \frac{\partial}{\partial \varepsilon} R(s, \varepsilon) \bigg|_{\varepsilon=0} + O(R_{\varepsilon,\varepsilon}(s, \varepsilon')\varepsilon^2). \]

We can prove that

\[ R_{\varepsilon,\varepsilon,s(1, \varepsilon')} = O(n) \quad \text{(IMPORTANT!)} \]

where \( R_{\varepsilon,\varepsilon,s(1, \varepsilon')} \) is the first derivative with respect to \( s \) at \( s = 1 \) of \( R_{\varepsilon,\varepsilon}(s, \varepsilon') \).

Thus

\[ H(Z^n_1) = H(X^n_1) - (\ln 2)\varepsilon \frac{\partial^2}{\partial s \partial \varepsilon} R(s, \varepsilon) \bigg|_{\varepsilon=0, s=1} + O(n\varepsilon^2) \]

\[ = H(X^n_1) - (\ln 2)\varepsilon \sum_{z^n_1} \frac{\partial}{\partial s} \frac{\partial}{\partial \varepsilon} P^s_Z(z^n_1) \bigg|_{\varepsilon=0, s=1} + O(n\varepsilon^2). \]
3. We introduce a decomposition of $M_i$ as follows

$$M_i = M(z_i, z_{i+1}) = \begin{bmatrix} (1 - \varepsilon) P_X(z_{i+1}|z_i) & \varepsilon P_X(\bar{z}_{i+1}|z_i) \\ (1 - \varepsilon) P_X(z_{i+1}|\bar{z}_i) & \varepsilon P_X(\bar{z}_{i+1}|\bar{z}_i) \end{bmatrix}$$

$$= \begin{bmatrix} P_X(z_{i+1}|z_i) & 0 \\ P_X(z_{i+1}|\bar{z}_i) & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} -P_X(z_{i+1}|z_i) & P(\bar{z}_{i+1}|z_i) \\ -P_X(z_{i+1}|\bar{z}_i) & P(\bar{z}_{i+1}|\bar{z}_i) \end{bmatrix}$$

$$\overset{\text{def}}{=} M_i^{(0)} + \varepsilon M_i^{(1)},$$

Then

$$P_Z(z_1^n) = P(Z_1^n = z_1^n) = p_0 M_1 M_2 \cdots M_{n-1} 1^t = (M_0^{(0)} + \varepsilon M_0^{(1)}) (M_1^{(0)} + \varepsilon M_1^{(1)}) (M_2^{(0)} + \varepsilon M_2^{(1)}) \cdots (M_{n-1}^{(0)} + \varepsilon M_{n-1}^{(1)}) 1^t.$$
4. To compute the derivative of $P_Z^s(z_1^n)$ at $\varepsilon = 0$, we first differentiate both sides of the above equation, obtaining

$$\left. \frac{\partial}{\partial \varepsilon} P_Z(z_1^n) \right|_{\varepsilon=0} = \sum_{i=0}^{n-1} M_0^{(0)} M_1^{(0)} \cdots M_{i-1}^{(0)} M_i^{(1)} M_{i+1} \cdots M_{n-1}^{(0)} 1.$$ 

And after some algebra we arrive at

$$\left. \frac{\partial}{\partial \varepsilon} P_Z(z_1^n) \right|_{\varepsilon=0} = P_X(z_1^n) \sum_{i=0}^{n-1} (g_i(z_1^n) - 1),$$

where

$$g_i(z_1^n) = \frac{P_X(\bar{z}_{i+1}|z_i) P_X(z_{i+2}|\bar{z}_{i+1})}{P_X(z_{i+1}|z_i) P_X(z_{i+2}|z_{i+1})} = \frac{P_X(z_i \bar{z}_{i+1} z_{i+2})}{P_X(z_i z_{i+1} z_{i+2})}.$$
5. Thus

\[
\frac{\partial}{\partial \varepsilon} P^s_z(z_1^n) \bigg|_{\varepsilon=0} = \left[ sP^{-1}_z(z_1^n)P_X(z_1^n) \sum_{i=0}^{n-1} (g_i(z_1^n) - 1) \right]_{\varepsilon=0}.
\]

thus, in a matrix form, we obtain

\[
\frac{\partial}{\partial \varepsilon} R(s, \varepsilon) \bigg|_{\varepsilon=0} = s \pi(s) \sum_{i=1}^{n-1} P^{i-1}(s) \left( Q_1(s)Q_2(s) - P^2(s) \right) P^{n-i-2}(s) 1^t
\]

where

\[
Q_1(s) = \begin{bmatrix}
\pi_{00} s^{-1} & \pi_{01} s^{-1} \\
\pi_{10} s^{-1} & \pi_{11} s^{-1}
\end{bmatrix}, \quad Q_2(s) = \begin{bmatrix}
\pi_{00} s^{-1} & \pi_{01} s^{-1} \\
\pi_{10} s^{-1} & \pi_{11} s^{-1}
\end{bmatrix}.
\]

and

\[
P(s) = \begin{bmatrix}
\pi_{00}^s & \pi_{01}^s \\
\pi_{10}^s & \pi_{11}^s
\end{bmatrix},
\]

Observe that

\[
Q_1(1)Q_2(1) = P^2(1)
\]
6. To find the linear term in the Taylor expansion for entropy, we use the spectral representation of the matrix $P(s)$. Let

$\lambda(s)$ be the main eigenvalue of $P(s)$

$r_1^t(s), l_1(s)$ be the corresponding right and left main eigenvectors,

$\mu(s)$ be the second eigenvalue,

$r_2^t(s), l_2(s)$ be the respective right and left eigenvectors.

The matrix spectral representation yields

$$P^k(s) = \lambda^k(s) r_1^t(s) l_1(s) + \mu^k(s) r_2^t(s) l_2(s).$$

Using this we finally obtain

$$\frac{\partial^2}{\partial \varepsilon \partial s} R(s, \varepsilon) \bigg|_{\varepsilon=0, s=1} = n \pi(1) r_1^t(1) l_1(1) l^t 1(1)$$

$$\times \left. \frac{\partial}{\partial s} \left( Q_1(s) Q_2(s) - P^2(s) \right) \right|_{s=1} r_1^t(1),$$

since $Q_1(1)Q_2(1) = P^2(1)$. 
Let \( H_s(Z^n_1) \) denote the Rényi’s entropy of order \( s \), that is,

\[
H_s(Z^n_1) = \log \frac{\sum z^n_1 P^s(z^n_1)}{1 - s}.
\]

Then the entropy rate is

\[
h_s(Z) = h_s(X) + \frac{\varepsilon}{(1 - s)\lambda(s)} l_1(s) \left( Q(s) - P^2(s) \right) r_1(s) + O(\varepsilon^2),
\]

where the Markov Rényi’s entropy rate is

\[
h_s(X) = \frac{1}{1 - s} \log \lambda(s).
\]