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STABILITY CONSIDERATIONS FOR NETWORKS

Project Summary

The main objective for this research was to establish effective methods for the study of the asymptotic behavior and fundamental limits that occur in networking. In the research, we restrict our interest to stability problems in a multidimensional environment. The variability inherent in most distributed systems is such that to make meaningful performance predictions, it is necessary to study the evolution of a stochastic model. Such a stochastic approach was assumed throughout this project. Our past experiences led us to the conclusion that Markovian techniques applied to multidimensional stability provide only a partial solution. During the course of the research, we proposed several new approaches to the stability analysis of networking that are based on a non-Markovian philosophy. We demonstrated that such an approach can rigorously provide stability criteria for such open problems as stability of multiaccess systems (e.g., ALOHA system), token passing rings, rings with spatial reuse, and so forth.

Introduction

Stability criteria for Markov chains and queueing systems representing networks have a long tradition. In recent years, resurgence of interest in these problem arose due to novel applications. It resulted in an excellent book by Meyn and Tweedie. This book as well as most research in this area are based on the so called Lyapunov or test function approach. Construction of this function is quite troublesome for multidimensional Markov chains. A general approach was suggested in 1981 by Malyshev and Mensikov. This general construction still fails for many important distributed systems, however, recently some progress has been achieved. Our approach is non-standard and it is based on a different philosophy, but it has some similarities with the *faces* and *induced Markov chains* of Malyshev and Menshikov. As mentioned above, in our analysis we apply mathematical induction (that recently became very popular in stability analysis), Loynes stability criteria, and stochastic monotonicity. Monotonicity was recently used in by several people (e.g., Meyn, Down, Sadowsky, etc.) to establish stability regions for other multidimensional queueing systems and computer networks. Finally, we should mention a recent new development in this area suggested by Dai, and Dai and Meyn who used a fluid approximation to derive general stability criteria for queueing networks. For more exhaustive discussions of the existing literature on stability criteria the reader is referred to the book of Meyn and Tweedie.

Throughout, we understand stability of a multidimensional process as follows: A multidimensional processes $\mathbf{N}^t = (N_1^t, \ldots, N_M^t)$ (not necessarily a Markovian process) is stable if for $\mathbf{x} \in I\!N_0^M$ the following holds

$$\lim_{t\to\infty} \Pr\{\mathbf{N}^t < \mathbf{x}\} = F(\mathbf{x}) \qquad \text{and} \qquad \lim_{\mathbf{x}\to\infty} F(\mathbf{x}) = 1$$

where $F(\mathbf{x})$ is the limiting distribution function, and by $\mathbf{x} \to \infty$ we understand that $x_j \to \infty$ for all $j \in \mathcal{M} = \{1, \ldots, M\}$. If a weaker condition holds, namely,

$$\lim_{\mathbf{x}\to\infty} \lim_{t\to\infty} \inf \ \Pr\{\mathbf{N}^t < \mathbf{x}\} = 1 \ ,$$

then the process is called *substable* by Loynes (1962), *tight* by Breiman (1968), and *bounded in probability* by Meyn and Tweedie (1992). Otherwise, the system is unstable.

In this report, we present a sample of our stability results. We concentrate on four papers: Namely, (i) Georgiadis and Szpankowski [3] who analyzed a class of token passing rings; (ii) Georgiadis, Szpankowski and Tssiulas [4] who established the maximal stability region for a ring network with spatial reuse and adaptive scheduling policy; (iii) Szpankowski [18] who gave necessary and sufficient conditions for stability of ALOHA system; (iv) finally our recent paper [5] where we analyzed ring networks with spatial reuse and with quota (this paper presents also a generalization of our stability methodology to a large class of queueing systems satisfying a weak monotonicity property).

Stability of a Class of Token Passing Rings

In our paper [3] we analyzed the following system:

(A1) There are M stations (queues) on a loop, each having infinite capacity buffer.

(A2) The maximum time customers are served during the token visit at a queue is limited to $\tau_i < \infty$ units of time. Only customers that are present at the instant of token arrival can be served. Moreover, we have nonpreemptive discipline, that is, the customer that is in the server when the time limit τ_i is reached, is served to completion before the token moves to the next queue.

(A3) The arrival process A_i^t, $t \in [0,\infty)$ to the ith queue is a Poisson process with parameter $\lambda_i > 0$. Here, A_i^t is the number of arrivals at queue i up to time t. The arrival process at a queue is independent of the arrival processes to other queues.

(A4) The service time process $\{S_i^k\}_{k=1}^{\infty}$ at queue i consists of i.i.d. random variables with mean $s_i = ES_i^1 > 0$. The service time process at a queue is independent of the arrival processes at all queues and independent of the service time processes at other queues.

(A5) The switchover times between i and $i+1 \bmod M$ queue, $\{U_i^k\}_{k=1}^{\infty}$, are i.i.d., independent of the switchover times $\{U_j^k\}_{k=1}^{\infty}$ for $j \neq i$, and independent of the arrival and the service time processes. The average total switchover time is defined as $u_0 = \sum_{i=1}^{M} EU_i^1$. To avoid inessential complications we assume that $P(U_i^n > 0) = 1, \quad i = 1, \cdots, M$.

For such a system, known also as *time-limited* token passing ring (introduced by Leung and Eisenberg), we established stability criteria. To formulate them, we need some notation. Define $\widetilde{L}_i = \min\{k : \sum_{j=1}^{k} S_i^j \geq \tau_i\}$ and $\ell_i = E\widetilde{L}_i$. Clearly, ℓ_i is the average of the maximum number of customers served during τ_i. In [3] we proved the following result:

Theorem 1 *Consider a token passing ring consisting of M stations with τ-limited service schedule for the ith station, and Poisson arrivals. Then the system is stable if and only if $\sum_{j=1}^{M} \rho_j < 1$ and*

$$\lambda_j < \frac{\ell_j}{u_0}(1-\rho_0) \quad \text{for all} \quad j \in \mathcal{M} = \{1, \ldots M\} ,$$

where $u_0 = \sum_{i=1}^{M} EU_i$ is the average total switchover time, and $\rho_0 = \sum_{i=1}^{M} \rho_i$ with $\rho_i = \lambda_i s_i$ and $s_i = ES_i$ being the average service time at the ith station. ■

More importantly, the proof offered in [3] can be applied virtually unchanged to the following class of monotonic and contractive policies. We assume that the number of customers served from queue i when there are n queued messages at the instant of token arrival to queue i is $f_i(n, \mathbf{X})$, where $\mathbf{X}$ is a possibly random quantity that depends on the policy. We assume that $f_i(\cdot, \mathbf{X})$ is a *nondecreasing* function of the number of customers in the ith queue. In addition, the following relation holds

$$f_i(n_1, \mathbf{X}) - f_i(n_2, \mathbf{X}) \leq n_1 - n_2 \qquad if \quad n_1 > n_2 \ .$$

At the kth token arrival to queue i the quantity $\mathbf{X}$ takes the value $\mathbf{A}_k$. The random variables $\{\mathbf{A}_k\}_{k=1}^{\infty}$ are i.i.d, independent of the past arrival times, past service times and past switchover times. Note that $f_i(n, \mathbf{A}_k)$ tends to a random variable, $f_i^*(\mathbf{A}_k)$, when $n \to \infty$. We assume that $f_i^*(\mathbf{A}_k)$ is finite and

$$\ell_i \stackrel{\text{def}}{=} E f_i^*(\mathbf{A}_k) < \infty.$$

We will see in the next section that the time-limited token ring policy falls within the class just described. Other special cases are the ℓ-limited gated discipline and Bernoulli gated disciplines. In the last discipline, $f_i(n, X) = \min\{n, X\}$, where X is a geometrically distributed random variable. The following result was established in [3]:

Theorem 2 *Consider a token passing ring that employs a monotonic and contractive service discipline described above. Then, the system is stable if and only if*

$$\lambda_i < \frac{\ell_i}{u_0}(1 - \rho_0) \qquad \text{for all} \quad i \in \mathcal{M}$$

where $\rho_0 < 1$. ∎

One policy that does not fall in the previously described category is the *preemptive* τ-limited token ring. In such as system the token interrupts his service immediately after the time limit τ expires and continues servicing the same customer in the next round. While the formal arguments of our methodology can be applied for this system, there are technical difficulties that need to be overcome for a complete proof. We believe that these technical arguments can be provided, but we do not have a complete proof yet, and therefore we express the following conjecture.

Conjecture. *The preemptive τ-limited token ring is stable if and only if*

$$\lambda_j < \frac{\tau_j}{s_j u_0}(1 - \rho_0) \quad \text{for all} \ \ j \in \mathcal{M} = \{1, \ldots M\} \ ,$$

and $\rho_0 < 1$. ∎

Maximal Stability Region in a Ring With Spatial Reuse

In [4] we analyzed the following system: The network consists of a number of nodes arranged in a circle. Each node i receives exogenous traffic with destination potentially any other node j of the ring. The packets are queued in the nodes until transmission. All the transmissions follow the unique direction of the ring. The packet transmission time is constant and the transmissions of all nodes are synchronized. The time is slotted. A packet transmitted by node i is transferred within one slot to its destination node j unless there is some node k preceding j in the transmission direction of the ring, which is transmitting another packet in the same slot. In the latter case, node k stores node's i packet for later transmission. Therefore, the parts of the ring traversed by different packets at the same slot, are non-overlapping. The decisions of whether a node will or will not transmit in each slot, as well as the service discipline among the different traffic streams in the node is determined by the transmission policy. The following policy is proposed here.

Transmission policy Π

The policy operates in cycles and is based on the idea of allocating quotas to the nodes. Let τ_k be the beginning of the kth cycle and set $\tau_1 = 1$. At time τ_k each node allocates itself "quota" $\nu_i(k) = Q_i(\tau_k)$, where $Q_i(t)$ is the queue size of node i at time t. Node i can transmit up to $\nu_i(k)$ packets during cycle k according to *any fixed nonidling* policy, i.e., the only restriction that is imposed on the transmissions is that the node transmits a packet in its outgoing link whenever either its queue is nonempty, or a message is received in the same slot in its incoming link with destination another node on the ring. Cycle k ends when *all* the quotas of all nodes are delivered to their destination.

We introduce some notation before we proceed. Let M be the number of nodes and set $\mathcal{M} = \{1, \ldots, M\}$. The operations $i \oplus j$ and $i \ominus j$ denote respectively, addition and subtraction modulo M, with the convention that index 0 refers to node M. Furthermore, when i, j refer to node indices we denote $\sum_{k=i}^{j} x_k := x_i + x_{i\oplus 1} + \ldots + x_{j\ominus 1} + x_j$. We assume that the nodes are arranged on the ring according to their index so that the outgoing link to node i is the incoming link for node $i \oplus 1$. Node i may receive external traffic with destination any other node j in the system. Let $R_{ij}(t)$ be the number of packets that arrive at node i from the outside with destination node j at time t. If $i = j$, then it is assumed that the packet has to cross all the nodes on the ring until it is received by the originating node, i.

Throughout this section we adopt the following assumption.

(A) The vector process $\{\mathbf{R}(t)\}_{t=1}^{\infty}$, where $\mathbf{R}(t) = \{R_{ij}(t),\ i, j \in \mathcal{M}\}$, consists of i.i.d. vectors. We denote $R_{ij} := R_{ij}(1)$. Note that do not make any independence assumptions for the work arriving in various nodes at the same slot. To avoid technical difficulties we will also assume that $\Pr(R_{ij}(t) = 0,\ i, j \in \mathcal{M}) > 0$.

It turns out that the above assumption can be relaxed in certain ways without affecting significantly the validity of our results. In order to formulate our main results in a compact form, we need some additional notation. Let

$$\rho_{ij} := \lim_{n\to\infty} \frac{\sum_{t=1}^{n} R_{ij}(t)}{n} = ER_{ij}$$

be the (end-to-end) arrival rate of packets that arrive to node i with destination node j. We also define $\alpha_{im} := \sum_{j=m\oplus 1}^{i} \rho_{ij}$ and $r_m = \sum_{i\in\mathcal{M}} \alpha_{im}$. Note that α_{im} is the average number of packets per slot that are generated by node i and have to cross node m in order to reach their destination. Therefore, r_m is the average number of packets that cross node m during a slot. Finally, we set $r = \max\{r_m :\ m \in \mathcal{M}\}$.

Since at most one packet can be transmitted in a slot by node m, the condition $r_m \leq 1,\ m \in \mathcal{M}$, is necessary for stability. Therefore, the stability region of any policy is a subset of the region

$$\mathcal{R} = \{\boldsymbol{\rho} = (\rho_{ij})_{i,j=1}^{M} :\ r = \max_{1\leq m\leq M} r_m \leq 1\}$$

In this paper we show that as long as the end-to-end arrival rates belong to

$$\mathcal{R}^o = \{\boldsymbol{\rho} :\ r = \max_{1\leq m\leq M} r_m < 1\},$$

policy Π stabilizes the network in a strong sense. Specifically we show that (i) the queue length $Q_i(t)$ possesses a limiting distribution; (ii) its lth moment $EQ_i^l(t)$ as $t \to \infty$ exists provided that $ER_{ij}^{l+1} < \infty$; (iii) the queue length $Q_i(t)$ as $t \to \infty$ has an exponential tail (i.e., large backlogs are very unlikely), provided that the same is true for R_{ij}. We summarize our main results in the following theorem.

Theorem 3 (i) *Under policy* Π*, the process of queue lengths* $\{Q_i(t),\ i \in \mathcal{M}\}_{t=1}^{\infty}$ *converges in distribution to a random vector* $\{\widetilde{Q}_i,\ i \in \mathcal{M}\}$ *having a honest distribution, if* $r < 1$.

(ii) *If* $r < 1$ *and* $ER_{ij}^{l+1} < \infty$ *for some* $l \geq 1$ *and all* $i, j \in \mathcal{M}$*, then*

$$\lim_{t \to \infty} EQ_i^l(t) = E\widetilde{Q}_i^l < \infty \tag{1}$$

(iii) *If* $r < 1$ *and for* $\vartheta > 0$ *the moment generating function of* R_{ij} *exists, that is,* $E\exp(\vartheta R_{ij}) < \infty$ *for all* $i, j \in \mathcal{M}$*, then there exists* $\vartheta' > 0$ *such that*

$$E\exp(\vartheta' \widetilde{Q}_i) < \infty \tag{2}$$

for every $i \in \mathcal{M}$. ∎

Stability of ALOHA System: A Nonlinear Stability Region

We start with a short description of the buffered ALOHA system. The system consists of M distributed users, each having an infinite buffer for storing fixed-length packets. The channel is slotted, and a slot duration is equal to a packet transmission time. The arrival process is i.i.d. with respect to slots, and arrival processes are independent from a user to a user. A nonempty user transmits a packet with probability r_i in a slot, where $i \in \mathcal{M}$ and $\mathcal{M} = \{1, 2, \ldots, M\}$ is the set of users. If two or more users transmit simultaneously, then a collision occurs and the packets must be retransmitted in future. When exactly one packet is transmitted in a slot, then a successful transmission takes place, the packet is removed from its queue, and another packet, if the queue is nonempty, gets its chance to be served. For technical reasons we additionally introduce slot availability. We assume that a slot is available to all users with probability $1 - b$, and blocked with probability b. In the standard ALOHA we have $b = 0$.

Let N_j^t represent the queue length in the jth user at the beginning of the tth slot, where $t = 0, 1, \ldots$. Under the above assumptions, the M-dimensional process $\mathbf{N}^t = (N_1^t, N_2^t, \ldots, N_M^t)$ is a *Markov chain* since

$$N_j^{t+1} = (N_j^t - Y_j^t)^+ + X_j^t$$

where X_j^t represents the number of customers arriving during the tth slot to the jth user with $EX_j^t = \lambda_j < \infty$, and

$$Y_j^t = (1 - B^t) R_j^t \left(1 - \sum_{k \in \mathcal{M} - \{j\}} R_k^t \chi(N_k^t) \right)^+ .$$

In the above: $1 - \Pr\{R_j^t = 0\} = \Pr\{R_j^t = 1\} = r_j$, and $\Pr\{B^t = 0\} = 1 - \Pr\{B^t = 1\} = 1 - b$, and for all $k \in I\!N_0$

$$\chi(k) = \begin{cases} 0 & \text{if } k = 0 \\ 1 & \text{if } k \geq 1 \end{cases}$$

where $I\!N_0$ is a set of nonnegative integers.

To explain our philosophy behind the stability analysis, we start with an informal overview of our approach. First of all, we construct a modified ALOHA system as follows. Let $\mathcal{P} = (\mathcal{S}, \mathcal{U})$ be a partition of $\mathcal{M}$ such that users in $\mathcal{S} \neq \mathcal{M}$ work exactly in the same manner as in the original ALOHA model, while users in $\mathcal{U}$ *persistently* attempt to send (dummy) packets even if their buffers are empty (e.g., dummy packets). We call users in $\mathcal{U}$ *persistent* (or jamming or saturated), and users in $\mathcal{S}$ *nonpersistent.* Note that the system consisting of users in $\mathcal{S}$ forms a smaller copy of the original ALOHA system with a new probability of slot availability, namely $1 - b = \prod_{j \in \mathcal{U}} (1 - r_j)$. In passing, we observe that our sets $\mathcal{U}$ and $\mathcal{S}$ resemble *faces* and *induced Markov chains* of Malyshev and Mensikov (1981).

For a given partition $\mathcal{P} = (\mathcal{S},\mathcal{U})$ of $\mathcal{M}$ such that $\mathcal{S} \neq \mathcal{M}$, let $\overline{\mathbf{N}}^t_{\mathcal{P}} = (\overline{\mathbf{N}}^t_{\mathcal{S}}, \overline{\mathbf{N}}^t_{\mathcal{U}})$ denote the queue lengths in the modified system where $\overline{\mathbf{N}}^t_{\mathcal{S}}$ (resp. $\overline{\mathbf{N}}^t_{\mathcal{U}}$) represents the queue lengths in $\mathcal{S}$ (resp. in $\mathcal{U}$). It can be proved (cf. Tsybakov and Mikhailov (1979) and Szpankowski (1986)) that $\overline{\mathbf{N}}^t_{\mathcal{P}}$ dominates the original queueing process $\mathbf{N}^t$, that is,

$$\mathbf{N}^t \leq_{st} \overline{\mathbf{N}}^t_{\mathcal{P}}$$

provided $\mathbf{N}^0 = \overline{\mathbf{N}}^0_{\mathcal{P}}$.

Note that by our construction, the process $\overline{\mathbf{N}}^t_{\mathcal{S}}$ is an $|\mathcal{S}|$-dimensional Markov chain that mimics the behavior of the ALOHA system. Since $|\mathcal{S}| < M$, we can apply mathematical induction to establish its stability. We further assume that the Markov chain $\overline{\mathbf{N}}^t_{\mathcal{S}}$ is stationary and ergodic. We denote this stationary version as $\overline{\mathbf{N}}^0_{\mathcal{S}}$ to indicate that the process starts from the stationary distribution.

Let $Y^t_j(\mathcal{S})$ be the output process from the jth queue in the dominant system. We write $P^j_{succ}(\mathcal{S}) = EY^t_j(\mathcal{S})$ to denote the probability of a successful transmission from the jth user in the dominant system $P^j_{succ}(\mathcal{S}) = EY^t_j(\mathcal{S})$ provided $Y^t_j(\mathcal{S})$ is a stationary sequence. This is assumed to hold as a simple consequence of our assumption regarding stationarity of $\overline{\mathbf{N}}^t_{\mathcal{S}}$. Then,

$$P^j_{succ}(\mathcal{S}) = r_j \prod_{k \in \mathcal{U}-\{j\}} (1-r_k) \sum_{\mathbf{z}_{\mathcal{S}} \in \Theta_{\mathcal{S}}} \left(\Pr\{\chi(\overline{\mathbf{N}}^0_{\mathcal{S}}) = \mathbf{z}_{\mathcal{S}}\} \prod_{k \in \mathcal{S}-\{j\}} (1-r_k)^{z_k} \right) , \tag{3}$$

where the summation is over the set

$$\Theta_{\mathcal{S}} = \{\mathbf{z}_{\mathcal{S}} : \mathbf{z}_{\mathcal{S}} = (z_1, \ldots, z_{|\mathcal{S}|}),\ z_j \in \{0,1\},\ j \in \mathcal{S}\} .$$

In the above, $\chi(\overline{\mathbf{N}}^0_{\mathcal{S}}) = (\chi(\overline{N}^0_{i_1}), \ldots, \chi(\overline{N}_{i_{|\mathcal{S}|}}))$ where $i_j \in \mathcal{S}$ for all $j = 1, \ldots, |\mathcal{S}|$, and by $k \in \mathcal{A}-\{j\}$ we mean that k belongs to $\mathcal{A}$ and it is *not* equal to j (even if $j \notin \mathcal{A}$). We set by convention $\Pr\{\chi(\overline{\mathbf{N}}^0_{\mathcal{S}}) = 1\} = 1$ in the case $\overline{\mathbf{N}}^t_{\mathcal{S}}$ is unstable.

Observe also that one can view (3) as the probability of success in an $\mathcal{S}$-copy of the ALOHA system with slot availability equal to $1-b_{\mathcal{U}} = \prod_{k \in \mathcal{U}}(1-r_k)$. Furthermore, if one considers a version of ALOHA with (external) slot availability equal to $1-b$, then the right-hand side of (3) must be multiplied by $1-b$.

To present our result in a compact form we introduce some more notation. Let $\mathcal{R}_{\mathcal{M}}$, $\mathcal{R}_{\mathcal{S}}$ and $\mathcal{R}_{\mathcal{U}}$ be respectively stability regions for the whole system, and the systems consisting of nonpersistent users $\mathcal{S}$ and persistent ones $\mathcal{U}$ in the modified ALOHA system under the partition $\mathcal{P} = (\mathcal{S},\mathcal{U})$. We set $\mathcal{R}_{\mathcal{P}} = \mathcal{R}_{\mathcal{S}} \cap \mathcal{R}_{\mathcal{U}}$. Sometimes, we write more explicitly $\mathcal{R}_{\mathcal{M}}[b]$ and/or $\mathcal{R}_{\mathcal{S}}[b]$ for the stability region of a system with slot availability $1-b$. We use below $\mathcal{R}_{\mathcal{M}}$ ($\mathcal{R}_{\mathcal{S}}$) and $\mathcal{R}_{\mathcal{M}}[b]$ ($\mathcal{R}_{\mathcal{S}}[b]$) interchangeable to simplify notations as long as it does not lead to a confusion. In addition, we define $\mathcal{M}_k = \mathcal{M}-\{k\}$ so that the corresponding partition is $\mathcal{P}_k = (\mathcal{M}_k, \{k\})$. Thus, $\mathcal{R}_{\mathcal{M}_k}[r_k]$ denotes the stability region of an $M-1$-dimensional ALOHA system with probability of blocking $b = r_k$. We also write $\boldsymbol{\lambda}_{\mathcal{S}} = \{\lambda_i\}_{i \in \mathcal{S}}$.

The next theorem is our main finding, and it provides sufficient and necessary condition for stability of the buffered ALOHA system. Below, we write iff* to mean: *if and only if* with possible exception of the boundary points of a stability region. (In fact, our technique usually does not establish stability of boundary points, however, in most cases only minor modifications are necessary to prove instability of boundary points; e.g., see Georgiadis and Szpankowski [3] and its references).

Theorem 4 *The stability region of the buffered ALOHA system with slot availability* $1 - b$ *is*

$$\mathcal{R}_{\mathcal{M}}[b] = \bigcup_{k=1}^{M} \{\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_M) : \; \lambda_k < P_{succ}^k(\mathcal{M}_k), \quad and \quad \boldsymbol{\lambda}_{\mathcal{M}_k} \in \mathcal{R}_{\mathcal{M}_k}[1 - (1-b)(1-r_k)]\} \; .$$

*In particular, the original ALOHA system is stable iff** $\boldsymbol{\lambda} \in \mathcal{R}_{\mathcal{M}}[0]$.

Stability of a Ring with Spatial Reuse and Quota

We report now our results from [5]. Undoubtly, these are our strongest and most general results. Although we concentrated in the paper on analyzing stability of a ring with spatial reuse and quota, the paper gives a general methodology to obtain sufficient and necessary conditions for stability of a class of queueing networks possessing a monotonicity property. Interestingly enough, obtained stability region is non-linear, i.e., it depends not only on the first moment of the inputs but the whole input distribution.

We start with a description of the system. We consider a unidirectional ring network consisting of a set of $\mathcal{M}$ nodes with cardinality, $|\mathcal{M}| = M$. Node $i \in \mathcal{M}$ transmits in its outgoing link, either packets arriving to this node from the outside world (i.e., "external" packets) or packets that were originated at some other node and have to cross node i in order to reach their destination. Time is divided in slots, packets are of fixed size and each slot is equal to the length of a packet. We assume zero propagation delay. A node can transmit a packet on the outgoing link at the same time that it receives another packet in the incoming link. A node receiving a packet whose destination is another node in the ring (ring packet) may relay the packet in the outgoing link *in the same slot*, i.e., the ring has *cut-through* capabilities. Moreover, a ring packet has non-preemptive priority over the packets that exist in the node queue. Packets are removed from the ring by their destination (not by the source as in standard token rings). We study the following policy, which is a generalization of a policy proposed in the literature.

(A1) The system works in cycles, and the kth cycle starts at time τ_k. We shall write $\mathbf{N}(k) = (N_1(k), \ldots, N_M(k))$ to denote the number of packets in the node buffers at the beginning of cycle $k = 1, \ldots$. The number of external packets that node i is allowed to transmit during cycle k is $Q_i(k) = \min\left\{f_i(N_i(k)), \widehat{Q}_i\right\}$, $\widehat{Q}_i > 0$, where $f_i(\cdot)$ is a nondecreasing and contractive function, i.e., $f_i(s_1) - f_i(s_2) \le s_1 - s_2$ whenever $s_1 > s_2$. The quantity $\widehat{Q}_i$ is called the (maximum) *quota*. At each time slot during a cycle, a node may transmit packets that are either generated locally or by some upstream node. The kth cycle ends when all $Q_i(k)$ packets, $1 \le i \le M$, are delivered to their destinations. Algorithmic and implementation details can be found in literature, however, of interest to our discussion here is only the statistics of the length of time needed to complete a cycle (cf. (4) below). The standard ring network operating with the quota allocation policy corresponds to the case when $f_i(s) = s$.

As part of the technique used in the proof of the stability conditions we need to analyze a system $\Theta^{\mathcal{M},\mathcal{U}}$, where in addition to the set, $\mathcal{M}$, of regular nodes there is also a set, $\mathcal{U}$, of "persistent nodes", which operate as follows.

(A2) (i) There are no external packet arrivals at node $i \in \mathcal{U}$, and (ii) a node $i \in \mathcal{U}$ participates in the policy described in (A1) by generating locally and transmitting exactly $\widehat{Q}_i$ "dummy" packets in a cycle (in addition to the packets that may have originated in some other node but have to be retransmitted by node i in its outgoing link in order to reach their destination).

While the nodes in $\mathcal{U}$ affect the duration of the cycles, by definition they do not have queues and we are interested only in the stability of the queue length process of the nodes in $\mathcal{M}$. As will be seen in the next section, the introduction of persistent nodes assures that when some regular nodes behave like persistent ones, the system consisting of the rest of the nodes is a copy of the original system, but of *lower dimension*. This property permits the application of mathematical induction. The case $\mathcal{U} = \emptyset$ corresponds to the ring we are interested in.

Next, we make an assumption regarding the statistics of external packet arrival process at the ring nodes.

(A3) We denote by $R_i(t)$ the number of external packets arriving at station $i \in \mathcal{M}$ in slot $t \geq 1$. The nth packet originated at node $i \in \mathcal{M} \cup \mathcal{U}$ has destination $D_i(n) \in \mathcal{M} \cup \mathcal{U}$. The processes $\{R_i(t)\}_{t=1}^{\infty}$, $i \in \mathcal{M}$ and $\{D_i(n)\}_{n=1}^{\infty}$, $i \in \mathcal{M} \cup \mathcal{U}$ consist of i.i.d. random variables and are independent of each other We set $\lambda_i = ER_i(1)$, $i \in \mathcal{M}$ and $p_{ij} = \Pr\{D_i(1) = j\}$, $i, j \in \mathcal{M} \cup \mathcal{U}$. Clearly, $\sum_{j \in \mathcal{M} \cup \mathcal{U}} p_{ij} = 1$ for $i \in \mathcal{M} \cup \mathcal{U}$.

Before proceeding, we must introduce some new notations. Boldface letters denote vectors, while calligraphic ones denote sets of nodes. Our main goal is to study the ergodicity of the embedded Markov chain $\mathbf{N}(k) = (N_1(k), \ldots, N_M(k))$ for $k = 0, 1, \ldots$. We write $\mathcal{M}_{\mathcal{A}} = \mathcal{M} - \mathcal{A}$ (while nonstandard, this notation simplifies the presentation significantly). We will often consider the partition $(\mathcal{M}_{\mathcal{V}}, \mathcal{V})$ of the set $\mathcal{M}$, where $\mathcal{V} \subseteq \mathcal{M}$. For a vector $\mathbf{x} = (x_1, \ldots, x_M)$ we set $\mathbf{x}^{\mathcal{A}} = \{x_i\}_{i \in \mathcal{A}}$. In particular, we write $\mathbf{N}(k) = (\mathbf{N}^{\mathcal{M}_{\mathcal{A}}}(k), \mathbf{N}^{\mathcal{A}}(k))$. For M-dimensional vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{x} \leq \mathbf{y}$ reads $x_i \leq y_i$ for all $1 \leq i \leq M$.

The behavior of the network depends crucially on the cycle length $T_k = \tau_{k+1} - \tau_k$ which is also called the *evacuation time*. Let $T(\mathbf{q})$ (or $T_k(\mathbf{q})$) be the length of cycle k when the quota vector is $\mathbf{Q}^{\mathcal{M} \cup \mathcal{U}}(k) = \mathbf{q} = (q_1, \ldots, q_M, \ldots, q_{M+|\mathcal{U}|})$. It was shown that[1]

$$T_k(\mathbf{q}) = \max\left\{\max_{i \in \mathcal{M} \cup \mathcal{U}} H_i(\mathbf{q}), 1\right\} \tag{4}$$

where $H_i(\mathbf{q})$ is the total number of packets out of $\sum_{i \in \mathcal{M} \cup \mathcal{U}} q_i$ originated in a cycle at any node (i.e., regular or persistent) that have to pass through the outgoing link of node i in order to reach their destination. Note that $H_i(\mathbf{q})$ includes the packets originated at node i. Also, note that the statistics of $T(\mathbf{q})$ depend only on the vector $\mathbf{q}$ and the packet destination probabilities p_{ij}.

Below we establish a monotonicity property of the cycle lengths. As we will see, this is a relevant property of the cycle lengths from the stability point of view. In fact, our analysis holds for any other system which, in addition to operating in cycles during which a certain quota can be transmitted by each node and satisfying the statistical assumptions in (A3), has the property that the cycle lengths are independent of the past history given $\mathbf{Q}^{\mathcal{M} \cup \mathcal{U}}(k) = \mathbf{q}$ and satisfy the monotonicity property presented in the next proposition.

Proposition 1 *Let* $\mathbf{q}_1 \leq \mathbf{q}_2$. *Then*

$$T_k(\mathbf{q}_1) \leq_{st} T_k(\mathbf{q}_2)$$

where $\leq_{st}$ *means "stochastically smaller".*

[1] In previous works, it was assumed that $q_i \geq 1$ for some node i. In our model, however, it is possible that all nodes have empty queues at the beginning of a cycle, in which case $T_k(\mathbf{0}) = 1$. For this reason we include the maximum with 1 in (4).

Proof. Follows easily from formula (4). ■

Let us now consider a modified system in which a set $\mathcal{V} \subset \mathcal{M}$ of users becomes persistent, that is, every user in $i \in \mathcal{V}$ transmits $\widehat{Q}_i$ packets (i.e., it transmits "dummy" packets when their queues are empty or possess less than $\widehat{Q}_i$ packets). From the point of view of the nodes in the set $\mathcal{M}_{\mathcal{V}}$, the nodes in $\mathcal{V}$ behave exactly as the *persistent* nodes in the set $\mathcal{U}$. Note, however, that there is a difference between the nodes in $\mathcal{U}$ and $\mathcal{V}$ in that the nodes in $\mathcal{V}$ receive external packets and therefore have queues formed. We denote such a system as $\overline{\Theta}^{(\mathcal{M}_{\mathcal{V}},\mathcal{V}),\mathcal{U}}$. Define $\widetilde{\mathbf{N}}^{(\mathcal{M}_{\mathcal{V}},\mathcal{V})}(k) = (\widetilde{\mathbf{N}}^{\mathcal{M}_{\mathcal{V}}}(k), \widetilde{\mathbf{N}}^{\mathcal{V}}(k))$ as the queue length vector in the system $\overline{\Theta}^{(\mathcal{M}_{\mathcal{V}},\mathcal{V}),\mathcal{U}}$. In the next result, we show that the queues in the modified system dominate stochastically the queue length in the original system. This property is crucial to applications of our method.

Proposition 2 *Consider two partitions $(\mathcal{M}_{\mathcal{V}_1}, \mathcal{V}_1)$ and $(\mathcal{M}_{\mathcal{V}_2}, \mathcal{V}_2)$ such that $\mathcal{V}_1 \subseteq \mathcal{V}_2$. Then for every $k = 0, 1, \ldots$*

$$\widetilde{\mathbf{N}}^{(\mathcal{M}_{\mathcal{V}_1},\mathcal{V}_1)}(k) \leq_{st} \widetilde{\mathbf{N}}^{(\mathcal{M}_{\mathcal{V}_2},\mathcal{V}_2)}(k) \tag{5}$$

provided $\widetilde{\mathbf{N}}^{(\mathcal{M}_{\mathcal{V}_1},\mathcal{V}_1)}(0) = \widetilde{\mathbf{N}}^{(\mathcal{M}_{\mathcal{V}_2},\mathcal{V}_2)}(0)$.

We now present a construction of the stability region. Consider the system $\Theta^{\mathcal{M},\mathcal{U}}$ consisting of a set $\mathcal{M}$ of regular nodes and a set $\mathcal{U}$ of persistent nodes. Our goal is to establish stability conditions for the queue length vector $\mathbf{N}^{\mathcal{M}}(k)$. By stability we mean the existence of the limiting distribution.

The process $\mathbf{N}^{\mathcal{M}}(k)$ is an embedded Markov chain. Indeed, we have for every $i \in \mathcal{M}$,

$$N_i(k+1) = N_i(k) - Q_i(k) + \sum_{t=\tau_k}^{\tau_{k+1}-1} R_i(t). \tag{6}$$

Under (A1)-(A3) and equation (4), the above set of stochastic equations forms an M-dimensional Markov chain defined on a countable state space.

In the sequel, we will use the following property of multidimensional Markov chains defined on a countable state space: *To establish ergodicity of $\mathbf{N}^{\mathcal{M}}(k)$ it suffices to show that every component $N_i(k)$, $i \in \mathcal{M}$, of $\mathbf{N}^{\mathcal{M}}(k)$ is substable* (i.e., the one dimensional process $N_i(k)$ is bounded in probability as $k \to \infty$). This fact is easy to prove on a countable state space, and the reader is referred to [3, 18]. On a general state space, the situation is more complicated, and one should consult Meyn and Tweedie. This fact, called *isolation lemma* in [18], permits the study of the stability of each queue in isolation.

We now begin the construction of the stability region (i.e., the set of node arrival rates) of system $\Theta^{\mathcal{M},\mathcal{U}}$ based on the knowledge of the stability region and the steady-state average cycle lengths of lower dimensional systems. We denote stability region of a whole system as $S^{\mathcal{M},\mathcal{U}}$. We write $\overline{S}^{(\mathcal{M}_{\{i\}},\{i\}),\mathcal{U}}$ to denote the stability region of the dominant system $\overline{\Theta}^{(\mathcal{M}_{\{i\}},\{i\}),\mathcal{U}}$ which arose by making the ith node behave like a persistent one. Note that while node i in the dominant system behaves like a persistent one, this node still has a queue formed, and therefore region $\overline{S}^{(\mathcal{M}_{\{i\}},\{i\}),\mathcal{U}}$ consists of M-dimensional (*not* of $(M-1)$-dimensional as the region $S^{\mathcal{M}_{\{i\}},\mathcal{U}\cup\{i\}}$) vectors. For simplicity, whenever there is no possibility for confusion, we omit the set $\mathcal{U}$ from the notation in Θ, $\overline{\Theta}$, S, or $\overline{S}$. For example, unless otherwise specified, $\Theta^{\mathcal{M}} \equiv \Theta^{\mathcal{M},\mathcal{U}}$, $\Theta^{\mathcal{M}_{\mathcal{V}},\mathcal{V}} \equiv \Theta^{\mathcal{M}_{\mathcal{V}},\mathcal{U}\cup\mathcal{V}}$, $\overline{S}^{(\mathcal{M}_{\{i\}},\{i\})} \equiv \overline{S}^{(\mathcal{M}_{\{i\}},\{i\}),\mathcal{U}}$.

The construction of the stability region follows the steps developed in [3, 18]. We will therefore skip the details of a rigorous derivation and instead we will explain in some detail the main idea behind each step. The construction is done inductively as follows:

Step 1: Derive the (sufficient) stability condition for a ring with one regular node and an arbitrary set $\mathcal{U}$ of persistent nodes. In this case, we have a single queue (at the regular node) and the derivation of the stability condition is easy. Specifically, let i be a single regular node in the ring. Then, since the queue length $N_i(k)$ is a Markov chain, using the Lyapunov test function method one directly proves that the chain is ergodic if,

$$\lambda_i < \frac{\widehat{Q}_i}{ET(\widehat{\mathbf{Q}})} \tag{7}$$

where, we recall that $T(\widehat{\mathbf{Q}})$ denotes the cycle length when $\mathbf{Q}^{\mathcal{M}\cup\mathcal{U}}(1) = \widehat{\mathbf{Q}}$ at the beginning of a cycle. That is, $ET(\widehat{\mathbf{Q}}) \equiv E[T_1|\mathbf{Q}^{\mathcal{M}\cup\mathcal{U}}(1) = \widehat{\mathbf{Q}}]$.

Step 2: Assume that we derived the stability region for a ring with $M-1$ regular nodes and an arbitrary set $\mathcal{U}$ of persistent nodes. We next seek to define the stability region of a ring with a set $\mathcal{M}$, $|\mathcal{M}| = M$, regular nodes, and an arbitrary set $\mathcal{U}$ of persistent nodes, in terms of the stability regions and the steady-state average cycle lengths of *lower dimensional systems.* This is done by taking a set $\mathcal{V} \subseteq \mathcal{M}$, $\mathcal{V} \neq \emptyset$, of regular nodes and making them behave like persistent ones, i.e., by considering system $\overline{\Theta}^{(\mathcal{M}_\mathcal{V},\mathcal{V})}$. By Proposition 2, $\mathbf{N}^{\mathcal{M}}(k) \equiv \widetilde{\mathbf{N}}^{(\mathcal{M},\emptyset)}(k) \leq_{st} \widetilde{\mathbf{N}}^{(\mathcal{M}_\mathcal{V},\mathcal{V})}(k)$, provided that $\mathbf{N}^{\mathcal{M}}(0) = \widetilde{\mathbf{N}}^{(\mathcal{M}_\mathcal{V},\mathcal{V})}(0)$. Therefore, $\Theta^{\mathcal{M}}$ is stable whenever $\overline{\Theta}^{(\mathcal{M}_\mathcal{V},\mathcal{V})}$ is, i.e., $S^{\mathcal{M}} \supseteq \overline{S}^{(\mathcal{M}_\mathcal{V},\mathcal{V})}$. Since $\mathcal{V}$ is arbitrary, we conclude that $S^{\mathcal{M}} \supseteq \bigcup_{\mathcal{V}\subseteq\mathcal{M}} \overline{S}^{(\mathcal{M}_\mathcal{V},\mathcal{V})}$, where $\mathcal{V} \neq \emptyset$. However, it follows again from Proposition 2 that if $i \in \mathcal{V}$, then $\overline{S}^{(\mathcal{M}_{\{i\}},\{i\})} \supseteq \overline{S}^{(\mathcal{M}_\mathcal{V},\mathcal{V})}$. Therefore,

$$S^{\mathcal{M}} \supseteq \bigcup_{i\in\mathcal{M}} \overline{S}^{(\mathcal{M}_{\{i\}},\{i\})}.$$

In fact, it turns out (see Theorem 6 below) that for the problem at hand, we have equality (with the possible exception of boundaries) in the previous subset relation. With a slight abuse of notation, to avoid the introduction of new symbols we will also denote the set $\bigcup_{i\in\mathcal{M}} \overline{S}^{(\mathcal{M}_{\{i\}},\{i\})}$ by $S^{\mathcal{M}}$.

Step 2a: Next, we determine the stability region $\overline{S}^{(\mathcal{M}_{\{i\}},\{i\})}$ of system $\overline{\Theta}^{(\mathcal{M}_{\{i\}},\{i\})}$, in terms of the stability region and the steady-state average cycle lengths of the system $\Theta^{\mathcal{M}_{\{i\}},\{i\}}$ which is of dimension $M-1$, and therefore, its stability region has been determined by the inductive assumption. It should be noted that this is the point where the "fixed but arbitrary" set $\mathcal{U}$ is used in the proof since now we can claim that $\Theta^{\mathcal{M}_{\{i\}},\{i\}}$ is a "smaller copy" of the original system. Specifically, in system $\overline{\Theta}^{(\mathcal{M}_{\{i\}},\{i\})}$ the set of persistent nodes is $\mathcal{U}$. However, in system $\Theta^{\mathcal{M}_{\{i\}},\{i\}}$, the set of persistent nodes is $\mathcal{U} \cup \{i\} \neq \mathcal{U}$. Therefore we could not have applied the inductive hypothesis if the assumption in this hypothesis did not involve an arbitrary set $\mathcal{U}$. To determine $\overline{S}^{(\mathcal{M}_{\{i\}},\{i\})}$ we apply the isolation lemma, i.e., we look for conditions under which each queue in the set $\mathcal{M}$, under system $\overline{\Theta}^{(\mathcal{M}_{\{i\}},\{i\})}$, is substable. For this, we look first at the queues in the set $\mathcal{M}_{\{i\}}$, which evolve exactly as in system $\Theta^{\mathcal{M}_{\{i\}},\{i\}}$. Therefore, these queues are stable as long as $\boldsymbol{\lambda}^{\mathcal{M}_{\{i\}}} \in S^{\mathcal{M}_{\{i\}},\{i\}}$, which by the inductive hypothesis is known.

Step 2b: It remains to determine conditions under which the queue at node $\{i\}$ is (sub)stable in system $\overline{\Theta}^{(\mathcal{M}_{\{i\}},\{i\})}$, which is done as follows. Assuming that $\boldsymbol{\lambda}^{\mathcal{M}_{\{i\}}} \in S^{\mathcal{M}_{\{i\}},\{i\}}$, we can construct a stationary and ergodic version of the queue length vector $\widetilde{\mathbf{N}}^{\mathcal{M}_{\{i\}}}(k)$ by starting it from the stationary distribution. Provided that this is done, the cycles in $\overline{S}^{(\mathcal{M}_{\{i\}},\{i\})}$, denoted as $T^{\mathcal{M}_{\{i\}}}(k)$, form a stationary and ergodic sequence (by similar arguments to the one presented in our papers

[3, 18]). We set $T^{\mathcal{M}_{\{i\}}} = T^{\mathcal{M}_{\{i\}}}(1)$. More generally, in the following we denote by $T^{\mathcal{M}_{\mathcal{V}}}$ the steady-state cycle length in the system $\Theta^{\mathcal{M}_{\mathcal{V}},\mathcal{V}}$, provided that this system is stable.[2] Since the queue length of node i satisfies (6) and the cycle lengths are stationary, an application of Loynes' criterion shows that the queue at node i is stable if

$$\lambda_i < \widehat{Q}_i / ET^{\mathcal{M}_{\{i\}}} \; .$$

which completes the construction of the stability region. □

In summary, system $\overline{\Theta}^{(\mathcal{M}_{\{i\}},\{i\})}$ is stable when

$$\boldsymbol{\lambda}^{\mathcal{M}} \in \overline{S}^{(\mathcal{M}_{\{i\}},\{i\})} = \left\{ \boldsymbol{\lambda} : \; \boldsymbol{\lambda}^{\mathcal{M}_{\{i\}}} \in S^{\mathcal{M}_{\{i\}},\{i\}} \text{ and } \lambda_i < \frac{\widehat{Q}_i}{ET^{\mathcal{M}_{\{i\}}}} \right\}.$$

Repeating the previous argument for all $i \in \mathcal{M}$, we finally have the following result.

Theorem 5 *Let*

$$S^{\mathcal{M}} = \bigcup_{i \in \mathcal{M}} \left\{ \boldsymbol{\lambda} : \quad \boldsymbol{\lambda}^{\mathcal{M}_i} \in S^{\mathcal{M}_{\{i\}},\{i\}} \qquad \textit{and} \qquad \lambda_i < \frac{\widehat{Q}_i}{ET^{\mathcal{M}_{\{i\}}}} \right\}. \tag{8}$$

Then, system $\Theta^{\mathcal{M}}$ is stable if $\boldsymbol{\lambda} \in S^{\mathcal{M}}$.

Using the stability condition of the one-dimensional system as described above and iterating the recursive formula (8), we obtain a more explicit form for the stability region.

Corollary 1 *Let Σ be the set of permutations of the set $\mathcal{M} = \{1, 2, \ldots, M\}$, and let $\sigma = (\sigma(1), \ldots, \sigma(M)) \in \Sigma$. System $\Theta^{\mathcal{M}}$ is stable if $\boldsymbol{\lambda} \in S^{\mathcal{M}}$, with*

$$S^{\mathcal{M}} = \bigcup_{\sigma \in \Sigma} \left\{ \boldsymbol{\lambda} : \; \lambda_{\sigma(l)} < \frac{\widehat{Q}_{\sigma(l)}}{ET^{\mathcal{M}_{\sigma}(l)}}, \;\; l \in \mathcal{M} \right\}, \tag{9}$$

where $\mathcal{M}_{\sigma}(l) = \bigcup_{n=1}^{l-1} \{\sigma(n)\}$ (by convention $\bigcup_{n=1}^{0} \{\sigma(n)\} = \emptyset$) .

Theorem 5 provides sufficient conditions for the stability region of the M-dimensional system in terms of the sufficient conditions for the stability region (through $S^{\mathcal{M}_{\{i\}},\mathcal{U}\cup\{i\}}$) and the steady-state average cycle lengths (through $ET^{\mathcal{M}_{\{i\}}}$) of $M-1$-dimensional systems. As we will see below, with the exception of the boundaries, these conditions are also necessary.

We now start discussing the necessity of the conditions in (8). The following decomposition is crucial for the analysis.

Proposition 3 *Let $S_c^{\mathcal{M}}$ be the complement of the stability region $S^{\mathcal{M}}$. Then, the following decomposition holds*

$$S_c^{\mathcal{M}} = \bigcup_{\mathcal{V} \subseteq \mathcal{M}} \left\{ \boldsymbol{\lambda} : \quad \boldsymbol{\lambda}^{\mathcal{M}_{\mathcal{V}}} \in S^{\mathcal{M}_{\mathcal{V}},\mathcal{V}} \quad , \quad \lambda_j \geq \frac{\widehat{Q}_j}{ET^{\mathcal{M}_{\mathcal{V}}}} \quad \text{for all } j \in \mathcal{V} \right\} \tag{10}$$

where $\mathcal{V}$ ranges over all nonempty subsets of $\mathcal{M}$.

[2]Note that by definition, $T^{\emptyset}$ is the steady-state cycle length in the system $\Theta^{\emptyset,\mathcal{M}}$, that is in the system where all nodes behave like persistent nodes, i.e., node $i \in \mathcal{U} \cup \mathcal{M}$ generates $\widehat{Q}_i$ packets during a cycle. Therefore, in this case we have $T^{\emptyset} \equiv T(\widehat{\mathbf{Q}})$.

In order to prove the necessary stability condition we need the following general result that is of its own interest.

Proposition 4 *Let $\mathbf{X}^{\mathcal{M}}(n)$, $n = 1, \ldots,$ be an M-dimensional Markov chain (not necessarily denumerable). Assume that it is known that if the process starts from state $\mathbf{u} \in \Re^M$, then for all $i \in \mathcal{V} \subseteq \mathcal{M}$,*

$$\lim_{n\to\infty} X_i(n) = \infty.$$

Then, given any bounded one-dimensional set A, there is a state $\mathbf{c} \in \Re^M$ such that $c_i \notin A$ for all $i \in \mathcal{V}$ and

$$\Pr\{X_i(n) \notin A,\ i \in \mathcal{V},\ n \geq 1 \mid \mathbf{X}(1) = \mathbf{c}\} > 0\ ,$$

that is, with positive probability all components of $\mathbf{X}(n)$ with indices belonging to $\mathcal{V}$ never return to the set A.

We are now ready to show that with the exception of the boundaries, condition (8) is necessary for the stability of the ring with spatial reuse. In addition, we provide a characterization of the instability region. Specifically, we show that with the exception of the boundaries, when the system is unstable, we can identify regions where some queues are substable and the remaining queues tend to infinity with positive probability. Note that while it is easy to show that instability of one queue leads to instability of the whole system (for a formal proof see for example [18]), in general, instability of a multidimensional Markov chain, does not imply that at least one of the components converges to infinity. It is easy to construct multidimensional systems where fluctuations of the queue lengths between large and small values occur when the system is unstable. Consider for example the case of two queues with packets of unit length, served by a single server and assume that the server serves exhaustively the queue that it visits. If $\lambda_i < 1$ for $i = 1, 2$ but $\lambda_1 + \lambda_2 > 1$, then the system is unstable while the queue sizes of both queues return to zero infinitely often with probability one, for all initial states.

The next theorem completes the construction of the stability region for the ring with spatial reuse.

Theorem 6 *System $\Theta^{\mathcal{M}}$ is unstable if $\boldsymbol{\lambda} \in \widehat{S}_c^{\mathcal{M}}$, where $\widehat{S}_c^{\mathcal{M}}$ is the complement of $S^{\mathcal{M}}$ minus the boundary points, that is,*

$$\widehat{S}_c^{\mathcal{M}} = \bigcup_{\mathcal{V}\subseteq\mathcal{M}} \left\{ \boldsymbol{\lambda} : \quad \boldsymbol{\lambda}^{\mathcal{M}_{\mathcal{V}}} \in S^{\mathcal{M}_{\mathcal{V}},\mathcal{V}} \quad , \quad \lambda_j > \frac{\widehat{Q}_j}{ET^{\mathcal{M}_{\mathcal{V}}}} \quad \text{for all} \quad j \in \mathcal{V} \right\} , \tag{11}$$

where $\mathcal{V}$ ranges over all nonempty subsets of $\mathcal{M}$. Furthermore, in the region

$$\widehat{S}_c^{\mathcal{M}}(\mathcal{V}) = \left\{ \boldsymbol{\lambda} : \quad \boldsymbol{\lambda}^{\mathcal{M}_{\mathcal{V}}} \in S^{\mathcal{M}_{\mathcal{V}},\mathcal{V}} \quad , \quad \lambda_j > \frac{\widehat{Q}_j}{ET^{\mathcal{M}_{\mathcal{V}}}} \quad \text{for all} \quad j \in \mathcal{V} \right\} \tag{12}$$

all queues $j \in \mathcal{M}_{\mathcal{V}}$ are substable while all queues $i \in \mathcal{V}$ tend to infinity with positive probability.

Proof. Consider the dominant system $\overline{\Theta}^{(\mathcal{M}_{\mathcal{V}},\mathcal{V})}$ and let $\boldsymbol{\lambda} \in \widehat{S}_c^{\mathcal{M}}(\mathcal{V})$. Since $\boldsymbol{\lambda}^{\mathcal{M}_{\mathcal{V}}} \in S^{\mathcal{M}_{\mathcal{V}},\mathcal{V}}$, the queue lengths $\widetilde{\mathbf{N}}^{\mathcal{M}_{\mathcal{V}}}(k)$ constitute an ergodic Markov chain and starting from any state we have

$$\lim_{k\to\infty} \frac{\sum_{m=1}^{k} T_m^{\mathcal{M}_{\mathcal{V}}}}{k} = ET^{\mathcal{M}_{\mathcal{V}}}.$$

Since, in addition $\lambda_j > \widehat{Q}_j/ET^{\mathcal{M}_\mathcal{V}}$, an application of Loynes method for instability shows that starting from any state, $\lim_{k\to\infty} \widetilde{N}_j(k) = \infty$ for all $j \in \mathcal{V}$. Setting $A = \left[0, \max_{j\in\mathcal{V}} \widehat{Q}_j\right]$ in Proposition 4, we conclude that there is a state $\mathbf{c} \in \Re^M$ such that if the process $\widetilde{\mathbf{N}}^{\mathcal{M}_\mathcal{V}}(k)$ starts from state $\mathbf{c}$, then there is a set of sample paths, Ω_c, of positive probability such that $\widetilde{N}_j(k) \geq \widehat{Q}_j,\ j \in \mathcal{V}$ for all $k = 1, 2, \ldots$. Observe now that by definition, on the set Ω_c the queues in the original system Θ and in the dominant system $\overline{\Theta}^{(\mathcal{M}_\mathcal{V},\mathcal{V})}$ are identical and therefore $\lim_{k\to\infty} N_j(k) = \infty$ for all $j \in \mathcal{V}$. This implies that the Markov chain $\mathbf{N}^{\mathcal{M}_\mathcal{V}}(k)$ is transient. The fact that all queues in $\mathcal{M}_\mathcal{V}$ are substable follows directly from Proposition 2 and the ergodicity of $\widetilde{\mathbf{N}}^{\mathcal{M}_\mathcal{V}}(k)$. ■

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