Analytic Pattern Matching: From DNA to Twitter

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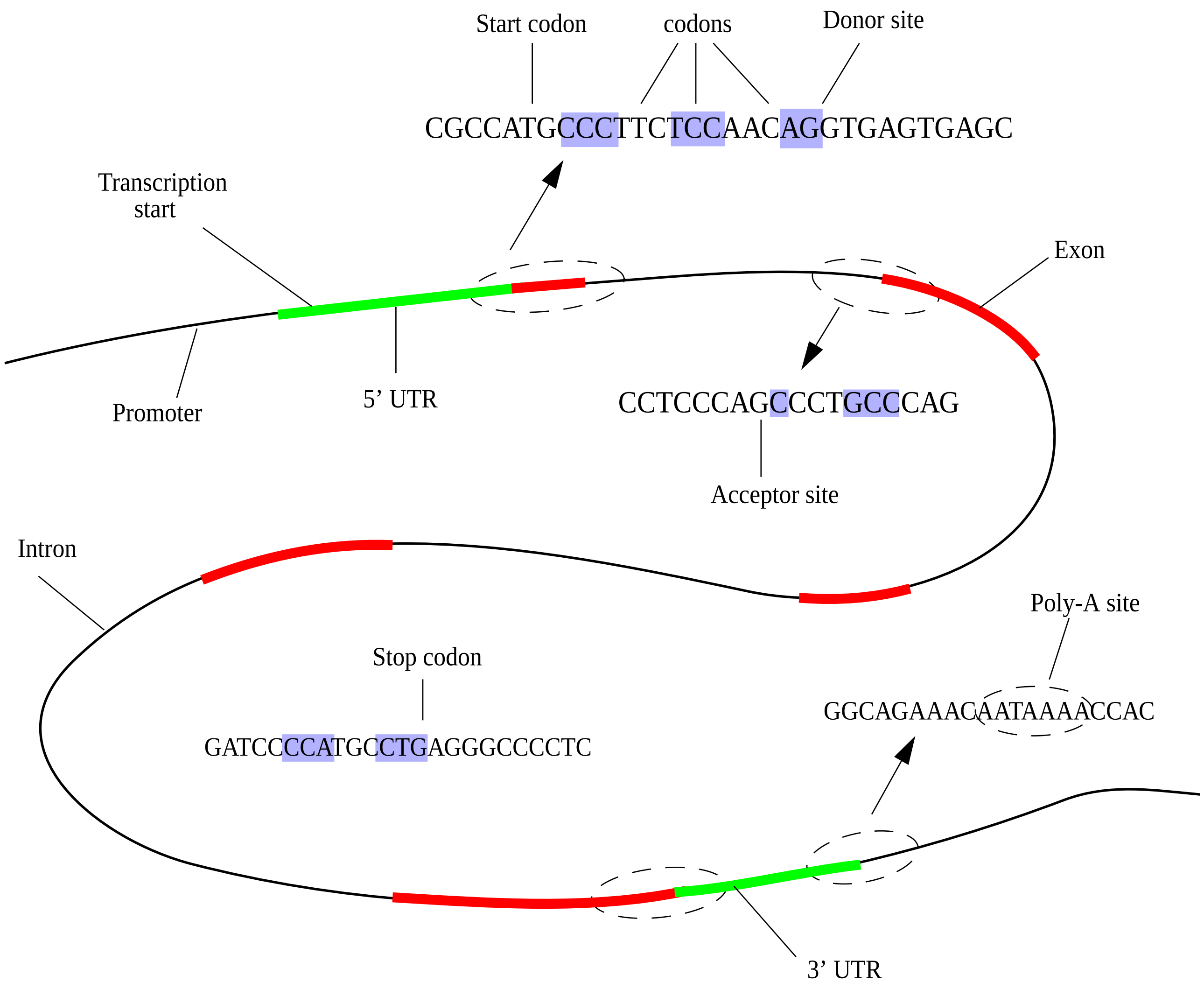
*Joint work with Philippe Jacquet

Outline

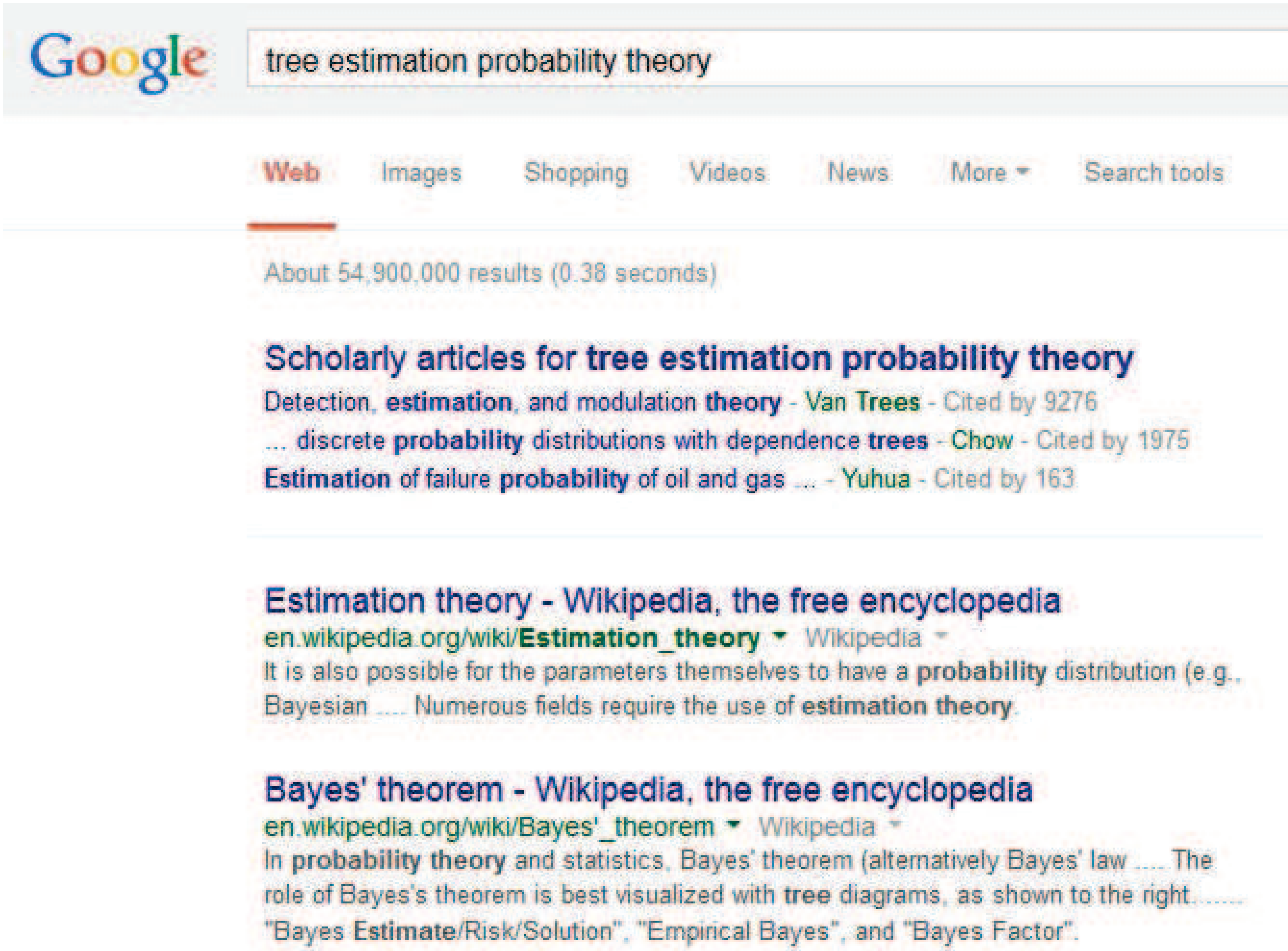
1. Motivations
   - Finding Biological Signals
   - Searching Google
   - Classifying Twitter
2. Pattern Matching Problems
   - Exact String Matching
   - Constrained String Matching
   - Generalized String Matching
   - Subsequence String Matching
   - String Complexity
3. Analysis & Applications
   - Exact String Matching & Finding Biological Motifs
   - Hidden Patterns & Intrusion Detection
   - Joint String Complexity & Classification of Twitter

Motivation – Biology & String Matching

Biological world is highly stochasticand inhomogeneous (S. Salzberg).



Motivation – Google & Subsequence Matching



Motivation – Intrusion Detection & Hidden Patterns

Convert all color commands to black or
Since PostScript files are often extremely large, it makes sense to try to compress them with either the zip or gzip programs. In such a case, the eps file is replace by a file with extension zip or eps gz, or eps-gz. Two problems now arise: first LATEX cannot read such files to obtain the bounding box information, and secondly, the driver needs to unpack such a file to include it in the final output. This can be accomplished with, for example: Declare-GraphicsRule.eps.gzeps.eps.bbgunzip which stablizes the graphics type as eps with the bounding box information in the file of the same name and extension. Con-vrt all color commands to black or white.

How to know whether this subsequence observed in an audit file **constitutes an attack** or is merely a result of **randomness?**

Goal: Minimize the number of false positives!

Motivation – Twitter & String Complexity

"allow users to download an entire movie in one second." I need this http://t.co/3fbNfKEkah
Green energy boss accuses Govt of obstructing renewable energy development http://t.co/v5Lq2Jx1GQ

Figure 1: Two similar twitter texts have many common words

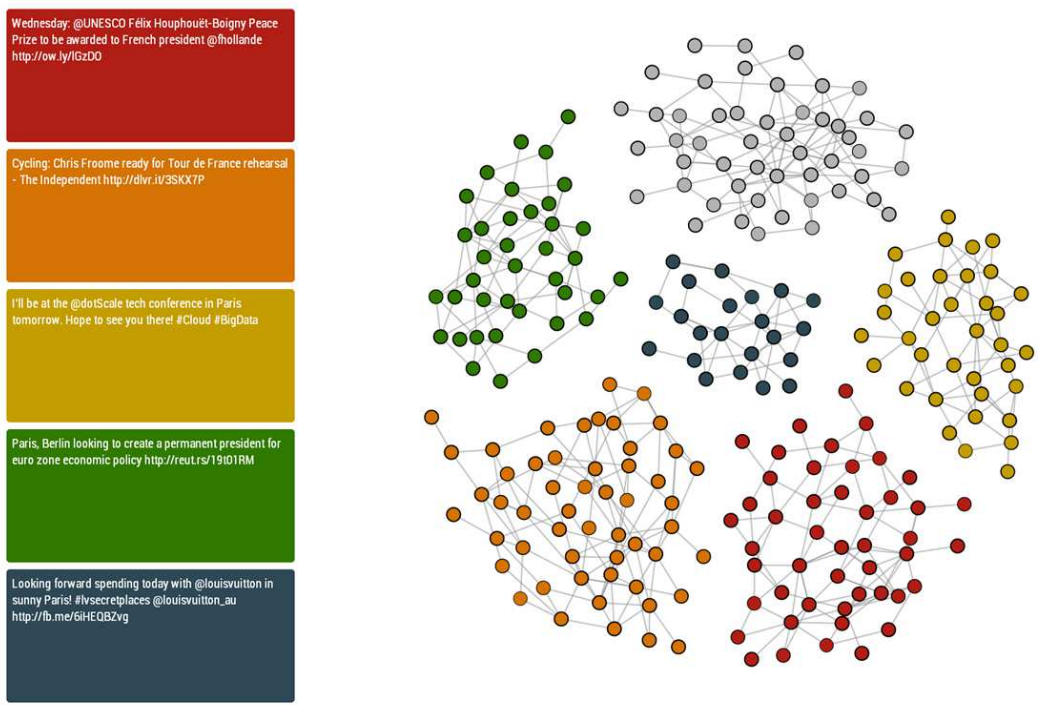


Figure 2: Twitters Classification

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3. Analysis & Applications

Pattern Matching

Let $\mathcal{W}$ and T be (set of) strings generated over a finite alphabet $\mathcal{A}$.

We call $\mathcal{W}$ the pattern and T the text. The text T is of length n and is generated by a probabilistic source.

Text: $T_m^n = T_m \dots T_n$.
Pattern: $\mathcal{W} = w_1 \dots w_m, \quad w_i \in \mathcal{A}$;
Set of patterns: $\mathcal{W} = \{\mathcal{W}_1, \dots, \mathcal{W}_d\}$ with $\mathcal{W}_i \in \mathcal{A}^{m_i}$.

Basic question:

how many times $\mathcal{W}$ occurs in T (or how long to wait until $\mathcal{W}$ occurs in T).

Define

$$O_n(\mathcal{W}) = \#\{i : \ T_{i-m+1}^i = \mathcal{W}, \ m \leq i \leq n\}$$

as the number of w occurrences in the text T_1^n.

Our goal: Study probabilistic behavior of $O_n(\mathcal{W})$ for various pattern matching problems using tools of analytic combinatorics.

Variations on Pattern Matching

(Exact) String Matching: In the exact string matching the pattern

$$\mathcal{W} = w_1 \ldots w_m$$

is a given string (i.e., consecutive sequence of symbols).

Generalized String Matching: In the generalized pattern matching a set of patterns (rather than a single pattern) is given, that is,

$$\mathcal{W} = (\mathcal{W}_0, \mathcal{W}_1, \ldots, \mathcal{W}_d), \quad \mathcal{W}_i \in \mathcal{A}^{m_i}$$

where $\mathcal{W}_i$ itself for $i \geq 1$ is a subset of $\mathcal{A}^{m_i}$.
The set $\mathcal{W}_0$ is called the forbidden set.

Three cases to be considered:

$\mathcal{W}_0 = \emptyset$ — interest in the number of patterns from $\mathcal{W}$ occurring in the text.

$\mathcal{W}_0 \neq \emptyset$ — we study the number of $\mathcal{W}_i$, $i \geq 1$ pattern occurrences under the condition that no pattern from $\mathcal{W}_0$ occurs in the text.

$\mathcal{W}_i = \emptyset$, $i \geq 1$, $\mathcal{W}_0 \neq \emptyset$ — restricted pattern matching.

Pattern Matching Problems

Hidden Words or Subsequence Pattern Matching: We search for a subsequence $\mathcal{W} = w_1 \ldots w_m$ rather than a string in a text.
That is, there are indices $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ such that

$$T_{i_1} = w_1, \; T_{i_2} = w_2, \cdots, \; T_{i_m} = w_m.$$

We also say that the word $\mathcal{W}$ is "hidden" in the text.

For example:

$$\begin{aligned} \mathcal{W} &= \textbf{date} \\ T &= \textbf{hidden pattern} \end{aligned}$$

occurs four times as a subsequence in the text as hidden pattern.

Joint String Complexity: For a given string X, we ask how many distinct subwords it contains. This is called string complexity.

For two strings X and Y, we want to know how many common and distinct subwords they contain. This is called joint string complexity.

New Book on Pattern Matching

How do you distinguish a cat from a dog by their DNA? Did Shakespeare really write all of his plays?

Pattern matching techniques can offer answers to these questions and to many others, from molecular biology, to telecommunications, to classifying Twitter content.

This book for researchers and graduate students demonstrates the probabilistic approach to pattern matching, which predicts the performance of pattern matching algorithms with very high precision using analytic combinatorics and analytic information theory. Part I compiles known results of pattern matching problems via analytic methods. Part II focuses on applications to various data structures on words, such as digital trees, suffix trees, string complexity and string-based data compression. The authors use results and techniques from Part I and also introduce new methodology such as the Mellin transform and analytic depoissonization.

More than 100 end-of-chapter problems help the reader to make the link between theory and practice.

Philippe Jacquet is a research director at INRIA, a major public research lab in Computer Science in France. He has been a major contributor to the Internet OLSR protocol for mobile networks. His research interests involve information theory, probability theory, quantum telecommunication, protocol design, performance evaluation and optimization, and the analysis of algorithms. Since 2012 he has been with Alcatel-Lucent Bell Labs as head of the department of Mathematics of Dynamic Networks and Information. Jacquet is a member of the prestigious French Corps des Mines, known for excellence in French industry, with the rank of “Ingenieur General”. He is also a member of ACM and IEEE.

Wojciech Szpankowski is Saul Rosen Professor of Computer Science and (by courtesy) Electrical and Computer Engineering at Purdue University, where he teaches and conducts research in analysis of algorithms, information theory, bioinformatics, analytic combinatorics, random structures, and stability problems of distributed systems. In 2008 he launched the interdisciplinary Institute for Science of Information, and in 2010 he became the Director of the newly established NSF Science and Technology Center for Science of Information. Szpankowski is a Fellow of IEEE and an Erskine Fellow. He received the Humboldt Research Award in 2010.

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Jacquet and Szpankowski Analytic Pattern Matching

CAMBRIDGE

Philippe Jacquet and Wojciech Szpankowski

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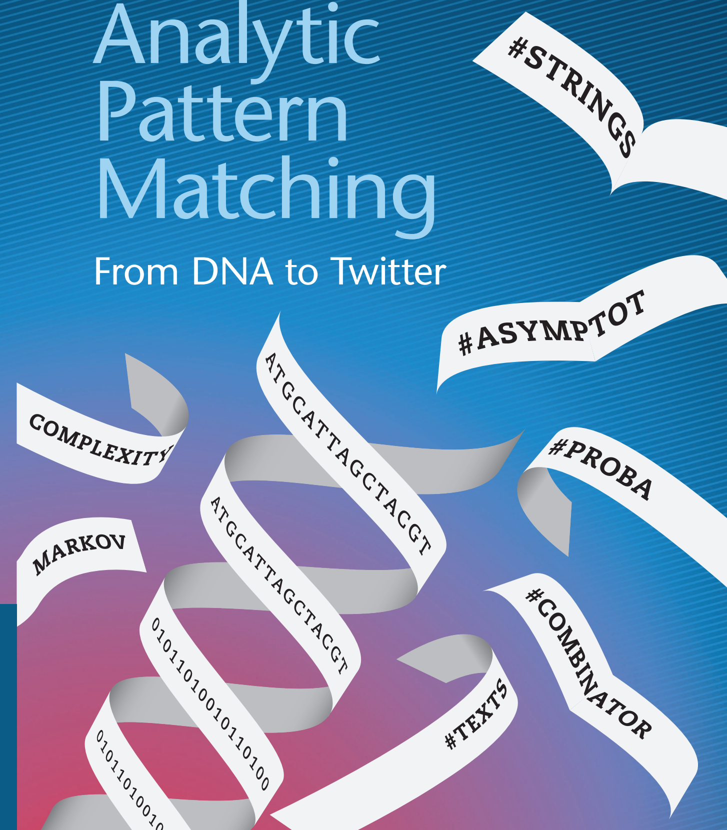

Book Contents

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Analysis: Exact String Matching

Memoryless Source: The text is generated by a memoryless source with probability of seeing $a \in \mathcal{A}$ equal to $P(a)$.

Tools. Symbolic calculus and analytic tools of languages:
(i) Language $\mathcal{L}$ is a collection of words satisfying some properties.
(ii) Generating function $L(z)$ of language $\mathcal{L}$ is defined as

$$L(z) = \sum_{u \in \mathcal{L}} P(u) z^{|u|}.$$

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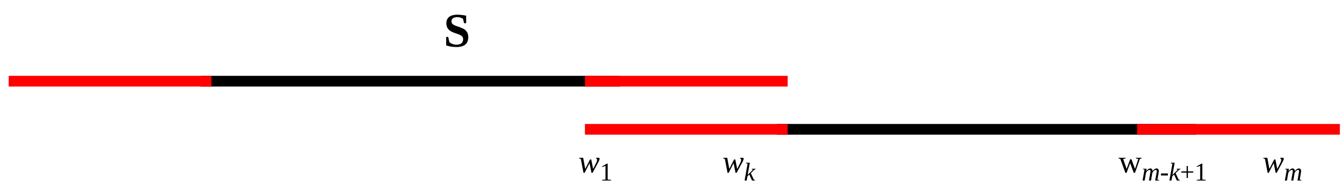
Autocorrelation Polynomial: For $\mathcal{W}$ define the autocorrelation set $\mathcal{S}$ as:

$$\mathcal{S} = \{w_{k+1}^m : \ w_1^k = w_{m-k+1}^m\},$$

and $\mathcal{WW}$ is the set of positions k satisfying $w_1^k = w_{m-k+1}^m$.



The generating function $S(z)$ of $\mathcal{S}$ is called the autocorrelation polynomial:

$$S(z) = \sum_{k \in \mathcal{WW}} P(w_{k+1}^m) z^{m-k}.$$

Example: For $\mathcal{W} = bab$, we have $\mathcal{WW} = \{1, 3\}$ and $\mathcal{S} = \{\epsilon, ab\}$.

Language $\mathcal{T}_r$ and Associated Languages

Define $\mathcal{T}_r$ as set of words containing exactly $r \geq 1$ occurrences of $\mathcal{W}$:

$$\mathcal{T}_r = \mathcal{R} \cdot \mathcal{M}^{r-1} \cdot \mathcal{U}.$$

which can be illustrated for $\mathcal{T}_4$ as follows

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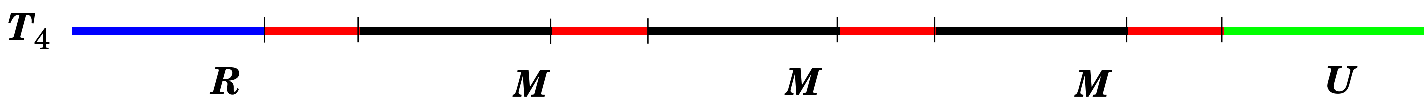

(i) Language $\mathcal{R}$: set of words containing only one occurrence of $\mathcal{W}$, located at the right end. **For example**: for $\mathcal{W} = aba$, we have $ccaba \in \mathcal{R}$.

(ii) Language $\mathcal{U}$:

$$\mathcal{U} = \{u : \ \mathcal{W} \cdot u \cdot \in \mathcal{T}_1\}$$

that is, a word $u \in \mathcal{U}$ if $\mathcal{W} \cdot u$ has exactly one occurrence of $\mathcal{W}$ at the left end of $\mathcal{W} \cdot u$. **For example**: $bba \in \mathcal{U}$ (since $ababba \in \mathcal{T}_1$) but $ba \notin \mathcal{U}$

(iii) Language $\mathcal{M}$:

$$\mathcal{M} = \{u : \ \mathcal{W} \cdot u \in \mathcal{T}_2 \text{ and } \mathcal{W} \text{ occurs at the right of } \mathcal{W} \cdot u\},$$

that is, $\mathcal{M}$ is a language such that $\mathcal{W}\mathcal{M}$ has exactly two occurrences of $\mathcal{W}$ at the left and right end of a word from $\mathcal{M}$.
For example: $ba \in \mathcal{M}$ since $ababa$).

Language Relations & Generating Functions

Lemma 1. (i) *The languages* $\mathcal{M}, \mathcal{U}$ *and* $\mathcal{R}$ *satisfy:*

$$\mathcal{U} \cdot \mathcal{A} = \mathcal{M} + \mathcal{U} - \{\epsilon\},$$

$$\bigcup_{k \geq 1} \mathcal{M}^k = \mathcal{A}^* \cdot \mathcal{W} + \mathcal{S} - \{\epsilon\}, \quad \mathcal{W} \cdot \mathcal{M} = \mathcal{A} \cdot \mathcal{R} - (\mathcal{R} - \mathcal{W}),$$

where $\mathcal{A}^*$ *is the set of all words.*

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\end{aligned}$$

where $\mathcal{A}^*$ *is the set of all words.*

(ii) *The generating functions associated with languages* $\mathcal{M}, \mathcal{U}$ *and* $\mathcal{R}$ *satisfy*

$$\begin{aligned}
U_{\mathcal{W}}(z) &= \frac{M(z) - 1}{z - 1} \\
\frac{1}{1 - M(z)} &= S_{\mathcal{W}}(z) + P(\mathcal{W}) \frac{z^m}{1 - z}, \quad R(z) = P(\mathcal{W}) z^m \cdot U_{\mathcal{W}}(z)
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(iii) *The generating functions* $T_r(z) = \sum_{n \geq 0} \Pr\{O_n(\mathcal{W}) = r\} z^n$ *and* $T(z, u) = \sum_{r=1}^{\infty} T_r(z) u^r$ *satisfy*

$$T_r(z) = R(z) M_{\mathcal{W}}^{r-1}(z) U_{\mathcal{W}}(z), \quad r \geq 1$$

$$T(z, u) = R(z) \frac{u}{1 - u M(z)} U_{\mathcal{W}}(z).$$

Main Results: Asymptotics

Theorem 1 (Regnier and W.S.)**.** **(i)** *Moments. for $n \geq m$ we have*

$$\mathbf{E}[O_n(\mathcal{W})] = P(\mathcal{W})(n-m+1), \quad \mathbf{Var}[O_n(\mathcal{W})] = nc_1 + c_2$$

with

$$c_1 = P(\mathcal{W})(2S(1) - 1 - (2m-1)P(\mathcal{W}). - (m-1)(2S(1) - 1) - 2S'(1)).$$

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$$\Pr\{O_n(\mathcal{W}) = r\} \sim \sum_{j=1}^{r+1} (-1)^j a_j \binom{n}{j-1} \rho_{\mathcal{W}}^{-(n+j)}$$

where $\rho_{\mathcal{W}}$ is the smallest root of $D_{\mathcal{W}}(z) = (1-z)S_{\mathcal{W}}(z) + z^m P(\mathcal{W}) = 0$.

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Central Limit Law: *For* $r = \mathbf{E}[O_n] + x\sqrt{\mathbf{Var}O_n}$ *for* $x = O(1)$

$$\Pr\{O_n(\mathcal{W}) = r\} = \frac{1}{\sqrt{2\pi c_1 n}} e^{-\frac{1}{2}x^2} \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right).$$

Large Deviations: *For Case* $r = (1+\delta)EO_n$ *with* $a = (1+\delta)P(\mathcal{W})$ *we have*

$$\Pr\{O_n(\mathcal{W}) \sim (1+\delta)EO_n\} = \frac{e^{-(n-m+1)I(a)+\delta_a}}{\sigma_a\sqrt{2\pi(n-m+1)}}$$

where $I(a) = a\omega_a + \rho(\omega_a)$ *and* $\rho(t)$ *to be the root of* $1 - e^t M_{\mathcal{W}}(e^{\rho}) = 0$*.*

Biology – Weak Signals and Artifacts

Denise and Regnier (2002) observed that in biological sequence whenever a word is overrepresented, then its subwords are also overrepresented.
For example, if $\mathcal{W}_1 = AATAAA$, then

$$\mathcal{W}_2 = ATAAAN$$

is also overrepresented.
Overrepresented subwords are called artifact, and it is important to disregard automatically noise created by artifacts.

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This harder question needs a new approach thru Generalized Pattern Matching to show that

$$\mathbf{E}[O_n(\mathcal{W}_2)|O_n(\mathcal{W}_1) = k] \sim \alpha n.$$

When $\mathcal{W}_1$ is overrepresented α differs significantly from $\mathbf{E}[O_n(\mathcal{W}_2)]$.

Polyadenylation Signals in Human Genes

Beaudoing et al. (2000) studied several variants of the well known AAUAAA polyadenylation signal in mRNA of humans genes.

Using our approach Denise and Regnier (2002) discovered/eliminated all artifacts and found new signals in a much simpler and reliable way.

Hexamer	Obs.	Rk	Exp.	Z-sc.	Rk	Cd.Exp.	Cd.Z-sc.	Rk
AAUAAA	3456	1	363.16	167.03	1			1
AAAUAA	1721	2	363.16	71.25	2	1678.53	1.04	1300
AUAAAA	1530	3	363.16	61.23	3	1311.03	6.05	404
UUUUUU	1105	4	416.36	33.75	8	373 .30	37.87	2
AUAAAU	1043	5	373.23	34.67	6	1529.15	12.43	4078
AAAAUA	1019	6	363.16	34.41	7	848.76	5.84	420
UAAAAU	1017	7	373.23	33.32	9	780.18	8.48	211
AUUAAA	1013	I	373.23	33.12	10	385.85	31.93	3
AUAAAG	972	9	184.27	58.03	4	593.90	15.51	34
UAAUAA	922	10	373.23	28.41	13	1233.24	–8.86	4034
UAAAAA	922	11	363.16	29.32	12	922.67	9.79	155
UUAAAA	863	12	373.23	25.35	15	374.81	25.21	4
CAAUAA	847	13	185.59	48.55	5	613.24	9.44	167
AAAAAA	841	14	353.37	25.94	14	496.38	15.47	36
UAAAUA	805	15	373.23	22.35	21	1143.73	–10.02	4068

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Hidden Patterns

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where we put additional constraints that

$$i_{j+1} - i_j \leq d_j.$$

The $I = (i_1, \ldots, i_m)$-tuple is called a position and $\mathcal{D} = (d_1, \ldots, d_m$ constitutes the constraints.
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Let $O_n(\mathcal{W})$ be the number of $\mathcal{W}$ occurrences in T. Observe that

$$O_n(\mathcal{W}) = \sum_I X_I$$

where

$$X_I := 1 \;\; if \;\; \mathcal{W} \text{ occurs at position } I \text{ in } T_n$$

and 0 otherwise.

How to Analyze It – De Bruijn Automata

1. The $(\mathcal{W}, \mathcal{D})$ **constrained subsequence problem** is viewed as the generalized string matching.

Example: If $(\mathcal{W}, \mathcal{D}) = a\#_2 b$, then $\mathcal{W} = \{ab, aab, abb\}$.

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2. de Bruijn Automaton. Let $M = \max\{length(\mathcal{W})\} - 1$. Define a de Bruijn automaton over $\mathcal{B}$ where

$$\mathcal{B} = \mathcal{A}^M.$$

De Bruijn automaton is built over $\mathcal{B}$.

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3. Let $b \in \mathcal{B}$ and $a \in \mathcal{A}$. Then the transition from the state b upon scanning symbol a of the text is to $\hat{b} \in \mathcal{B}$ such that

$$ba \mapsto \hat{b} = b_2 b_3 \cdots b_M a.$$

For example

$$\underbrace{abb}_{\mathcal{B}} \underbrace{a}_{\mathcal{A}} \mapsto \underbrace{bba}_{\mathcal{B}}.$$

4. The Transition Matrix: $\mathbf{T}(u)$ is a complex-valued transition matrix defined as:

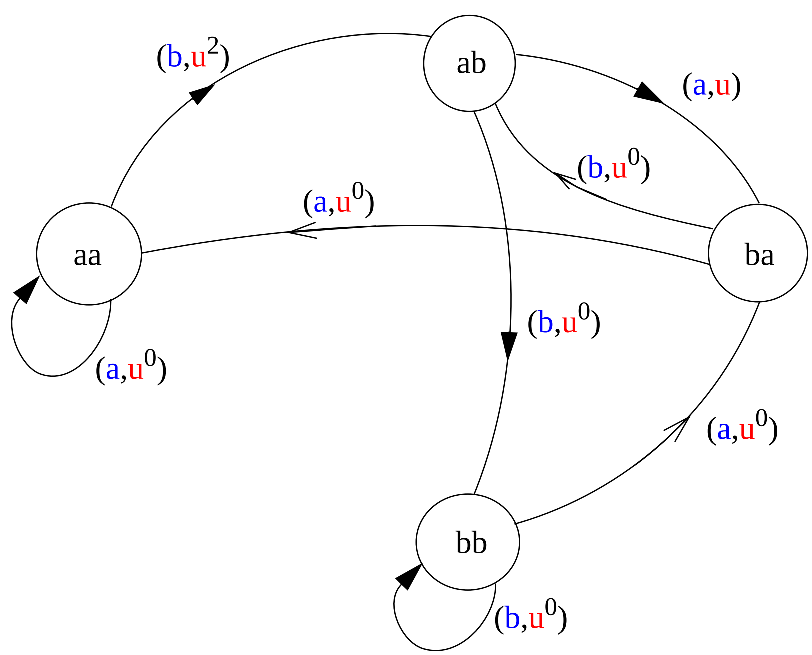
$$[\mathbf{T}(u)]_{b,\hat{b}} := P(a) u^{O_{M+1}(ba) - O_M(b)} [\![\hat{b} = b_2 b_3 \cdots b_M a]\!]$$

where $O_M(b)$ is the number of pattern occurrences $\mathcal{W}$ in the text b.

Example

5. Example. Let $\mathcal{W} = \{ab, aab, aba\}$. Then $M = 2$, and the the de Bruijn graph and matrix $\mathbf{T}(u)$ are shown below

$$\mathbf{T}(u) = \begin{array}{c} \\ aa \\ ab \\ ba \\ bb \end{array} \begin{array}{c} \begin{array}{cccc} aa & ab & ba & bb \end{array} \\ \begin{pmatrix} P(a) & P(b)\,u^2 & 0 & 0 \\ 0 & 0 & P(a)\,u & P(b) \\ P(a) & P(b) & 0 & 0 \\ 0 & 0 & P(a) & P(b) \end{pmatrix} \end{array} .$$



Generating Functions

6. Using properties of product of matrices we conclude that

$$O_n(u) = \mathbf{E}[u^{O_n(\mathcal{W})}] = \mathbf{b}^t(u)\mathbf{T}^n(u)\vec{1}$$

where $\mathbf{b}^t(u)$ is an initial vector and $\vec{1} = (1, \ldots, 1)$.

7. Spectral Decomposition
Let $\lambda(u)$ be the largest eigenvalue of $\mathbf{T}(u)$. Then

$$O_n(u) = c(u)\lambda^n(u)(1 + O(A^n))$$

for some $A < 1$. This proves that the generating function $O_n(u)$ satisfies the so called quasi-power law.

8. The above formula suggests that we deal with weakly dependent random variables described by the generating function $\lambda(u)$. Hence, for example

$$\mathbf{E}[O_n] = n\lambda'(1) + O(1).$$

Final Results

Mean and Variance

$$\begin{aligned}\mathbf{E}[O_n(\mathcal{W})] &= n\Lambda'(0) + O(1) = nP(\mathcal{W}) + O(1),\\ \mathbf{Var}[O_n(\mathcal{W}) &= n\Lambda''(0) + O(1) = n\sigma^2(\mathcal{W}) + O(1)\end{aligned}$$

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Central Limit Theorem

$$\Pr\left\{\frac{O_n - nP(\mathcal{W})}{\sigma(\mathcal{W})\sqrt{n}} \leq x\right\} \sim \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}$$

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Large Deviations

If $\mathbf{T}(u)$ is primitive, then

$$\Pr\{O_n(\mathcal{W}) = a\mathbf{E}[O_n]\} \sim \frac{1}{\sigma_a\sqrt{2\pi n}} e^{-nI(a)+\theta_a}$$

where $I(a)$ can be explicitly computed, and θ_a is a known constant.

Based on: Flajolet, W.S. and Vallee, JACM, 2006.

Some Experiments

The complete works of Shakespeare are found under

`http://the-tech.mit.edu/Shakespeare/.`

We first extracted the full text of Hamlet stripped of all the comments and searched for

$$\mathcal{W} = thelawisgaussian$$

		w = thelawisgaussian		$\tilde{w}$ = naissuagsiwaleht	
d	Expected (E)	Occurred (O_n)	O_n/E	Occurred (O_n)	O_n/E
13	9.195E+01	0	0.00	18	0.19
14	2.794E+02	693	2.47	371	1.32
15	7.866E+02	1,526	5.46	2,379	3.02
18	1.211E+04	31,385	2.58	14,123	1.16
20	5.886E+04	124,499	2.11	41,066	0.69
25	1.673E+06	2,527,148	1.51	1,277,584	0.76
30	2.577E+07	40,001,940	1.55	25,631,589	0.99
40	1.928E+09	2,757,171,648	1.42	2,144,491,367	1.11
50	5.482E+10	76,146,232,395	1.38	48,386,404,680	0.88
∞	1.330E+48	1.36554E+48	1.03	1.38807E+48	1.04

Figure 3: Observed occurrences (O_n) versus predicted values (expectations, E) in the alphabetical characters of Hamlet.

Reliable Threshold for Intrusion Detection

We argued that one needs a reliable threshold for intrusion detection. If false alarms are to be avoided, the problem is of finding a threshold $\alpha_0 = \alpha_0(\mathcal{W}; n, \beta)$ such that

$$P(O_n(\mathcal{W}) > \alpha_{th}) \leq \beta(= 10^{-5}).$$

Our results shows that

$$\alpha_{th} = nP(\mathcal{W}) + x_0\sigma(\mathcal{W})\sqrt{n}, \quad \beta = \frac{1}{\sqrt{2\pi}}\int_{x_0}^{\infty} e^{-t^2/2}dt \sim \frac{1}{x_0}e^{-x_0^2/2}.$$

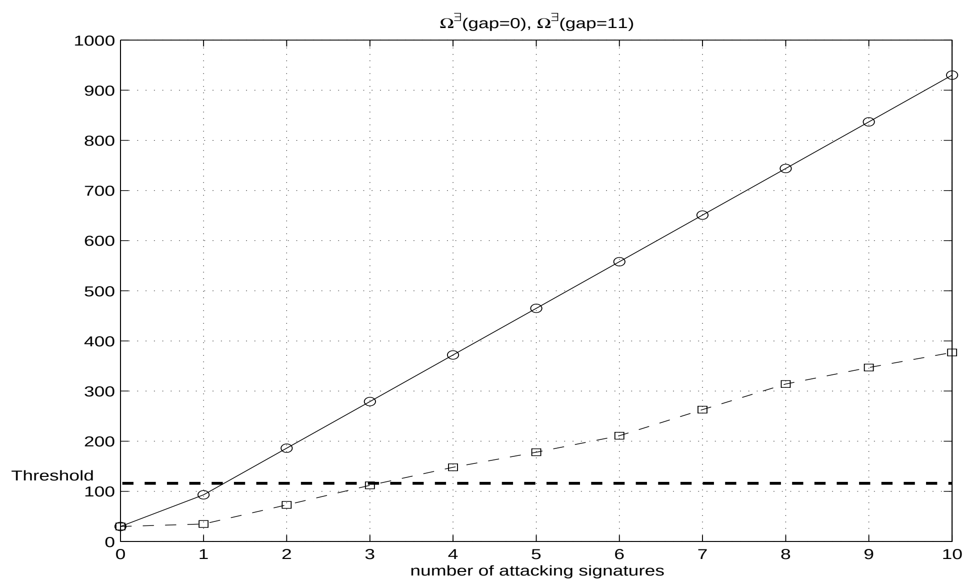

Figure 4: Pattern=wojciech, window=100 (cf. Gwadrea at al. (2004).

Outline Update

1. Motivations
2. Pattern Matching Problems
3. Analysis & Applications
   - Exact String Matching & Finding Biological Motifs
   - Hidden Patterns & Intrusion Detection
   - Joint String Complexity & Classification of Twitter

Some Definitions

String Complexity of a single sequence is the number of distinct substrings.

Throughout, we write X for the string and denote by $I(X)$ the set of *distinct substrings* of X over alphabet $\mathcal{A}$.

Example. If $X = aabaa$, then

$$I(X) = \{\epsilon, a, b, aa, ab, ba, aab, aba, baa, aaba, abaa, aabaa\},$$

so $|I(X)| = 12$. But if $X = aaaaa$, then

$$I(X) = \{\epsilon, a, aa, aaa, aaaa, aaaaa\},$$

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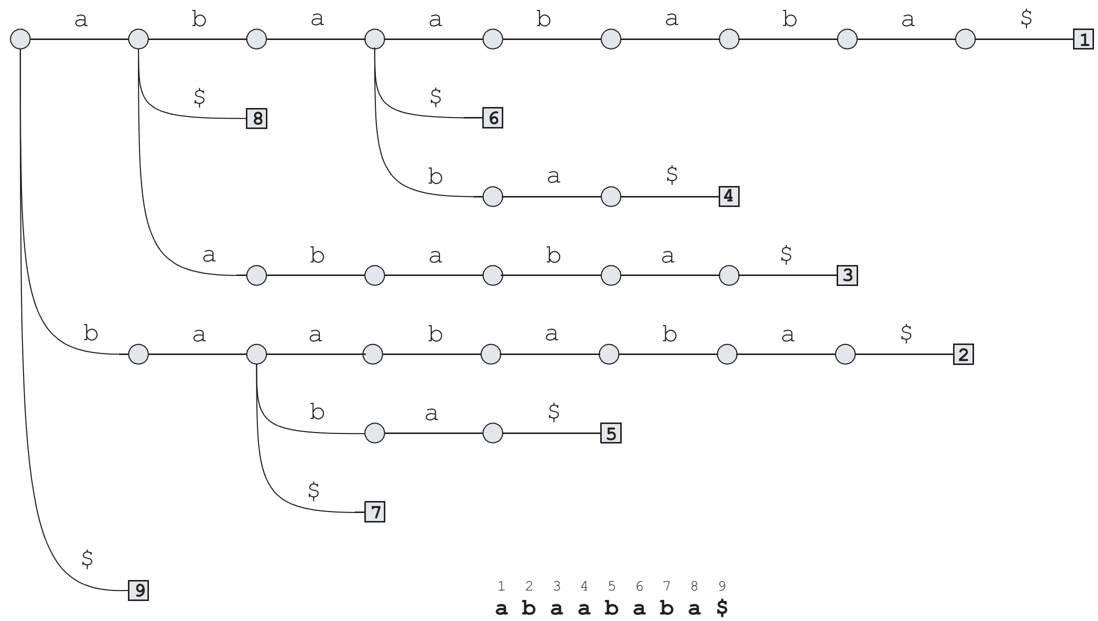
so $|I(X)| = 6$.

The **string complexity** is the cardinality of $I(X)$ and we study here the *average* string complexity

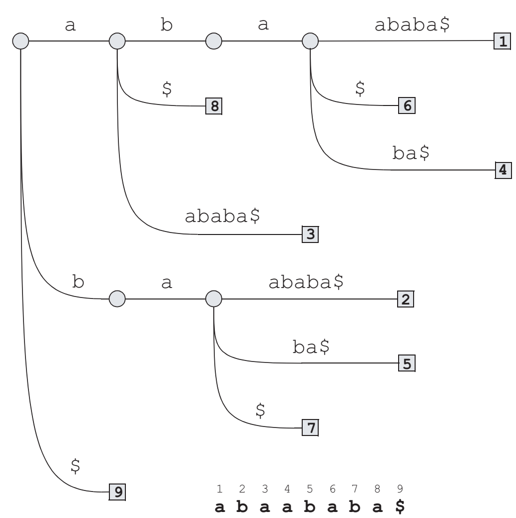
$$\mathbf{E}[|I(X)|] = \sum_{X \in \mathcal{A}^n} P(X)|I(X)|.$$

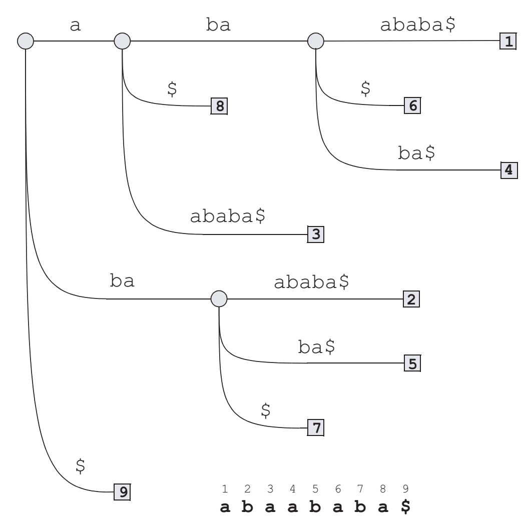
where X is generated by a memoryless/Markov source.

Suffix Trees and String Complexity



Non-compact suffix trie for $X = \texttt{abaababa}$ and string complexity $I(X) = 24$.





String Complexity = # internal nodes in a non-compact suffix tree.

Some Simple Facts

Let $O(w)$ denote the number of times that the word w occurs in X. Then

$$|I(X)| = \sum_{w \in \mathcal{A}^*} \min\{1, O(w)\}.$$

Since between any two positions in X there is one and only one substring:

$$\sum_{w \in \mathcal{A}^*} O(w) = \frac{(|X|+1)|X|}{2}.$$

Hence

$$|I(X)| = \frac{(|X|+1)|X|}{2} - \sum_{w \in \mathcal{A}^*} \max\{0, O(w) - 1\}.$$

Define: $C_n := \mathbf{E}[|I(X)| \mid |X| = n]$. Then

$$C_n = \frac{(n+1)n}{2} - \sum_{w \in \mathcal{A}^*} \sum_{k \geq 2} (k-1) P(O_n(w) = k).$$

We need to study probabilistically $O_n(w)$: that is:

number of w occurrences in a text X generated a probabilistic source.

Methodology and Some Results

Last expression allows us to write

$$C_n = \frac{(n+1)n}{2} + \mathbf{E}[S_n] - \mathbf{E}[L_n]$$

where $\mathbf{E}[S_n]$ and $\mathbf{E}[L_n]$ are, respectively, the average size and path length in the associated (compact) suffix trees.

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Poissonization → Mellin → Inverse Mellin (SP) → De-Poissonization

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We know that (Jacquet & Regnier, 1989; W.S., 2001)

$$\mathbf{E}[S_n] = \frac{1}{h}(n + \Psi(\log n)) + o(n), \quad \mathbf{E}[L_n] = \frac{n \log n}{h} + n\Psi_2(\log n) + o(n),$$

where $\Psi(\log n)$ and $\Psi_2(\log n)$ are periodic functions. Therefore,

$$C_n = \frac{(n+1)n}{2} - \frac{n}{h}(\log n - 1 + Q_0(\log n) + o(1))$$

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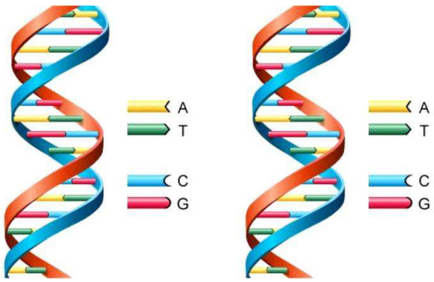
Theorem 2 (Janson, Lonardi, W.S., 2004). *For unbiased memoryless source:*

$$C_n = \binom{n+1}{2} - n \log_{|\mathcal{A}|} n + \left(\frac{1}{2} + \frac{1-\gamma}{\ln |\mathcal{A}|} + Q_1(\log_{|\mathcal{A}|} n)\right) n + O(\sqrt{n \log n})$$

where $\gamma \approx 0.577$ is Euler's constant and Q_1 is a periodic function.

Joint String Complexity

For X and Y, let $J(X, Y)$ be the set of common words between X and Y.



The joint string complexity is
$|J(X, Y)| = |I(X) \cap I(Y)|$

Example. If $X = aabaa$ and $Y = abbba$, then $J(X, Y) = \{\varepsilon, a, b, ab, ba\}$.

Goal. Estimate

$$J_{n,m} = \mathbf{E}[|J(X, Y)|]$$

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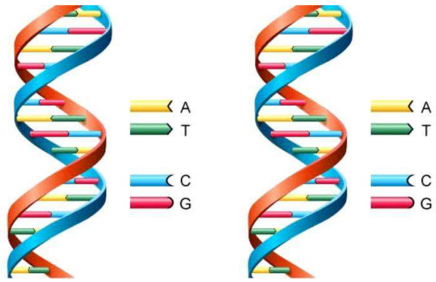

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Some Observations. For any word $w \in \mathcal{A}^*$

$$|J(X,Y)| = \sum_{w \in \mathcal{A}^*} \min\{1, O_X(w)\} \cdot \min\{1, O_Y(w)\}.$$

When $|X| = n$ and $|Y| = m$, we have

$$J_{n,m} = \mathbf{E}[|J(X,Y)|] - 1 = \sum_{w \in \mathcal{A}^* - \{\varepsilon\}} P(O_n^1(w) \geq 1) P(O_m^2(w) \geq 1)$$

where $O_n^i(w)$ is the number of w-occurrences in a string of generated by source $i = 1, 2$ (i.e., X and Y) which we assume to be memoryless sources.

Independent Joint String Complexity

Consider two sets of n independently generated (memoryless) strings.

Let ${\Omega_n}^i(w)$ be the number of strings for which w is a prefix when the n strings are generated by a source $i = 1, 2$ define

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Theorem 3. *There exists $\varepsilon > 0$ such that*

$$J_{n,m} - C_{n,m} = O(\min\{n, m\}^{-\varepsilon})$$

for large n.

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Recurrence for $C_{n,m}$

$$C_{n,m} = 1 + \sum_{a \in \mathcal{A}} \sum_{k,\ell \geq 0} \binom{n}{k} P_1(a)^k (1 - P_1(a))^{n-k} \binom{m}{\ell} P_2(a)^\ell (1 - P_2(a))^{m-\ell} C_{k,\ell}$$

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Main Results

Assume that $\forall a \in \mathcal{A}$ we have $P_1(a) = P_2(a) = p_a$.

Theorem 4. *For a biased memoryless source, the joint complexity is asymptotically*

$$C_{n,n} = n\frac{2\log 2}{h} + Q(\log n)n + o(n),$$

where $Q(x)$ is a small periodic function (with amplitude smaller than 10^{-6}) which is nonzero only when the $\log p_a$, $a \in \mathcal{A}$, are rationally related, that is, $\log p_a / \log p_b \in \mathbb{Q}$.

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Assume that $P_1(a) \neq P_2(a)$.
Theorem 5. *Define $\kappa = \min_{(s_1,s_2)\in\mathcal{K}\cap\mathbb{R}^2}\{(-s_1 - s_2)\} < 1$, where s_1 and s_2 are roots of*

$$H(s_1, s_2) = 1 - \sum_{a\in\mathcal{A}} (P_1(a))^{-s_1}(P_2(a))^{-s_2} = 0.$$

Then

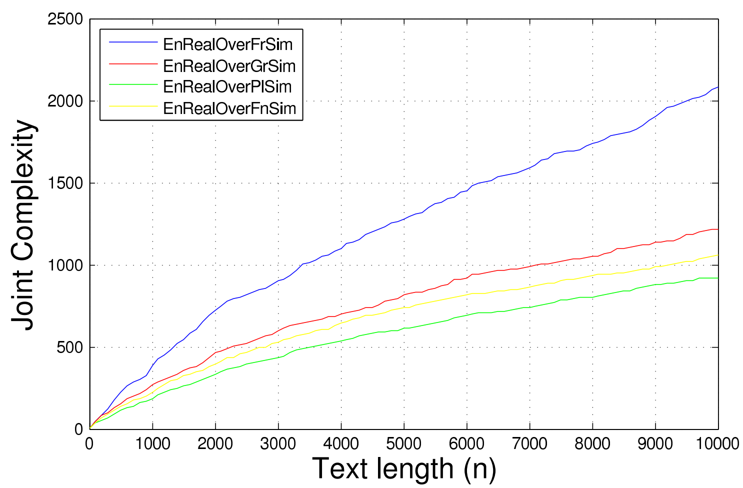
$$C_{n,n} = \frac{n^{\kappa}}{\sqrt{\log n}}\left(\frac{\Gamma(c_1)\Gamma(c_2)}{\sqrt{\pi\Delta H(c_1,c_2)\nabla H(c_1,c_2)}} + Q(\log n) + O(1/\log n)\right),$$

where Q is a double periodic function.

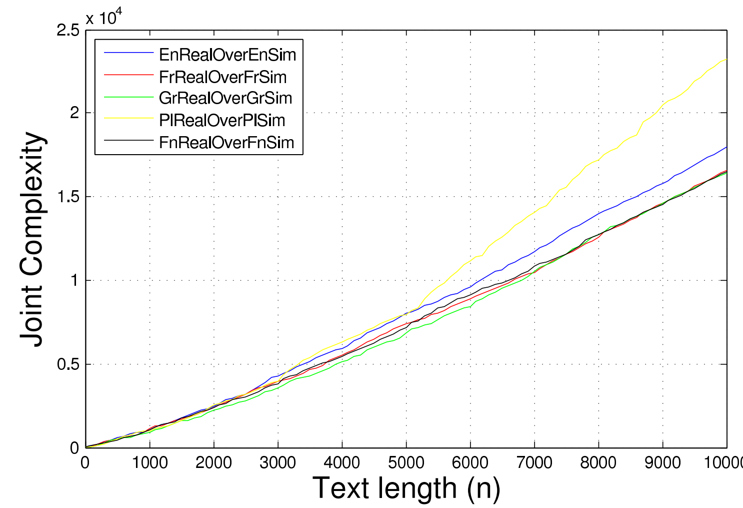
Classification of Sources

The growth of $C_{n,n}$ is:

- $\Theta(n)$ for identical sources;
- $\Theta(n^{\kappa}/\sqrt{\log n})$ for non identical sources with $\kappa < 1$.



(a)



(b)

Figure 5: Joint complexity: (a) English text vs French, Greek, Polish, and Finnish texts; (b) real and simulated texts (3rd Markov order) of English, French, Greek, Polish and Finnish language.

Acknowledgments

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Analytic Combinatorics

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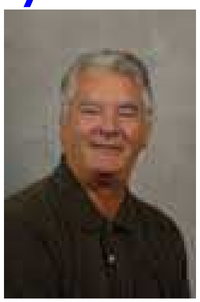

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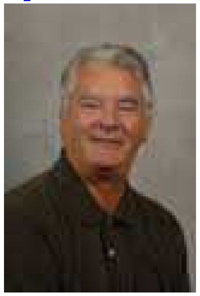

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. . . and **ALL** my co-authors.

That’s It

THANK YOU