

# Compression and Symmetry of Small-World Graphs and Structures

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## Abstract

For various purposes and, in particular, in the context of data compression, a graph can be examined at three levels. Its structure can be described as the unlabeled version of the graph; then the labeling of its structure can be added; and finally, given then structure and labeling, the contents of the labels can be described. Determining the amount of information present at each level and quantifying the degree of dependence between them, requires the study of symmetry, graph automorphism, entropy, and graph compressibility. In this paper, we focus on a class of small-world graphs. These are geometric random graphs where vertices are first connected to their nearest neighbors on a circle and then pairs of non-neighbors are connected according to a distance-dependent probability distribution. We establish the degree distribution of this model, and use it to prove the model's asymmetry in an appropriate range of parameters. Then we derive the relevant entropy and structural entropy of these random graphs, in connection with graph compression.

**Keywords** — Entropy, information, lossless compression, random graph, random network, small-world graph, graphical structure, symmetry

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# 1 Introduction

Our main aim in this work is to develop mathematical results on structural properties that are fundamental to statistical and information-theoretic problems involving the information shared between the labels and the structure of a random graph, specifically within the class of *small-world graphs*. For various statistical and signal processing tasks and, in particular, in the context of data compression, the information present in a graph can be examined at three levels. First, its *structure* can be described, that is, the unlabeled version of the graph. Second, its labeling can be described given its structure. And third, the actual contents of the labels can be described, given the structure and the labeling. In some problems, for example in recovering the node arrival order of dynamic networks [29], the goal is to first *recover* label information by examining a graph structure, and then to explain the structural properties (such as symmetry) involved in their analysis.

More formally, the labeled and unlabeled graph compression problems can be described as follows. Fix a graph model on the collection  $\mathcal{G}(n)$  of all simple, undirected, labeled graphs on  $n$  vertices. First, we aim to understand the best achievable performance of efficiently computable *source codes* for this model [15]. A source code  $(C_n, D_n)$  here consists of an encoder  $C_n$  mapping graphs in  $\mathcal{G}(n)$  to finite-length bit strings, and of a decoder  $D_n$  that inverts  $C_n$ . The goal is to make the (expected) length of the output bit string as short as possible. Of particular interest to us here is the related problem of the compression of graph *structures*. In this case, the encoder  $C_n$  is presented with a graph  $G_n$  isomorphic to a sample from  $\mathcal{G}(n)$ , and  $D_n(C_n(G_n))$  is only required to be a labeled graph isomorphic to  $G_n$ , so that only the structural information is preserved. We again seek to characterize efficient source codes with minimal code lengths. This optimal compression performance is characterized by the entropy of the distribution on unlabeled graphs induced by the model, which we call its *structural entropy*.

**Structural properties.** Several interesting structural properties and quantities arise naturally in connection with graph compression. As we describe next, determining the structural entropy often involves computing the size of the automorphism group of a graph, as well as the typical number of positive-probability *labeled representatives* (re-labelings or permutations) of a given structure.

In general, given a labeled graph  $G_n$  generated by some model on  $\mathcal{G}(n)$ , all  $n!$  label permutations lead to the same *structure*  $S_n := S_n(G_n)$ ; *however*, not all permutations may be permissible under the model, and some permutations may lead to the exact same graph. The latter property is well characterized by the automorphism group,  $\text{Aut}(G_n)$ , of  $G_n$ . When the cardinality of the automorphism group is one, then the graph is *asymmetric* since every feasible permutation is distinct (in term of the labeled graph) and gives the same structure. In some cases, such as the Erdős-Rényi (ER) model [11] and preferential attachment graphs (PAG) [8, 27], all permutations lead to the same graph with high probability. In other words, these models are invariant under isomorphism. Furthermore, in the ER model every permutation is feasible, unlike under the PAG model. For PAG graphs, the number of distinct re-labelings can be computed as the ratio of the number of feasible permutations,  $|\Gamma(G_n)|$ , and the size of the automorphism group,  $|\text{Aut}(G_n)|$ . As a consequence, the structural entropy of the unlabeled graph is a function of  $\log |\Gamma(G)|/|\text{Aut}(G)|$  as well as of the (labeled) graph entropy. However, when the model is not invariant under isomorphism, we need to actually estimate the *conditional entropy* of the (labeled) graph under a given structure.

One such class of models is the family of the small-world graphs [41, 24] on the circle. In this paper we focus on the symmetry, entropy and automorphism properties of graphs generated by this model.

**Contributions.** We study the small-world model [41, 24] where  $n$  vertices are arranged on the circle in increasing order, and each node is connected to its two nearest neighbors. Then different pairs of nodes are (independently) connected with probability proportional to  $1/k^a$ , where  $k$  is their distance and  $a \in (0, 1)$  is fixed parameter; precise definitions are given in Section 3. Such a model does not satisfy the two properties discussed above: It is not invariant under isomorphism, and not every permutation is feasible.

For the small-world model we first compute the mean degree of a node (Proposition 3.1), and in Theorem 3.4 we prove that it is asymmetric with high probability. This allows us to derive very accurate asymptotic estimates for the graph entropy and structural entropy; these are presented in Theorem 4.2 and Corollary 4.3. Finally, in Theorem 4.4 we give a precise upper bound on the conditional entropy of a small-world graph given its structure, a result which is of independent interest from the combinatorial point of view.

**Prior work.** There is a long history of very detailed results on the problem of determining the fundamental limits of the best achievable compression performance for sequential data; see, e.g., [38, 25, 26] and the references therein. However, the study of the compression problem for graph and tree models, in both the information theory and the computer science literature, is more recent [42, 7, 1, 14, 16, 12]. In 1990, Naor [33] proposed an efficiently computable representation for unlabeled graphs (answering Turán’s [39] open question), and showed that this representation is optimal up to the second leading term of the entropy when all unlabeled graphs are equally likely. Naor’s result is, asymptotically, a special case of corresponding expansions developed later in [14], where general ER graphs were analyzed. Further extensions to PAG graphs were derived in [28].

An approach based on automata, was used in [32] to design an optimal graph compression scheme. Recently, the authors of [17] proposed a general universal lossless source coding algorithm for graphs, and there are also a number of heuristic methods for real-world graph compression, including a grammar-based scheme for data structures [12, 31, 36]. A comprehensive survey of lossless graph compression algorithms can be found in [10].

There are a number of studies of the compression problem for trees [20, 22, 42, 30, 21]. For binary, plane-oriented trees, rigorous information-theoretic results were obtained in [30], and a universal, grammar-based lossless coding scheme was proposed in [21].

In the computer science literature, the focus has been almost exclusively on algorithmic complexity, and very little attention seems to have been given to comparisons with fundamental information-theoretic compression measures – which is the main focus of this paper. Also, work in both communities has largely been restricted to labeled graphs, or graphs with strong edge independence assumptions (with the exception of [2, 30]). As we show, interesting additional complications arise when the goal is to compress graphical *structures*.

**Paper organization.** In the next section we review some known symmetry and structural entropy properties of the ER and PAG models. The small-world graph model is introduced in Section 3, where its degree distribution is determined and its asymmetry established. Our main results on the graph entropy and structural entropy of small-world graphs are stated and proved in Section 4.

## 2 Preliminaries: Random Graphs and Entropy

### 2.1 Graphs, structures, labels, and symmetry

Let  $\mathcal{G}(n)$  denote the class of all (undirected, simple, labeled) graphs  $G = (V, E)$  on  $n = |V|$  vertices, where for simplicity we take  $V = \{1, 2, \dots, n\}$  throughout. Let  $P_n$  denote a *model* for such graphs, that is, a discrete probability mass function (PMF) on  $\mathcal{G}(n)$ . For example, under the classical Erdős-Rényi (ER) model [19, 11] with parameter  $p$ , for  $G \in \mathcal{G}(n)$ ,

$$P_n(G) = p^{|E|}(1-p)^{\binom{n}{2}-|E|},$$

where  $|E|$  is the number of edges of  $G = (V, E)$ . On the other hand, for the preferential attachment graphs  $\text{PA}(m, n)$  studied, e.g., in [8], where a new node connects to  $m$  existing nodes with probability which is proportional to their degree, the probability  $P_n(G)$  does only depend on  $|E|$ .

Any such model  $P_n$  induces a probability distribution  $Q_n$  on structures. Let  $\mathcal{S}(n)$  denote the class of all unlabeled graphs of  $n$  vertices. Then the induced probability of a structure  $S \in \mathcal{S}(n)$  is the sum of the probabilities of all graphs  $G$  with the same structure  $S$ ,

$$Q_n(S) = \sum_{G \in \text{Iso}(S)} P_n(G), \quad (1)$$

where  $\text{Iso}(S) \subset \mathcal{G}(n)$  is the isomorphism equivalence class consisting of all graphs in  $\mathcal{G}(n)$  with structure  $S$ .

Some standard models  $P_n$ , such as the simple ER model and  $\text{PA}(m, n)$  graphs [27], are invariant under isomorphism, that is,  $P_n(G) = P_n(G')$ , whenever  $G, G' \in \text{Iso}(S)$  for some  $S$ . In such cases we simply have,

$$Q_n(S) = P_n(G) \cdot |\text{Iso}(S)|,$$

where  $G$  is any graph in  $\text{Iso}(S)$ , i.e., with structure  $S$ . If, in addition, every permutation is permissible (i.e., it leads to a nonzero probability graph) by a given model – as in ER model – then the number of graphs isomorphic to a given  $G$  is equal to the number of permutations of the labels,  $n!$ , divided by the number of such permutations that lead to exactly the same graph, namely, the size of the automorphism group  $\text{Aut}(G)$  of  $G$ . Therefore,

$$Q_n(S) = P_n(G) \cdot \frac{n!}{|\text{Aut}(G)|}. \quad (2)$$

More generally, in cases like the  $\text{PA}(m, n)$  model [27], where not all permutations are permissible, we have,

$$Q_n(S) = P_n(G) \cdot \frac{|\Gamma(G)|}{|\text{Aut}(G)|}. \quad (3)$$

where  $\Gamma(G)$  is the set of permissible permutations. For example, for  $\text{PA}(m, n)$  we know that  $E[\log |\Gamma(G)|] = n \log n - O(n \log \log n)$  [27]. [Throughout the paper,  $\log$  denotes the natural logarithm  $\log_e$ .] As we will see later, the small-world model considered here is not invariant under isomorphism, and not every permutation is permissible.

It is of interest to know how much symmetry a given graph has. In particular, in some applications one needs to know if a graph is asymmetric, as defined below.

**Definition 2.1** A graph  $G$  is called asymmetric if  $|\text{Aut}(G)| = 1$ .

It is known that, in appropriate parameter ranges, the ER [23] and PAG [27] models generate asymmetric graphs with high probability:

**Theorem 2.2 (ER asymmetry [23])** (i) For a sequence of random graphs  $\{G_n\}$  under the ER model with parameters  $\{p_n\}$ , such that, as  $n \rightarrow \infty$ ,

$$p_n \gg \frac{\log n}{n} \quad \text{and} \quad 1 - p_n \gg \frac{\log n}{n},$$

we have, for any  $t > 0$ ,

$$\Pr(G_n \text{ is symmetric}) = O(n^{-t}),$$

as  $n \rightarrow \infty$ .

**(PAG asymmetry [27])** (ii) For a sequence of random graphs  $\{G_n\}$  under the  $\text{PA}(m, n)$  model with  $m \geq 3$ , we have that, for some  $\delta > 0$ ,

$$\Pr(G_n \text{ is symmetric}) = O(n^{-\delta}),$$

as  $n \rightarrow \infty$ .

One of our main results below will be the development of a statement analogous to Theorem 2.2 for a class of small-world random graphs.

## 2.2 Entropy and compressibility

Detailed asymptotic expansions for the graph entropy  $H(G_n)$  under the ER and PAG models are known as we review below. First we note, without proof, a simple expression for the binary entropy function.

**Lemma 2.3** As  $p \rightarrow 0$ , the binary entropy function  $h(p) = -p \log p - (1-p) \log(1-p)$  satisfies,

$$h(p) = p \log \left( \frac{1}{p} \right) + p - \frac{1}{2}p^2 + O(p^3).$$

Moreover, the error term always satisfies  $-(1/2)p^3 \leq O(p^3) \leq 0$ , and for  $p \leq 1/4$  it also satisfies  $O(p^3) \leq -(1/10)p^3$ .

**Lemma 2.4** (i) **(ER graph entropy)** For a sequence of ER random graphs  $\{G_n\}$  with parameters  $\{p_n\}$ ,

$$H(G_n) = \frac{n(n-1)}{2} h(p_n),$$

and if  $p_n \rightarrow 0$  as  $n \rightarrow \infty$ ,

$$H(G_n) = \frac{n(n-1)}{2} \left[ p_n \log \left( \frac{1}{p_n} \right) + p_n - \frac{1}{2}p_n^2 + O(p_n^3) \right].$$

(ii) **(PAG graph entropy [37, 27])** For a sequence of  $\text{PA}(m, n)$  random graphs  $\{G_n\}$ , we have, as  $n \rightarrow \infty$ ,

$$H(G) = mn \log n + m (\log 2m - 1 - \log m! - A) n + o(n), \quad (4)$$

where,

$$A = \sum_{d=m}^{\infty} \frac{\log d}{(d+1)(d+2)}.$$

PROOF. We only sketch the proof for the ER model. By definition,  $G_n \sim P_n$  describes  $\binom{n}{2}$  independent  $\text{Bern}(p_n)$  random variables, so,  $H(G_n) = \binom{n}{2} h(p_n)$  and using Lemma 2.3 gives the result. The prove of (4) can be found in [27].  $\square$

For a random graph  $G_n$  with structure  $S_n$ , the chain rule for entropy implies that,

$$H(S_n) = H(G_n) - H(G_n|S_n). \quad (5)$$

Using this identity together with relation (2) in combination with Theorem 2.2, Choi and Szpankowski [14] for the ER model and Luczak et al. [27] for PAG graphs, establish the following asymptotic expansions for the entropy of ER and PAG random structures. An analogous expansion for a class of small-world graphs is established in this paper.

**Theorem 2.5 (ER structural entropy)** (i) *For a sequence of ER random graphs  $\{G_n\}$  with parameters  $\{p_n\}$  that satisfy,*

$$p_n \gg \frac{\log n}{n} \quad \text{and} \quad 1 - p_n \gg \frac{\log n}{n},$$

*as  $n \rightarrow \infty$ , we have, for some  $\beta > 0$ ,*

$$H(S_n) = \frac{n(n-1)}{2} h(p_n) - \log n! + O\left(\frac{\log n}{n^\beta}\right).$$

**(PAG structural entropy)** (ii) *For a sequence of  $\text{PA}(m, n)$  random graphs  $\{G_n\}$  with  $m \geq 3$  we have, as  $n \rightarrow \infty$ ,*

$$H(S(G)) = (m-1)n \log n + R(n), \quad (6)$$

*where  $R(n)$  satisfies,*

$$Cn \leq |R(n)| \leq O(n \log \log n),$$

*for some nonzero constant  $C = C(m)$ .*

**Definition 2.6** *The compressibility of a random graph  $G_n = (V_n, E_n)$  is measured by the average number of bits (or, rather, nats) per edge used in its best possible description, that is,  $C_n = H(G_n)/E(|E_n|)$ . We say that the sequence of random graphs  $\{G_n\}$  is compressible, if  $C_n = O(1)$ .*

Recent studies indicate that many real-world examples of large graphs, including web graphs and social media graphs, are compressible. For an extensive discussion of compressibility in different models see [13].

For the ER model we note that each node has  $\text{Bin}(n-1, p_n)$  edges, which is  $\approx \text{Po}(np_n)$  for large  $n$ , as long as  $p_n = o(1)$ ; to see this, recall Theorem 1 of [9]. Also,  $E(|E_n|) = n(n-1)p_n/2$ , so by Lemma 2.4 in this case,

$$C_n = \frac{H(G_n)}{E(|E_n|)} \sim -\log p_n,$$

which is unbounded. Therefore, in the above sense, the ER model with parameters  $p_n = o(1)$  is incompressible. Similarly, for  $\text{PA}(m, n)$  graphs, we have  $C_n \sim \log n$ , for  $m \geq 3$ .

### 3 A small-world model

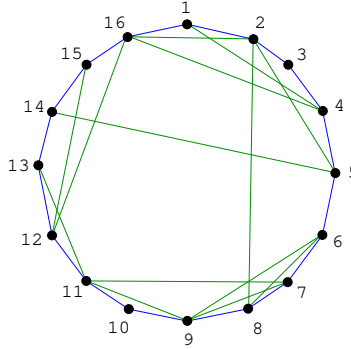
Here we examine a small-world-type model, similar to those introduced in [41, 24]; also see [34, 13, 2] and [18, Section 20.7].

Consider the vertex set  $V = \{1, 2, \dots, n\}$  arranged on the circle, with each vertex connected by an edge to its two nearest neighbors. For each one of the remaining  $\binom{n}{2} - n$  pairs of vertices  $(u, v)$ , we add an edge between them with probability  $p(|u - v|)$ , where  $|u - v|$  is the discrete distance on the circle and  $p_n(k) = c_n k^{-a}$ , for some  $a \in (0, 1)$  and with,

$$c_n = b_n(1 - a) \left( \frac{2}{n} \right)^{1-a},$$

where  $\{b_n\}$  is a nondecreasing, unbounded sequence of positive real numbers, with  $b_n = o(n^{1-a})$ , as  $n \rightarrow \infty$ . In all the results and discussion below we implicitly assume that  $n$  is large enough so that all the  $p_n(k)$  are less than one, which is always possible by the assumptions on  $b_n$ .

The graph shown below is an example of a small world graph with  $n = 16$ ; nearest neighbor edges are shown in blue and random edges are green. Note that there are more edges between nearby nodes and fewer between distant ones.



We call a random undirected graph  $G_n$  generated by this model a *random small-world graph with parameters  $a$  and  $b_n$* , and we write  $G_n \sim \text{SW}(a, b_n)$ . It is assumed throughout that  $a \in (0, 1)$  and  $b_n = o(n^{1-a})$ .

Before examining the  $\text{SW}(a, b_n)$  class further, we remark that small-world models are an important class of *geometric* random graphs in that, unlike in the ER model, the connectivity of a small-world graph depends on the actual locations of the nodes. Although it will not play a role in our analysis, we mention that another important characteristic of such graphs is the “small-world property.” This means that the graph distance between any two nodes is much smaller than in a purely random graph, with high probability; see the above references or the texts [35, 40] for details.

#### 3.1 Degree distribution

**Proposition 3.1 (SW mean degree)** *The mean degree  $\mu_n$  of an arbitrary node in a random graph  $G_n \sim \text{SW}(a, b_n)$  satisfies,*

$$\mu_n = 2b_n + 2 + O\left(\frac{b_n}{n^{1-a}}\right),$$

as  $n \rightarrow \infty$ .

For the proof we need the following lemmas. The expansions in Lemma 3.2 follow from straightforward applications of Euler-Maclaurin summation; see, e.g., [5, 6].

**Lemma 3.2** *As  $n \rightarrow \infty$ ,*

$$\sum_{k=1}^n \frac{1}{k} = \log n + \gamma + \frac{1}{2n} + O\left(\frac{1}{n^2}\right),$$

*where  $\gamma$  is Euler's constant and the error term is bounded in absolute value by  $\frac{1}{6n^2}$ , for all  $n \geq 2$ . Also, as  $n \rightarrow \infty$ , for any  $s > 0$ ,  $s \neq 1$ ,*

$$\sum_{k=1}^n \frac{1}{k^s} = \frac{1}{(1-s)n^{s-1}} + \zeta(s) + \frac{1}{2n^s} + O\left(\frac{1}{n^{s+1}}\right),$$

*where  $\zeta$  is the Riemann zeta function,*

$$\zeta(s) := \begin{cases} \sum_{k=1}^{\infty} \frac{1}{k^s}, & \text{if } s > 1, \\ \lim_{M \rightarrow \infty} \left[ \sum_{k=1}^M \frac{1}{k^s} - \frac{M^{1-s}}{(1-s)} \right], & \text{if } s \in (0, 1), \end{cases}$$

*and the error term is bounded in absolute value by  $\frac{s}{6n^{s+1}}$ .*

Next we will apply Lemma 3.2 to get some simple estimates regarding the probabilities  $p_n(k)$ .

**Lemma 3.3** *For odd  $n$ , let:*

$$S_{n,1} = 2 \sum_{k=2}^{(n-1)/2} p_n(k), \quad S_{n,2} = 2 \sum_{k=2}^{(n-1)/2} p_n(k)^2.$$

*Similarly, for even  $n$ , let:*

$$S'_{n,1} = 2 \sum_{k=2}^{(n-2)/2} p_n(k) + p_n(n/2), \quad S'_{n,2} = 2 \sum_{k=2}^{(n-2)/2} p_n(k)^2 + p_n(n/2)^2.$$

*Then, as  $n \rightarrow \infty$ ,*

$$S_{n,1} = 2b_n + O\left(\frac{b_n}{n^{1-a}}\right), \quad S_{n,2} = \begin{cases} \left(\frac{4(1-a)^2}{1-2a}\right) \frac{b_n^2}{n} + O\left(\frac{b_n^2}{n^{2-2a}}\right), & a \in (0, 1/2), \\ \frac{b_n^2 \log n}{n} + O\left(\frac{b_n^2}{n}\right), & a = 1/2, \\ 2^{3-2a}(1-a)^2 \zeta(2a) \frac{b_n^2}{n^{2-2a}} + O\left(\frac{b_n^2}{n}\right), & a \in (1/2, 1), \end{cases}$$

*and the same results hold with  $S'_{n,1}$  in place of  $S_{n,1}$ , and  $S'_{n,2}$  in place of  $S_{n,2}$ .*

**PROOF.** We only give the proof for odd  $n$ ; the case of even  $n$  is similar.

For  $S_{n,1}$ , by the definition of the  $p_n(k)$  we have,

$$S_{n,1} = 2b_n(1-a) \left(\frac{2}{n}\right)^{1-a} \left[ \sum_{k=1}^{(n-1)/2} \frac{1}{k^a} - 1 \right],$$

and by Lemma 3.2 this is,

$$S_{n,1} = 2b_n(1-a)\left(\frac{2}{n}\right)^{1-a} \left[ \frac{[(n-1)/2]^{1-a}}{1-a} + O(1) \right] = 2b_n + O\left(\frac{b_n}{n^{1-a}}\right).$$

For  $S_{n,2}$ , we similarly have,

$$S_{n,2} = 2b_n^2(1-a)^2\left(\frac{2}{n}\right)^{2-2a} \left[ \sum_{k=1}^{(n-1)/2} \frac{1}{k^{2a}} - 1 \right],$$

and we apply Lemma 3.2 in three cases. For  $a \in (0, 1/2)$ ,

$$\begin{aligned} S_{n,2} &= 2b_n^2(1-a)^2\left(\frac{2}{n}\right)^{2-2a} \left[ \frac{[(n-1)/2]^{1-2a}}{1-2a} + O(1) \right] \\ &= \frac{2(1-a)^2}{1-2a} b_n^2 \left(\frac{2}{n}\right) \left(\frac{2}{n}\right)^{1-2a} [(n-1)/2]^{1-2a} + O(1) \\ &= \left(\frac{4(1-a)^2}{1-2a}\right) \frac{b_n^2}{n} + O\left(\frac{b_n^2}{n^{2-2a}}\right). \end{aligned}$$

For  $a = 1/2$ ,

$$S_{n,2} = \frac{b_n^2}{n} \left[ \sum_{k=1}^{(n-1)/2} \frac{1}{k} - 1 \right] = \frac{b_n^2 \log n}{n} + O\left(\frac{b_n^2}{n}\right).$$

And for  $a \in (1/2, 1)$ ,

$$\begin{aligned} S_{n,2} &= 2b_n^2(1-a)^2\left(\frac{2}{n}\right)^{2-2a} \left[ \zeta(2a) + O\left(\frac{1}{n^{2a-1}}\right) \right] \\ &= 2^{3-2a}(1-a)^2 \zeta(2a) \frac{b_n^2}{n^{2-2a}} + O\left(\frac{b_n^2}{n}\right), \end{aligned}$$

as claimed.  $\square$

**PROOF OF PROPOSITION 3.1.** The edges of  $G_n$  can be described as  $\binom{n}{2} - n$  independent Bernoulli random variables. Choose and fix  $n \geq 5$  be arbitrary.

Suppose  $n$  is odd. Considering, without loss of generality, the node  $u = 1$ , let  $X_k, Y_k$ , for  $k = 2, 3, \dots, (n-1)/2$ , denote binary random variables, where each  $X_k$  and each  $Y_k$  describe whether node  $u = 1$  is connected to a different node at distance  $k$  from node  $u = 1$ . Then  $\{X_k, Y_k\}$  are independent Bernoulli random variables with corresponding parameters  $\{p_n(k)\}$ , and the degree of node  $u = 1$ ,  $W_n$ , say, can be expressed as  $W_n = [2 + \sum_k (X_k + Y_k)]$ . Therefore, the mean degree of any vertex is,

$$\mu_n := E(W_n) = 2 + 2 \sum_{k=2}^{(n-1)/2} p_n(k) = 2 + S_{n,1}. \quad (7)$$

Similarly, if  $n$  is even, there are  $n$  possible edges between pairs of nodes at each distance  $k = 2, 3, \dots, \frac{n-2}{2}$ , and  $n/2$  possible edges between pairs of nodes at distance  $n/2$ . Here, the mean degree of a vertex is,

$$\mu_n = 2 + 2 \sum_{k=2}^{(n-2)/2} p_n(k) + p_n(n/2) = 2 + S'_{n,1}. \quad (8)$$

Combining (7) and (8) with Lemma 3.2 completes the proof.  $\square$

### 3.2 Asymmetry

Let  $G = (V, E)$  be an arbitrary undirected graph on  $V = \{1, 2, \dots, n\}$ , with no self loops. We first make a series of definitions following the terminology of [23].

The set of neighbors of a vertex  $u \in V$  is denoted,

$$N(u) = \{v \in V : (u, v) \in E\}.$$

Let  $\pi$  be any permutation on  $V$ . The *defect of a vertex*  $u \in V$  under the permutation  $\pi$  is,

$$D_\pi(u) = |N(\pi(u)) \Delta \pi(N(u))|,$$

which can also be expressed as,

$$D_\pi(u) = \sum_{v \neq \pi(u)} [\mathbb{I}_{\{(\pi(u), v) \in E, (u, \pi^{-1}(v)) \notin E\}} + \mathbb{I}_{\{(\pi(u), v) \notin E, (u, \pi^{-1}(v)) \in E\}}]. \quad (9)$$

The *defect of the graph*  $G$  under  $\pi$  is,

$$D_\pi(G) = \max_{u \in V} D_\pi(u),$$

and the *total defect of*  $G$  is,

$$D(G) = \min_{\pi \neq \text{id}} D_\pi(G),$$

where  $\text{id}$  denotes the identity permutation. Note that  $G$  is asymmetric iff  $D(G) \neq 0$ .

**Theorem 3.4 (SW asymmetry)** *Let  $\{G_n\}$  be a sequence of small-world random graphs,  $G_n \sim \text{SW}(a, b_n)$ ,  $n \geq 1$ . If,*

$$b_n = o(n^{1-a}), \quad \text{and} \quad \frac{b_n}{\log n} \rightarrow \infty, \quad \text{as } n \rightarrow \infty,$$

*then, for any  $t > 0$ ,*

$$\Pr(G_n \text{ is symmetric}) = O(n^{-t}),$$

*as  $n \rightarrow \infty$ .*

Following [23], we base part of the proof on an application of the following simple concentration bound.

**Proposition 3.5** [3, 4] *Let  $Z = f(\xi_1, \xi_2, \dots, \xi_m)$  be a function of the independent Bernoulli random variables  $\{\xi_i\}$ , and suppose that  $f$  has the bounded difference property that, for some  $c > 0$ ,*

$$\max_{j, \{\xi_i\}} |f(\xi_1, \dots, \xi_{j-1}, \xi_j, \xi_{j+1}, \dots, \xi_m) - f(\xi_1, \dots, \xi_{j-1}, 1 - \xi_j, \xi_{j+1}, \dots, \xi_m)| \leq c. \quad (10)$$

*Let  $p_i = E(\xi_i)$  for each  $i$ , and  $\sigma^2 = c^2 \sum_i p_i(1 - p_i)$ . Then, for all  $0 < t < 2\sigma/c$ :*

$$\Pr[|Z - E(Z)| > t\sigma] \leq 2e^{-t^2/4}.$$

PROOF OF THEOREM 3.4. In view of the discussion preceding the theorem, if  $G_n \sim \text{SW}(a, b_n)$ , the probability that it is symmetric can be bounded above as,

$$\Pr(G_n \text{ is symmetric}) = \Pr(D(G_n) = 0) \leq \sum_{\pi \neq \text{id}} \Pr(D_\pi(G_n) = 0) = \sum_{\pi \neq \text{id}} \Pr\left(\max_{u \in V} D_\pi(u) = 0\right),$$

and defining, for any  $\pi \neq \text{id}$ ,

$$Z_\pi = \sum_{u: u \neq \pi(u)} D_\pi(u),$$

we have,

$$\Pr(G_n \text{ is symmetric}) \leq \sum_{\pi \neq \text{id}} \Pr(Z_\pi = 0). \quad (11)$$

To further bound the probability that  $Z_\pi = 0$ , we will use Proposition 3.5. To that end, first observe that, after ignoring the first term in (9), we have, for any  $\pi$  and any  $u$  such that  $u \neq \pi(u)$ ,

$$E[D_\pi(u)] \geq \sum_{v \neq u, \pi(u)} \Pr((u, \pi^{-1}(v)) \in E, (\pi(u), v) \notin E).$$

Under the assumptions that  $u \neq \pi(u)$  and  $v \neq u, \pi(u)$ , the two events in the last probability above always refer to two distinct edges, so they are independent, and hence,

$$E[D_\pi(u)] \geq \sum_{v \neq u, \pi(u)} \Pr((u, \pi^{-1}(v)) \in E) [1 - \Pr((\pi(u), v) \in E)].$$

Each term in the last sum is of the form  $p_n(k)[1 - p_n(k')]$  for some  $k, k'$ . Therefore, since  $p_n(k)$  is decreasing in  $k$  for each  $n$ , for odd  $n$ ,

$$E[D_\pi(u)] \geq [1 - p_n(2)] \sum_{v \neq u, \pi(u)} \Pr((u, \pi^{-1}(v)) \in E) \geq [1 - p_n(2)][S_{n,1} - p_n(2)],$$

with  $S_{n,1}$  defined in Lemma 3.3. So, by the result of the lemma, we have,

$$\begin{aligned} E[D_\pi(u)] &\geq [1 - p_n(2)] \left[ 2b_n + O\left(\frac{b_n}{n^{1-a}}\right) - p_n(2) \right] \\ &= \left[ 1 - O\left(\frac{b_n}{n^{1-a}}\right) \right] \left[ 2b_n + O\left(\frac{b_n}{n^{1-a}}\right) \right] \\ &= 2b_n[1 + o(1)], \end{aligned}$$

since  $b_n = o(n^{1-a})$ . A similar computation shows that the same result holds for even  $n$ . And letting  $d(\pi)$  denote the degree of a permutation  $\pi$ , i.e., the number of  $u$  such that  $\pi(u) \neq u$ , we have, by the above bound and the definition of  $Z_\pi$ , that:

$$E(Z_\pi) \geq 2d(\pi)b_n[1 + o(1)]. \quad (12)$$

Recall that all  $D_\pi(u)$  and  $Z_\pi$  can be expressed as functions of the independent Bernoulli random variables introduced in the proof of Proposition 3.1. From the expression in (9) it is clear that, changing the value of any one of the edges corresponding to these random variables

can only change the value of  $D_\pi(u)$  by at most 1, and adding or deleting any such edge only affects at most four of the terms in the sum  $Z_\pi$ . Therefore,  $Z_\pi$  considered as a function of these Bernoulli random variables satisfies the bounded difference property (10) with  $c = 4$ .

For the variance  $\sigma^2$  we note that each of the  $d(\pi)$  many terms in the sum defining  $Z_\pi$  depends on  $(n-3)$  of the corresponding binary variables. Therefore, since some of them may influence  $D_\pi(u)$  for more than one  $u$ , we can bound, for odd  $n$ ,

$$\sigma^2 \leq 4^2 \times 2 \times d(\pi) \times \sum_{k=2}^{(n-1)/2} p_n(k)[1 - p_n(k)] = \bar{\sigma}^2 := 16d(\pi)[S_{n,1} - S_{n,2}],$$

where  $S_{n,2}$  is defined in Lemma 3.3. By the result of the lemma, under the present assumptions we have  $S_{n,2} = o(b_n)$  for all  $a \in (0, 1)$ , and hence,

$$\sigma^2 \leq \bar{\sigma}^2 := 32d(\pi)b_n[1 + o(1)]. \quad (13)$$

On the other hand, for each  $u$  in the definition of  $Z_\pi$ , considering the influence on  $D_\pi(u)$  of only those  $v \neq \pi(u)$  that lie on the “right” of  $\pi(u)$  on the circle (in order to avoid double-counting edges), and arguing exactly as above, we obtain a corresponding lower bound,

$$\sigma^2 \geq \underline{\sigma}^2 := 16d(\pi)b_n[1 + o(1)]. \quad (14)$$

Analogous computations show that the bounds (13) and (14) also hold for even  $n$ , and we are now in a position to apply Proposition 3.5.

Let  $N$  be large enough so that, for all  $n \geq N$ , we have  $28d(\pi)b_n \leq \bar{\sigma}^2 \leq 33d(\pi)b_n$  by (13),  $\underline{\sigma}^2 \geq 15d(\pi)b_n$  by (14), and the lower bound in (12) is at least  $d(\pi)b_n$ . Note that  $N$  can be chosen independently of  $\pi$ , since the  $o(1)$  terms in each of these bounds do not depend on  $\pi$ .

Let  $s = \lambda\bar{\sigma}$ , for a fixed  $\lambda \in (0, 1/33)$ . Then, for any  $\pi \neq \text{id}$  and all  $n \geq N$ , we have, by the choice of  $\lambda$  and the upper bound on  $\bar{\sigma}^2$ ,

$$\begin{aligned} \Pr(Z_\pi = 0) &\leq \Pr(Z_\pi < d(\pi)b_n(1 - 33\lambda)) \\ &\leq \Pr(Z_\pi < d(\pi)b_n - \lambda\bar{\sigma}^2) \\ &= \Pr(Z_\pi < d(\pi)b_n - s\bar{\sigma}). \end{aligned}$$

And by the definition of  $\bar{\sigma}^2$  and the lower bound on  $E(Z_\pi)$ ,

$$\Pr(Z_\pi = 0) \leq \Pr(Z_\pi < d(\pi)b_n - s\sigma) \leq \Pr(Z_\pi < E(Z_\pi) - s\sigma).$$

Therefore, by the bound in Proposition 3.5, we obtain,

$$\Pr(Z_\pi = 0) \leq \Pr(|Z_\pi - E(Z_\pi)| > s\sigma) \leq 2e^{-s^2/4} = 2e^{-\lambda^2\bar{\sigma}^2/4} \leq 2e^{-7\lambda^2d(\pi)b_n}, \quad (15)$$

as long as,

$$\lambda < \frac{15}{66} = \frac{15d(\pi)b_n}{2 \times 33d(\pi)b_n} \leq \frac{\sigma}{2\bar{\sigma}} \leq \frac{\sigma}{2\sigma},$$

which implies  $s < \sigma/2 = 2\sigma/c$ .

Finally, we will sum all the probabilities in (15) as in (11). Since there are no more than  $n!/(n-d)! \leq n^d$  permutations that fix  $(n-d)$  vertices, we have, that,

$$\Pr(G_n \text{ is symmetric}) \leq \sum_{\pi \neq \text{id}} \Pr(Z_\pi = 0) \leq 2 \sum_{d=1}^n n^d e^{-7\lambda^2db_n} = 2 \sum_{d=1}^n e^{d[\log n - 7\lambda^2b_n]},$$

and since  $b_n/\log n \rightarrow \infty$ , the right-hand side above is  $O(n^{-t})$ , for any  $t > 0$ , as claimed.  $\square$

## 4 Entropy of the small-world model

### 4.1 Graph entropy

As with Lemma 3.2, the expansions in Lemma 4.1 below are easy applications of Euler-Maclaurin summation [5, 6].

**Lemma 4.1** *As  $n \rightarrow \infty$ ,*

$$\sum_{k=1}^n \frac{\log k}{k} = \frac{1}{2}(\log n)^2 + \gamma' + \frac{1}{2} \frac{\log n}{n} + O\left(\frac{\log n}{n^2}\right),$$

where  $\gamma'$  is defined, in analogy to Euler's constant, as,

$$\gamma' = \lim_{n \rightarrow \infty} \left[ \sum_{k=1}^n \frac{\log k}{k} - \frac{1}{2}(\log n)^2 \right],$$

and the error term is bounded in absolute value by  $\frac{1+\log n}{6n^2}$ , for all  $n \geq 2$ . Also, as  $n \rightarrow \infty$ , for any  $s > 0$ ,  $s \neq 1$ ,

$$\sum_{k=1}^n \frac{\log k}{k^s} = \frac{\log n}{(1-s)n^{s-1}} - \frac{1}{(1-s)^2 n^{s-1}} - \zeta'(s) + \frac{\log n}{2n^s} + O\left(\frac{\log n}{n^{s+1}}\right),$$

where the error term is bounded in absolute value by  $\frac{1+s \log n}{6n^{s+1}}$ , for all  $n \geq 2$ .

**Theorem 4.2 (SW graph entropy)** *Let  $G_n \sim \text{SW}(a, b_n)$ ,  $n \geq 1$ , be a sequence of small-world random graphs with,*

$$b_n = o(n^{1-a}), \quad \text{and} \quad \frac{b_n}{\log n} \rightarrow \infty, \quad \text{as } n \rightarrow \infty.$$

*The entropy of this small-world model is,*

$$H(G_n) = nb_n [\log n - \log b_n - C_a + o(1)], \tag{16}$$

where,

$$C_a = a \left( \frac{1 + \log 2}{1-a} \right) + \log \left( (1-a)2^{1-a} \right) - 1. \tag{17}$$

**PROOF.** In the notation of Proposition 3.1, the edges connecting each node on the circle is described by a collection of independent Bernoulli random variables. Therefore, considering all  $n$  nodes and accounting for double-counting, when  $n$  is odd (the case when  $n$  is even is similar),

$$H(G_n) = \frac{n}{2} H(\{X_k, Y_k\}) = n \sum_{k=2}^{(n-1)/2} h(p_n(k)).$$

A weaker version of Lemma 2.3 is that, for small  $p$ , we have,  $h(p) = p \log(1/p) + p - O(p^2)$ , where the error term is between 0 and  $p^2$  for  $p < 1/2$ . Therefore, taking  $n$  large enough so that all  $p_n(k) < 1/2$ , we have that,

$$\begin{aligned} H(G_n) &= -n \sum_{k=2}^{(n-1)/2} p_n(k) \log p_n(k) + n \sum_{k=2}^{(n-1)/2} p_n(k) - n \sum_{k=2}^{(n-1)/2} \Delta_{n,k} p_n(k)^2 \\ &= anc_n \sum_{k=2}^{(n-1)/2} \frac{\log k}{k^a} - \frac{n}{2} (\log c_n) S_{n,1} + \frac{n}{2} S_{n,1} - \Delta_n \frac{n}{2} S_{n,2}, \end{aligned} \quad (18)$$

for appropriate constants  $\Delta_{n,k}, \Delta_n$  in  $[0, 1]$ , where  $S_{n,1}$  and  $S_{n,2}$  are defined as in Lemma 3.3.

Using Lemma 4.1, the first term in (18) can be expressed as,

$$\begin{aligned} &a(1-a)2^{1-a}n^ab_n \left[ \frac{\log((n-1)/2)}{(1-a)((n-1)/2)^{a-1}} - \frac{1}{(1-a)^2((n-1)/2)^{a-1}} + O(1) \right] \\ &= n^ab_n \left[ an^{1-a} \log n - an^{1-a} \left( \frac{1+\log 2}{1-a} \right) + O(1) \right] \\ &= ab_n n \log n - ab_n n \left( \frac{1+\log 2}{1-a} \right) + o(n) \end{aligned} \quad (19)$$

The sum of the second and third terms in (18), using Lemma 3.3, are,

$$\begin{aligned} &-\frac{n}{2} \left[ \log \left( (1-a)2^{1-a} \right) + \log b_n - (1-a) \log n - 1 \right] \left[ 2b_n + O\left( \frac{b_n}{n^{1-a}} \right) \right] \\ &= n^ab_n \left[ (1-a)n^{1-a} \log n + n^{1-a} - n^{1-a} \log b_n - n^{1-a} \log \left( (1-a)2^{1-a} \right) \right] + o(n \log n) \\ &= (1-a)b_n n \log n - b_n n \log b_n - b_n n \log \left( (1-a)2^{1-a} \right) + b_n n + o(nb_n). \end{aligned} \quad (20)$$

And the last term in (18), by Lemma 3.3, is  $o(nb_n)$  for all  $a \in (0, 1)$ . Substituting this together with (19) and (20) into (18), yields,

$$nb_n \left\{ \log n - \log b_n - \left[ a \left( \frac{1+\log 2}{1-a} \right) + \log \left( (1-a)2^{1-a} \right) - 1 \right] + o(1) \right\},$$

as required.  $\square$

**Remark.** Note that, combining the above expansion for the entropy  $H(G_n)$  with the expression for the mean degree of an arbitrary node in  $G_n$  given in Proposition 3.1, we have that, as  $n \rightarrow \infty$ ,

$$\frac{H(G_n)}{E(|E_n|)} = \frac{nb_n \log n (\delta_n + o(1))}{2nb_n + O(n)} \sim \frac{\delta_n}{2} \log n,$$

where the positive sequence  $\{\delta_n\}$  is bounded above and bounded away from zero. Therefore, the average number of “bits per edge” in  $G_n$  is unbounded, so in the terminology of [13] the  $\text{SW}(a, b_n)$  model under our assumptions is incompressible.

## 4.2 Structural entropy

Having an estimate for the entropy of a random graph  $G_n \sim \text{SW}(a, b_n)$  in Theorem 4.2, it is easy to get a corresponding estimate for the entropy of the random structure  $S_n$  associated with  $G_n$ . Since, given  $S_n$ , there are at most  $n! \leq n^n$  possible graphs  $G_n$  with structure  $S_n$ , we have that,

$$H(G_n|S_n) \leq n \log n. \quad (21)$$

And since  $H(S_n) = H(G_n) - H(G_n|S_n)$  as noted in (5), combining (21) with (16) immediately yields:

**Corollary 4.3 (SW structural entropy)** *Under the assumptions of Theorem 4.2, the entropy of the structures  $S_n$  associated with the small-world random graphs  $G_n \sim \text{SW}(a, b_n)$  satisfies,*

$$H(S_n) = nb_n [\log n - \log b_n - C_a + o(1)],$$

where the constant  $C_a$  is given in (17).

Finally we examine the conditional entropy  $H(G_n|S_n)$ , which describes the degree of uncertainty that remains about the graph  $G_n$  after knowing its structure  $S_n$ . In Theorem 4.4 we obtain a slightly more refined estimate than the crude upper bound in (21), which gives a tighter result when  $b_n = o(n^t)$  for all  $t > 0$ .

**Theorem 4.4 (SW conditional entropy)** *Let  $G_n \sim \text{SW}(a, b_n)$  be a sequence of small-world random graphs with associated structures  $S_n$ ,  $n \geq 1$ . Suppose that,*

$$b_n = o(n^{1-a}), \quad \text{and} \quad \frac{b_n}{\log n} \rightarrow \infty, \quad \text{as } n \rightarrow \infty.$$

*Then the conditional entropy of the graph  $G_n$  given its structure  $S_n$  has:*

$$H(G_n|S_n) \leq n \log b_n + (\log 5)n + \log \left( \frac{n}{b_n} \right) + O(1).$$

First we establish a simple, general upper bound. As in Section 2, we write  $P_n$  for the PMF of  $G_n$  on  $\mathcal{G}(n)$  and similarly  $Q_n$  for the induced PMF of  $S_n$  in  $\mathcal{S}(n)$ . We also write  $\mathcal{G}_a(n) \subset \mathcal{G}(n)$  for the support of  $P_n$ , and we call graphs  $G \in \mathcal{G}_a(n)$  *admissible*.

**Lemma 4.5** *For any graph  $G \in \mathcal{G}_a(n)$  with structure  $S$ , let  $\tau(G)$  denote the number of admissible graphs  $G'$  that are isomorphic to  $G$ ,*

$$\tau(G) = |\text{Iso}(S) \cap \mathcal{G}_a(n)|.$$

*Then:*

$$H(G_n|S_n) \leq \sum_{G \in \mathcal{G}(n)} P_n(G) \log \tau(G).$$

PROOF. First observe that  $\tau(G)$  is the same for all  $G \in \text{Iso}(S) \cap \mathcal{G}_a(n)$ . Therefore, with only a slight abuse of notation, we will write  $\tau(S)$  for  $\tau(G)$  if  $S$  is the structure of some  $G \in \mathcal{G}_a(n)$ . In analogy with  $\mathcal{G}_a(n)$ , let  $\mathcal{S}_a(n)$  denote the set of admissible structures, i.e., those  $S \in \mathcal{S}(n)$  for which there is an admissible  $G$  with structure  $S$ . Then we have,

$$\begin{aligned}
H(G_n|S_n) &= \sum_{S \in \mathcal{S}_a(n)} Q_n(S) H(G_n|S_n = S) \\
&\stackrel{(a)}{\leq} \sum_{S \in \mathcal{S}_a(n)} Q_n(S) \log \tau(S) \\
&\stackrel{(b)}{=} \sum_{S \in \mathcal{S}_a(n)} \sum_{G \in \text{Iso}(S)} P_n(G) \log \tau(S) \\
&= \sum_{G \in \mathcal{G}(n)} P_n(G) \log \tau(G),
\end{aligned}$$

where (a) follows from the elementary fact that the entropy of a random variable with  $m$  values is at most  $\log m$ , and (b) follows from the basic observation (1).  $\square$

Next we obtain a simple bound on the tails of the degree of the nodes of  $G_n \sim \text{SW}(a, b_n)$ .

**Proposition 4.6** *Under the assumptions of Theorem 4.4, the probability that there is at least one node in  $G_n$  with degree greater than  $9b_n/2$  is  $O(n^{-t})$ , for any  $t > 0$ .*

PROOF. We give the proof for odd  $n$ ; the case of even  $n$  is similar.

Let  $W_n(i)$  denote the (random) degree of node  $i$  in  $G_n$ , so that  $W_n(1) = W_n$  as in the proof of Proposition 3.1. We will apply Proposition 3.5 to bound the tails of  $W_n$ . Note that,  $E(W_n) = 2 + S_{n,1} = 2b_n + 2 + o(1)$ , by Lemma 3.3. Also, as a function of the Bernoulli variables  $\{X_k, Y_k\}$ ,  $W_n$  satisfies the assumptions of Proposition 3.5 with  $c = 1$ . And in this case, in the notation of Lemma 3.3, the variance  $\sigma^2$  is,

$$\sigma^2 = 2 \sum_{k=2}^{(n-1)/2} p_n(k)(1 - p_n(k)) = S_{n,1} - S_{n,2} = b_n[2 + o(1)].$$

Now consider  $N$  large enough such that, for all  $n \geq N$ ,

$$2b_n \leq E(W_n) \leq 3b_n, \quad \text{and} \quad b_n \leq \sigma^2 \leq 3b_n.$$

Then by the union bound and symmetry, we have that, for any  $\lambda \in (0, 1/2)$  and  $n \geq N$ ,

$$\begin{aligned}
\Pr \left( \max_{1 \leq i \leq n} W_n(i) > 2(\lambda + 2)b_n \right) &\leq n \Pr (W_n > 2(\lambda + 2)b_n) \\
&\leq n \Pr (W_n > E(W_n) + (2\lambda + 1)b_n) \\
&\leq n \Pr (|W_n - E(W_n)| > (2\lambda + 1)b_n) \\
&= n \Pr (|W_n - E(W_n)| > s\sigma),
\end{aligned}$$

where we took  $s = (2\lambda + 1)b_n/\sigma$ . Since  $0 < s < 2\sigma$  for  $n \geq N$  by our assumptions, we can apply Proposition 3.5 with  $\lambda = 1/4$  to obtain that,

$$\Pr \left( \max_{1 \leq i \leq n} W_n(i) > 9b_n/2 \right) \leq 2n \exp(-s^2/4) = 2 \exp \left\{ \log n - \frac{9b_n^2}{16\sigma^2} \right\} \leq 2 \exp \left\{ \log n - \frac{3b_n}{16} \right\},$$

and since  $b_n/\log n \rightarrow \infty$  as  $n \rightarrow \infty$ , the result follows.  $\square$

We are now in a position to prove the theorem.

PROOF OF THEOREM 4.4. Let  $\mathcal{B}_n$  be the collection of ‘bad’ graphs  $G \in \mathcal{G}_a(n)$  in the sense of Proposition 4.6, that have at least one node with degree greater than  $d := 9b_n/2$ . For any ‘good’ graph  $G \in \mathcal{B}_n^c$ , we can estimate  $\tau(G)$  as follows. Suppose  $G$  has structure  $S$  and let  $G' \in \text{Iso}(S) \cap \mathcal{G}_a(n)$  be not identical to  $G$ . Let  $\pi \neq \text{id}$  be the permutation on  $V = \{1, 2, \dots, n\}$  that maps  $G$  to  $G'$ . For  $G'$  to be admissible it must contain the cycle of edges  $1 - 2 - \dots - n - 1$ , which means that  $G$  must contain the cycle,

$$\pi^{-1}(1) - \pi^{-1}(2) - \dots - \pi^{-1}(n) - \pi^{-1}(1).$$

So to bound  $\tau(G)$  it suffices to get an upper bound on the number of permutations  $\pi$  with this property.

Fix an arbitrary  $i \in V$  as  $i = \pi^{-1}(1)$ . Since  $G \in \mathcal{B}_n^c$ , the degree of  $i$  is at most  $d$ , so there are at most  $d$  choices for the node  $\pi^{-1}(2)$ , and similarly, there are then at most  $d$  choices for  $\pi^{-1}(3)$ . Continuing this way, there are at most a total of  $d^{n-1}$  choices for the values of  $\pi^{-1}(j)$ , for  $j = 2, 3, \dots, n$ , and an additional  $n$  choices for the initial value of  $i = \pi^{-1}(1)$ . Therefore, there are at most  $nd^{n-1}$  possible such permutations, and so,

$$\tau(G) \leq n(5b_n)^{n-1}.$$

Finally, we can substitute this in Lemma 4.5 to obtain that,

$$\begin{aligned} H(G_n|S_n) &\leq \sum_{G \in \mathcal{B}_n^c} P_n(G) \log \tau(G) + \sum_{G \in \mathcal{B}_n} P_n(G) \log \tau(G) \\ &\leq \log \left[ n(5b_n)^{n-1} \right] + P_n(\mathcal{B}_n) \log n!, \end{aligned}$$

and using the elementary bound  $n! \leq n^n$ , and Proposition 4.6 with  $t = 2$ , we obtain,

$$\begin{aligned} H(G_n|S_n) &\leq \sum_{G \in \mathcal{B}_n^c} P_n(G) \log \tau(G) + \sum_{G \in \mathcal{B}_n} P_n(G) \log \tau(G) \\ &\leq n \log b_n + (\log 5)n + [\log n - \log b_n] + O(1), \end{aligned}$$

as claimed. □

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