Multicast Tree Structure and the Power Law

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Abstract

In this paper we investigate structural properties of multicast trees that give rise to the so called multicast power law. The law asserts that the ratio $R(n)$ of the average number of links in a multicast tree connecting the source to $n$ destinations to the average number of links in a unicast path, satisfies asymptotically $R(n) \approx cn^\phi$, $0 < \phi < 1$. In order to obtain a better insight we first analyze some simple multicast tree topologies, which under appropriately chosen parameters give rise to the multicast power law. The asymptotic analysis of $R(n)$ in this case, indicates that it is very difficult to infer the validity of power law by observing experimental graphs of $R(n)$ alone. Next we introduce a new easy to measure metric of multicast tree topology which is applicable to general networks where the multicast trees are constructed as subtrees of a given spanning tree (global multicast tree). We show that this metric provides a more reliable means for inferring the validity of the power law. Finally, we perform experiments on real and simulated networks to demonstrate the use of the new metric.

1 Introduction

Multicast communication in the Internet was proposed more than a decade ago in [2, 8] (cf. also [7]), and the experimental MBone network has been operational since 1992. In multicast communication senders transmit to a logical address and receivers join a logical group. Multicast routing ensures that only a single copy of a packet destined to multiple destinations traverses each link, so that the overall traffic load is reduced. Also, multicast alleviates the overhead on senders who can reach an entire group by the transmission of a single packet. The trade-off is that multicast requires extra control and routing overhead at the routing nodes.

In this paper we concentrate on the quantification of the main advantage of multicast routing. That is, we address the question of what is the expected traffic load reduction due to multicasting, when compared to unicast communication. Recently, efforts were undertaken to address this question. Motivated by the problem of pricing multicast communications, Chuang and Sirbu [6] performed experiments on a number of real and generated network

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topologies. They measured the average number of links, $L(n)$, in the multicast tree needed to reach $n$ randomly selected destination hosts from a given source; $L(n)$ represents the average cost of multicast tree per unit of bandwidth (it is assumed that all destinations require the same bandwidth). If unicast communication is employed, then the communication cost per unit of bandwidth is $Un$ where $U$ is the average number of links in a unicast path. The efficiency gain of multicast versus unicast is reflected in how far $L(n)$ deviates from the (unicast) linear growth. Chuang and Sirbu \cite{6}, after extensive simulations, concluded experimentally that for small $n$, $L(n)/U = \Theta(n^{0.8})$, i.e., the ratio increases as a power of $n$. This naturally raises the following questions. Is this behavior to be expected, or is it just an experimental approximation? Is the behavior specific to the chosen topologies, or should it be expected of other topologies as well? Can one identify conditions under which the power law relation holds? This paper attempts to provide theoretical answers to some of these questions.

The basic experimental result in the Chuang and Sirbu paper is reproduced here in Figure 1. The authors considered a multicast tree with $N$ routing nodes. Multiple destination hosts may be connected to the network through a routing node (e.g., each routing node may have a number of dial-in ports, or may have a LAN connected to one of its ports). A source and a number $n$ of multicast destination hosts, is picked randomly. A multicast tree consisting of shortest paths from the source to each of the destinations is constructed. We denote by $L(n)$ the average number of links of the multicast trees created by the above procedure. Note that the number of destination hosts, $n$, can be either smaller or larger that the number of routing nodes $N$. Therefore, the ratio $a = n/N$ can vary from zero to infinity. Figure 1 shows the Chuang and Sirbu findings concerning the ratio $R(n) = L(n)/U$. For $n$ small relative to $N$, i.e., when $a \ll 1$, the power law $R(n) = L(n)/U \approx n^{0.8}$ seems to be exhibited. Deviation from the power law and a phase transition appears around $a = 1$, and saturation occurs for $a \gg 1$.

The first attempt to obtain an analytical insight to the empirical multicast power law of Chuang and Sirbu was undertaken by Phillips, Shenker and Tangmunarunkit in \cite{20}. The authors provided an approximate analysis of $R(n)$ for a regular $V$-ary tree topology. Their analysis indicated that $R(n)$ grows according to $n (A - B \ln n)$ where $A$ and $B$ may depend on $N$ but are independent of $n$ (cf. (17) and (18) of \cite{20}). While this growth does not obey the power law, it turns out that on a log-log scale it looks very much like it for the network sizes considered and for $n$ small relative to $N$. Another work that explores analytically the possibility of the appearance of the power law related to multicast communication is the work of Mieghem, Hooghiemstra and Hofstad \cite{15}. Their conclusion (Corollary 6 in \cite{15}) is that for networks for which it holds $U/N^{0.2} \to 0$, the multicast power law cannot hold if $n = aN$ with $0 < a \leq 1$ and large $N$. However, this conclusion does not exclude the possibility that the power law still holds for realistic Internet topologies when $a = n/N \to 0$ and $n$ and $N$ are large. This limiting regime is of significant interest since the Internet is growing fast and numerous multicast groups that are small relative to $N$, are likely to exist. In this paper we explore the possibility of appearance of the power law under these latter conditions.

As in \cite{6}, \cite{20}, \cite{15}, we assume that multicast is performed on a spanning tree joining a given source to all the network nodes, which we call global multicast tree. Nodes of a multicast group are connected to the source using the appropriate subtree of the global multicast tree. Multicast trees of this type usually are shortest-path trees and are, or can be
used by several Internet multicast protocols such as DVMPRP [19], MOSPF [16], PIM-DM [9].

We address the following question. Can one determine general conditions on global multicast tree structures, based on which it can be inferred that the multicast power law holds or not? In order to get a better insight, we first concentrate on generalizations of the simple tree structures considered in [20]. For these structures, detailed asymptotic results can be obtained. While we do not claim that these structures provide close approximate models of multicast trees encountered in practice, as will be seen (see Section 3) they are very useful since they provide a means for understanding more clearly the problem and the difficulties involved.

First, to make sure the approximations of [20] did not tilt final results, we provide precise analysis of the full \( V \)-ary global multicast tree topology considered in [20] (i.e., a tree for which all the nodes except the leaf nodes have outdegree \( V \), and all leaf nodes are at depth \( D \)), which confirms that their result (cf. Theorem 1) indeed gives a good approximation for the leading term for \( R(n) \) when \( a \) is small. Interestingly enough, for small \( a \) we discover some small oscillations of the coefficient in front of \( n \). Next, we consider a more general global multicast tree topology. Specifically, we start with a full \( V \)-ary multicast tree; between two successive branchings of the multicast tree we add several concatenated relay (otherwise called unary) nodes, i.e., nodes at which no branching occurs. The average number of these concatenated nodes decreases exponentially as the distance (in number of branchings) from the source increases. More precisely, a node in such a tree at level \( k \), \( k \geq 1 \), is replicated \( V^{(D-k)\theta} \) times, where \( D \) is the depth of the tree and \( 0 \leq \theta \leq 1 \). A tree with such a property is shown in Figure 2. Note that for such a tree, each of the subtrees that have as root one of the nodes at level 1 has \( 1/V \) number of nodes as the original tree and is of the same form at the original tree. For this reason, we appropriately call such a tree self-similar. Notice that this definition of self-similar tree differs from other ones in the literature [17], [19]. When \( \theta = 0 \), we have the full \( V \)-ary tree. When \( 1 > \theta > 0 \), we show that the ratio \( R(n) \) exhibits the power law. More precisely, we show that for \( a = n/N \to 0 \), it holds \( R(n) \sim (c + \psi(n))n^{1-\theta} \) where \( c \) is an explicitly computable constant and \( \psi(n) \) is an oscillating function of rather small amplitude for small values of \( V \). (cf. Corollary 2(i)).

While self-similar trees provide concrete examples where the power law appears, they also show that it is rather difficult to confirm experimentally, by simply measuring and plotting multicast tree costs, whether the power law holds. This is so, since as will be seen in Section 3, even with the exact asymptotics we obtain it is very easy to confuse the multicast tree cost of a regular \( V \)-ary tree with that of a “corresponding” self-similar tree. Since this confusion may lead to erroneous inferences with significant errors for large \( N \), we look next for general structural properties of the global multicast tree that may provide us with a more reliable means of deciding whether the multicast power law holds or not. It turns out (see Section 4) that such structural properties do exist and are easy to obtain for general networks. These properties refer to the number of routing nodes on the global multicast tree that can be reached by a node on that tree. We define a metric \( F(k) \) (see Section 3 for the exact definition) that incorporates these structural properties of the global multicast tree and show that its behavior determines the existence or non-existence of the power law. Finally, we perform experiments on random and real network topologies that demonstrate the use of the metric we propose.

The paper is organized as follows. In the next section we consider a full \( V \)-ary tree
Figure 1: Figure 7 from Chunag and Sirbu [6] showing the phase transition of the ratio of
the number of links traverse in multicast and the average path length in unicast versus the
number of destinations $n$.

Figure 2: A Self-Similar Tree with $D = 3$ and $\theta = 1$. 
network topology and provide a precise analysis of $L(n)$ for regular and self-similar trees (cf. Section 2). In Section 4 we provide general conditions under which the existence or absence of the power law can be inferred. We provide experimental results in Section 5. Finally, the Appendix contains derivations of our theoretical results. In passing, we should mention that our findings have been established by analytical techniques of the precise analysis of algorithms such as Mellin transform and complex asymptotics (cf. [14, 21]). We give a brief survey of these methods since they may be useful for the study of other problems of similar nature.

2 Tree Topologies

In this section we present results concerning regular and self-similar tree topologies.

2.1 Regular Trees

As in [20], we consider a $V$-ary global multicast tree where the source is located at the root of the tree and all the potential destination hosts of the multicast tree are connected to the network through the leaf nodes of the tree. In Section 2.3 we consider the possibility that destination hosts may be connected to the network through other tree nodes, not just through the leaf nodes. We assume that behind each leaf node there may be multiple destination hosts. This is the case when, for example, each of the leaf nodes connects a LAN to the network.

Let $D$ be the depth of the tree, i.e., its longest (in terms of hops) path. We assume that the $V$-ary tree is full (all nodes but the leaves have outdegree $V$ and all the leaves are at depth $D$). If $N$ is the number of leaf nodes then clearly,

$$N = V^D.$$  \hfill (1)

Let the multicast group consist of $n$ hosts and

$$a = \frac{n}{N} > 0.$$  \hfill (2)

Note that since more than one destination host may be behind each node, it is possible that $a > 1$. However, the possibility for the power law to appear arrises only when $a \ll 1$, which is the most interesting case from an analytical point of view, and very likely to occur in practice. We assume that the probability of a destination host being connected to the network through a given leaf node is uniform and independent of the way the rest of the hosts are connected.

Following Chuang and Sirbu [6], to quantify the reduction of traffic load in multicast over unicast, we shall analyze the average number of links $L(n)$ in the multicast tree that connects $n$ randomly selected hosts. Then $U = L(1)$ denotes the average number of links in the path between the source and a host in unicast transmission. The reduction ratio $R(n)$ is defined as

$$R(n) = \frac{L(n)}{U}.$$  \hfill (3)

Observe that for the full $V$-ary tree we have $U = D$. 

To estimate the average number of links in the multicast tree connecting \( n \) nodes, we observe that at level \( k \) of the tree, \( 1 \leq k \leq D \), there are \( V^k \) links; the probability that a particular link is in the multicast tree when \( n \) destination hosts have been selected is

\[
1 - (1 - 1/V^k)^n.
\]

Thus the average number of links in the multicast tree is

\[
L(n) = \sum_{k=1}^{D=\log_V N} V^k \left( 1 - (1 - V^{-k})^n \right). \tag{4}
\]

Our goal is to estimate \( L(n) \) asymptotically as \( n \to \infty \) for \( a \to 0 \), \( a \to 1 \) and \( a \to \infty \). In the Appendix we prove the following result.

**Theorem 1** (i) For \( a \to 0 \) the quantity \( L(n) \) attains the following asymptotics

\[
L(n) = n \left( D + \frac{1}{\ln V} - \frac{\ln n}{\ln V} + \left( \frac{1}{2} - \frac{\gamma}{\ln V} \right) + \psi_1(\ln a) \right)
- \frac{V}{V-1} - \frac{1}{2 \ln V} + \frac{1}{2} \psi_2(\ln a) + O \left( \frac{1}{\ln n} \right), \tag{5}
\]

where \( \gamma \approx 0.57721 \) is the Euler constant, and \( \psi_1(x), \psi_2(x) \), are oscillating periodic functions of small amplitude for small \( V \) that can be expressed as

\[
\psi_1(x) = \sum_{k=-\infty}^{\infty} \frac{\Gamma(-1-2\pi ik/\ln V)}{\ln V} \exp \left( 2\pi ik \frac{x}{\ln V} \right), \tag{6}
\]

\[
\psi_2(x) = 2\pi i \sum_{k=-\infty}^{\infty} \frac{k \Gamma(-2\pi ik/\ln V)}{\ln^2 V} \exp \left( 2\pi ik \frac{x}{\ln V} \right). \tag{7}
\]

In fact, \( |\psi_1(x)| < 0.0000001725, 0.00041227, 0.0085, 0.068, 0.153 \) for \( V = 2, 3, 5, 100, 1000 \), respectively.

(ii) For \( a \to 1 \),

\[
L(n) = N \left( \frac{V}{V-1} - C_1 - C_2 (a-1) + C_3 (a-1)^2 \right) - \frac{V}{V-1} - C_3 a + O \left( \frac{1}{\log n} \right), \tag{8}
\]

where

\[
C_1 = \sum_{l=0}^{\infty} V^{-l} e^{-V}, \quad C_2 = \sum_{l=0}^{\infty} e^{-V^l}, \quad C_3 = \sum_{l=0}^{\infty} V^l e^{-V^l}.
\]

(iii) For \( a \to \infty \),

\[
L(n) = N \left( \frac{V}{V-1} - e^{-a} \right) - \frac{V}{V-1} - \frac{1}{2} (ae^{-a} + ae^{-V}) + O \left( \frac{1}{\ln n} \right). \tag{9}
\]
We can now compare the theoretical results of Theorem 1 to the experimental results of Chuang and Sirbu presented in Figure 1. In particular, taking into account that $\ln N = D \ln V$ we see from (5) that the ratio $R(n)$ for $a \to 0$ can be approximated by

$$R_N(n) \approx n \left( 1 + \frac{1 - \gamma}{\ln N} + \frac{\ln V}{2 \ln N} - \frac{\ln n}{\ln N} + \frac{\ln V}{\ln N} \psi_1(\ln a) \right),$$

which is not of the power law form. However, one can argue that for small $a$ and fixed $N$ formulas (8) and (9) explain respectively the transition and saturation region of Figure 1.

We note that the approximate analysis in [20] lead to the approximation

$$L(n) \approx n \left( D + \frac{1}{\ln V} - \frac{\ln n}{\ln V} \right),$$

or

$$R_N(n) \approx n \left( 1 + \frac{1}{\ln N} - \frac{\ln n}{\ln N} \right).$$

(11)

Comparing (10) and (11) we see that the main term that the approximate analysis missed, is $d = 0.5/D - 0.5772/(D \ln V)$ plus the oscillating function $\psi_1(\ln a)$ which for small to medium $V$ is of small amplitude. The term $d$ is small, i.e., $-0.33272 \leq d < 0.5$. Therefore the approximate analysis provided in [20] is fairly accurate. However, as seen from the analysis above, the approximation is valid only for $a \ll 1$.

### 2.2 Self-Similar Trees

According to the results of the previous section, the multicast power law does not hold for $V$-ary global multicast trees. In this section we show that if the global multicast tree has a “self-similar” structure in the sense to be discussed below, then we indeed have the power law behavior for $a \to 0$.

As in the previous subsection, consider a $V$-ary tree where all possible hosts are located at the leaves of the tree. However, we assume now that the link connecting a node at level $k$ and a node at level $k-1$ consists of a concatenation of a random number of links. Let $\ell_k$ be the average number of these links. We postulate that $\ell_k$ is a fraction of $\ell_{k-1}$, that is, for some $A$ we have $\ell_1 = A$ and

$$\ell_k = \phi \ell_{k-1}, \quad 0 \leq \phi \leq 1, \quad k \geq 2.$$ 

Therefore, $\ell_k = \phi^{k-1} A$. Setting $\phi = V^{-\theta}$ we find,

$$\ell_k = V^{-\theta(k-1)} A = \ell_D V^{(D-k)\theta}, \quad \theta > 0.$$

In the rest of the paper, we assume for simplicity and without loss of generality that $\ell_D = 1$. The last equality suggests another interpretation of $\theta$: observe that there are $K = V^{D-k}$ leaves hanging from a node at level $k$; thus we reproduce such a node $K^\theta$ times.

We call a tree with the above structure, a self-similar $V$-ary tree with similarity factor $\theta$. Note that when $\theta = 0$, we have the regular $V$-ary tree. In the following we assume that $0 \leq \theta < 1$. 

7
We analyze now $L_\theta(n)$ and $R_\theta(n)$ for self-similar trees. In particular, as before, we derive

$$L_\theta(n) = \sum_{k=1}^{D} V^{(D-k)\theta} V^{-k} \left(1 - \left(1 - V^{-k}\right)^n\right),$$

and for the average path length in a unicast connection we find

$$U_\theta = \sum_{k=1}^{D} V^{(D-k)\theta} = \frac{N^\theta - 1}{V^\theta - 1}. \quad (12)$$

In the Appendix we prove the following asymptotic expansions for $L_\theta(n)$.

**Theorem 2**

(i) For $a \to 0$ the quantity $L(n)$ attains the following asymptotics

$$L_\theta(n) = N^\theta \left(\frac{\theta}{\theta \ln V} - \psi_3(\ln a)\right) - \frac{V^\theta}{V^\theta - 1} - \frac{1}{2} \frac{\theta \Gamma(\theta)}{n^\theta \ln V} - \frac{1}{\pi^\theta \psi_4(\ln a)} + O\left(\frac{1}{\ln n}\right), \quad (13)$$

where $\bar{\theta} = 1 - \theta$, $\gamma = 0.5772\ldots$ is the Euler constant, $\Gamma(\theta)$ is the Gamma function, $\psi_3(a)$ and $\psi_4(a)$ are oscillating periodic functions of small amplitude for small $V$ that can be expressed as

$$\psi_3(x) = \sum_{k=-\infty}^{\infty} \frac{\Gamma(-1 + \theta - 2\pi i k / \ln V)}{\ln V} \exp\left(2\pi i k \frac{x}{\ln V}\right), \quad (14)$$

$$\psi_4(x) = \sum_{k=-\infty}^{\infty} \frac{\Gamma(\theta - 2\pi i k / \ln V)}{\ln V} \exp\left(2\pi i k \frac{x}{\ln V}\right). \quad (15)$$

(ii) For $a \to 1$,

$$L(n) = N\left(\frac{V^\theta}{V^\theta - 1} - C_1(\theta) - C_2(\theta) (a - 1) + C_3(\theta) (a - 1)^2\right) - \frac{N^\theta V^\theta}{V^\theta - 1} - \frac{C_3(\theta)}{2} a + O\left(\frac{1}{\log n}\right), \quad (16)$$

where

$$C_1(\theta) = \sum_{l=0}^{\infty} V^{-\theta l} e^{-V^l}, \quad C_2(\theta) = \sum_{l=0}^{\infty} V^{\theta l} e^{-V^l}, \quad C_3(\theta) = \sum_{l=0}^{\infty} V^{(1+\theta) l} e^{-V^l}. \quad (16)$$

(iii) For $a \to \infty$,

$$L_\theta(n) = N\left(\frac{V^\theta}{V^\theta - 1} - e^{-a}\right) - \frac{N^\theta V^\theta}{V^\theta - 1} - \frac{1}{2} \left(a e^{-a} + a V^{\theta + 1} e^{-a V}\right) + O\left(\frac{1}{\ln n}\right). \quad (17)$$
Observe that for $a \to 0$, Theorem 2(i) and (12) suggest the following approximation.

$$R(n, \theta) \approx \frac{L(n, \theta)}{U \theta} \approx n^{1-\theta} \left( V^\theta - 1 \right) \left( \frac{\Gamma(\theta)}{(1-\theta) \ln V - \psi_3(\ln a)} \right) - \frac{V - V^{1-\theta}}{V^{1-\theta} - 1}. \quad (18)$$

Thus, we obtain the power law with exponent of $n$ equal to $1-\theta$. We see from Corollary 2(iii) that for $a \to \infty$ with $N$ fixed the ratio $L(n)$ tends to $(NV^{1-\theta} - N^\theta V^{1-\theta})/(V^{1-\theta} - 1)$. Of course this is to be expected, since in this case all nodes belong to the multicast tree and hence $L(n)$ is equal to the number of links in the global multicast tree. Finally, around $a = 1$ we have a transitive behavior depicted in Theorem 2(ii). Therefore, the dependence on $n$ of the multicast tree cost for self-similar multicast trees, is similar to the one shown in Figure 1.

### 2.3 Destination Hosts Located at Non-relay Nodes

In the previous sections we assumed that destination hosts are located at the leaves of the global multicast tree. If destination hosts can also be located at any of the non-relay nodes of the global multicast tree, then in order to find the average cost of multicast we argue as in Section 2 and find

$$L_\theta(n) = \sum_{k=1}^{D} \left( V^{(D-k)\theta} V^k \left( 1 - \left( 1 - \frac{V^{D-k+1} - 1}{V^{D+1} - V} \right)^n \right) \right) = V^D \frac{V^{(D+1)} - V^\theta}{V^\theta - 1} - V^D \sum_{k=0}^{D-1} V^{-\theta k} \left( 1 - \frac{a V^{k+1} - 1}{n} \right)^n.$$

The term of $L_\theta(n)$ that needs to be analyzed asymptotically is

$$\hat{L}_\theta(n) = \sum_{k=0}^{D-1} V^{-\theta k} \left( 1 - \frac{a V^{k+1} - 1}{n} \right)^n.$$

This term has the same form as the one analyzed before and hence its asymptotic expansion has the same form. Therefore, the results are qualitatively the same as in the case where destination hosts are located at nodes at the leaves of the tree. We do not dwell into more details since, as explained in the introduction, our main motivation for studying regular and self-similar trees is to get a concrete example where the multicast power law appears and complete analysis is provided, in order to get a better insight.

### 3 Comparison Between Regular and Self-Similar Trees.

From the discussion in the previous sections we have the following two asymptotic formulas for $N$ and $n$ large and as $n/N \to 0$. For self similar trees ($\theta > 0$),

$$R(n, \theta) \approx n^{1-\theta} \left( V^\theta - 1 \right) \frac{\Gamma(\theta)}{(1-\theta) \ln V},$$

while for regular $V$-ary trees,

$$R_N(n) = R(n, 0) \approx n \left( 1 + \frac{1 - \gamma}{\ln N} + \frac{\ln V}{2 \ln N} - \frac{\ln n}{\ln N} \right).$$
In the graph in Figure 3 we plot the asymptotic form of $R_N(n) = R(n, 0)$ in a log-log scale for $V = 2$, $N = 4096 = 2^{12}$, 40960, 409600 and for $n$ in the range 10 to 50. As observed in [20] and [15], in this scale, the graphs look almost linear. The slopes, $s_N$, for $N = 4096, 40960, 409600$ are, 0.832, 0.879, .905, respectively. Let us turn our attention now to $R(n, \theta)$. Clearly, plotting $R(n, \theta)$ for small $n$ in a log-log scale provides a linear curve with slope $1 - \theta$. Therefore, one is tempted from the plots in Figure 3 to infer that the power law holds with $\theta = 1 - s_N$, i.e., respectively, .168, .121, 0.095. Plotting $R(n, \theta)$ in Figure 3 for these values of $\theta$, provides curves that are indistinguishable from those already present. In fact, even if one plots the curves in a linear scale as proposed in [5], the difference between the two curves is small. To emphasize this point, in Figure 4 we plot the ratio $r_N(n) = R_N(n) / R(n, 1 - s_N)$ for $V = 2$ and $N = 4096, 40960, 409600$. We see that the relative error is below 2.3%, well within the limits that can be achieved by experimental verification.

From the discussion above we see that it is very difficult to decide experimentally by observing the slopes of multicast gain alone whether for the graphs under consideration the multicast power law holds. The question therefore arises whether there are metrics, applicable to general networks, based on which the inference can be made easier and more robust. Besides its theoretical interest, this issue is important in practice. To make this point clear, assume that we infer by experimenting with networks of size 4096 that the power law holds with $\theta = .168$ and that we intend to make inferences on multicast resource consumption for networks of the same type, but of size 409600, that are not available for experimentation. Assume also for the sake of argument that the global multicast tree topologies are either regular $V$-ary trees or self-similar trees. If instead of the assumed self-similar tree with $\theta = .168$ the tree is actually a regular $V$-ary tree, then the values we are seeking are $R_{409600}(n)$, while we are estimating $R(n, .168)$. In Figure 5 we plot the ratio $R_{409600}(n) / R(n, .168)$. We see that with this procedure the estimate, i.e., $R(n, .163)$, underestimates the actual values, $R_{409600}(n)$, by as much as 23%.

We now examine the possibility of introducing a general and robust metric for global
Figure 4: Ratio $r_N(n) = R_N(n)/R(n, 1 - s_N)$.

Figure 5: Plot of Actual Versus Estimated Multicast Tree Performance.
multicast trees, based on which it can be inferred easier whether the multicast power law holds or not for small ratio $n/N$. For a node $t$ on the global multicast tree, define by $r(t)$ the number of destination host nodes that can be reached by $t$, using the global multicast tree links (the tree links are considered unidirectional). For example, in the self-similar tree in Figure 2, if destination hosts are located at leaf nodes, then we have $r(t) = 1$ if $t$ is a leaf node or if $t$ is a relay node between levels 2 and 3. If $t$ is located at level 1 or if $t$ is a relay node between levels 0 and 1 we have $r(t) = 4 = 2^{3-1}$. We call $r(t)$ the “reachability degree of $t$”. Intuitively the reachability degree of a node should play an important role in determining this node’s role in multicast tree resource consumption. If $r(t)$ is large, there is a high likelihood that node $t$ will participate in the multicast tree to be formed. As we will see in the next section, $r(t)$ plays also an important role in determining analytically the multicast tree performance in general networks.

Next, we need some definitions that are based on node reachability degree, $r(t)$. Let $Q(k)$ be the number of nodes not including the root node, with reachability degree $k$. The reason for the exclusion of the root node in the definition of $Q(k)$ is technical and will become clear from the analysis in Section 4. If destination nodes are located at the leaves of the global multicast tree, it is easy to see that for the regular $V$-ary tree, we have $Q(k) = V^D k^{-1}$ if $k = V^l$, $1 \leq l \leq D - 1$, and $Q(k) = 0$ for all other values of $k$. In general, taking into account that there are $V^\theta(D-1) - 1$ relay nodes between levels $l-1$ and $l$, where $0 \leq \theta \leq 1$, we find that for the self-similar $V$-ary tree (including the regular $V$-ary tree) we have $Q(k) = k^{-1+\theta} V^D$, if $k = V^l$, $1 \leq l \leq D - 1$, and $Q(k) = 0$ for all other values of $k$.

Now, let us define by $F(k)$ the number of tree nodes other than the root node, with reachability degree at least $k$, $1 \leq k \leq N$. That is,

$$F(k) = \sum_{l=k}^{N} Q(k).$$

From the previous discussion about self-similar trees, we have that $F(k) = 0$ for $k > V^{D-1}$ and for $1 \leq k \leq V^{D-1}$,

$$F(k) = \sum_{l=[\log_{V} k]}^{D-1} Q(V^l) = N \sum_{l=[\log_{V} k]}^{D-1} V^{l(1+\theta)} = NV^{[\log_{V} k](1+\theta)} \frac{1 - V^{(D-[\log_{V} k])(1+\theta)}}{1 - V^{1+\theta}},$$

where $[x]$ is defined as the smallest integer larger than or equal to $x$. In Figure 6 we plot in log-log scale $F(k)$ for a regular tree with $N = 2^{12} = 4096$, $V = 2$, as well as its “corresponding” self-similar tree, i.e. the self-similar tree with $\theta = .168$. We observe that the two curves are now clearly distinguishable even for small $k$: not only their values are different, but the slopes we get by linear interpolation are different as well. The linear interpolation slope for the regular tree is about -1, while that of the corresponding self-similar tree is about -.835. In fact, decaying (with respect to $k$) bounds for $F(k)$, for a wide range of $k$, can be easily developed as follows.
Using the fact that for $\theta \leq 1$ it holds,

$$( -1 + \theta) \log_V k + (-1 + \theta) \leq \lceil \log_V k \rceil (-1 + \theta) \leq (-1 + \theta) \log_V k,$$

we have,

$$NV(-1+\theta)k^{-1+\theta} \frac{1 - V^{(D-\lceil \log_V k \rceil)(-1+\theta)}}{1 - V^{-1+\theta}} \leq F(k) \leq \frac{N}{1 - V^{-1+\theta}} k^{-1+\theta}$$

and since for $k \leq V^{D-1} = N/V$ it holds,

$$\frac{1 - V^{(D-\lceil \log_V k \rceil)(-1+\theta)}}{1 - V^{-1+\theta}} \geq 1,$$

we finally arrive at the inequalities

$$NV^{-1+\theta}k^{-1+\theta} \leq F(k) \leq \frac{N}{1 - V^{-1+\theta}} k^{-1+\theta}, \ 1 \leq k \leq N/V. \quad (19)$$

Note that given a general global multicast tree, $F(k)$ is very easy to measure, and therefore it is easy to obtain experimental results based on $F(k)$. It is not needed to experiment with, or simulate further, the multicast groups themselves. In addition, it turns out that under the assumption that destination host nodes are selected with uniform probability, any global multicast tree that has polynomially decaying bounds of $F(k)$ as in (19) gives rise to the multicast power law. This issue is undertaken in the next section.

## 4 Generalization

Let us assume that $n$ destination hosts are forming a multicast group. As in Section 2 we assume that the probability that a destination host is located behind a given node is $1/N$,
independent of where the rest of the destination hosts are located. Under these assumptions, given $F(k)$, $L(n)$ can be derived as follows. Consider a node $t$ on the global multicast tree $T$, other than the source node $s$, with reachability degree $r(t)$. Let $I_t$ be 1 if the link entering this node belongs to the created multicast tree and 0 otherwise. If $L$ is the random variable representing the number of links in the multicast tree $T$, then it holds,

$$L = \sum_{t \in T - \{s\}} I_t,$$
and hence,

$$L(n) = \sum_{t \in T - \{s\}} E[I_t].$$

Now, observe that $I_t = 1$ if and only if at least one of the $r(t)$ nodes that can be reached by $t$ using the global multicast tree links, is selected by the $n$ destination hosts. The probability that one of the destination hosts picks some of the $r(t)$ nodes is $r(t)/N$ and the probability than none of the $n$ destination hosts picks some of the $r(t)$ nodes is $(1 - r(t)/N)^n$. Hence,

$$E[I_t] = 1 - \left(1 - \frac{r(t)}{N}\right)^n,$$
and, recalling that $Q(k)$ is the number of nodes (not including the source node) for which $r(t) = k$,

$$L(n) = \sum_{t \in T - \{s\}} \left(1 - \left(1 - \frac{r(t)}{N}\right)^n\right) = \sum_{k=1}^{N} Q(k) \left(1 - \left(1 - \frac{k}{N}\right)^n\right).$$

Note that this formula holds for any global multicast tree. We next transform this formula in order to make depend on $F(k)$.

Using Abel’s partial summation formula (cf. [21]) we observe that for two real-valued sequences $u_k$ and $v_k$ it holds,

$$\sum_{k=1}^{N} (u_k - u_{k+1})v_k = u_1v_1 - u_{N+1}v_N + \sum_{k=2}^{N} u_k(v_k - v_{k-1}).$$

Using this and taking into account that $F(N + 1) = 0$, we proceed as follows:

$$L(n) = \sum_{k=1}^{N} Q(k) \left(1 - \left(1 - \frac{k}{N}\right)^n\right) = \sum_{k=1}^{N} (F(k) - F(k + 1)) \left(1 - \left(1 - \frac{k}{N}\right)^n\right)$$

$$= \sum_{k=2}^{N} F(k) \left(\left(1 - \frac{k-1}{N}\right)^n - \left(1 - \frac{k}{N}\right)^n\right) + F(1) \left(1 - \left(1 - \frac{1}{N}\right)^n\right)$$

$$= \sum_{k=1}^{N} F(k) \left(\left(1 - \frac{k-1}{N}\right)^n - \left(1 - \frac{k}{N}\right)^n\right).$$

(20)
Notice that if the global multicast tree is randomly selected, then (20) still holds with \( F(k) \) replaced by \( F'(k) = E[F(k)] \). Based on (20), we can now show that in general, a decay of \( F(k) \) (or \( F'(k) \)) according to (19) with \( \theta > 0 \), gives rise to the multicast power law, while a decay according to (19) with \( \theta = 0 \) does not give rise to such law.

**Theorem 3** Assume that for the global multicast tree and \( 0 \leq \theta < 1 \) it holds for large \( N \) and for \( k \leq \beta N \), \( 0 < \beta \leq 1 \),

\[
a_Nk^{-1+\theta} \leq F(k) \leq A_Nk^{-1+\theta}. \tag{21}
\]

a) If \( \theta = 0 \), it holds,

\[
\frac{a_N}{A_N}n \left( 1 - \frac{\ln n}{\ln N} + \Theta \left( \frac{1}{\ln N} \right) + \Theta \left( \frac{n}{N} \right) \right) \leq R(n),
\]

\[
R(n) \leq \frac{A_N}{a_N}n \left( 1 - \frac{\ln n}{\ln N} + \Theta \left( \frac{1}{\ln N} \right) + \Theta \left( \frac{n}{N} \right) \right).
\]

b) If \( \theta > 0 \), it holds,

\[
\frac{a_N}{A_N}n^{\beta^{-\theta}} \Gamma(\theta) \theta n^{1-\theta} \left( 1 + \Theta \left( \frac{1}{n} \right) + \Theta \left( \frac{n}{N} \right) \right) \leq R(n),
\]

\[
R(n) \leq \frac{A_N}{a_N}n^{\beta^{-\theta}} \Gamma(\theta) \theta n^{1-\theta} \left( 1 + \Theta \left( \frac{1}{n^{1-\theta}} \right) + \Theta \left( \frac{n}{N} \right) \right).
\]

The proof of Theorem 3 is given in the appendix. We see that under the generalized assumptions of Theorem 3 the power law may appear again, albeit in a weaker form than in self-similar trees. If \( \theta = 0 \), i.e. \( F(k) \) decays as \( 1/k \), then for large \( N \) and small ratios \( n/N \), \( R(n) \) behaves as \( B_Nn(1 - \ln n/N) \) and the power law does not hold. If \( \theta > 0 \), then the power law for \( R(n) \) appears. Hence measuring \( F(k) \) can provide a means to infer about the validity of power law for multicast trees.

## 5 Experiments

In this section we examine the behavior of \( R(n) \) and \( F(k) \) on real and simulated networks and compare the obtained graphs with the theoretical results obtained in Section 4 and the corresponding behavior of self-similar trees in Section 3.

We performed two sets of experiment. The first set was based on data from real internet topologies, obtained by the Scan project of ISI, with their software "mercator" implementing the ideas described [13]. We used the "Scan+Lucent" map which also merged data from the Internet Mapping project [4]. The graph is an approximation of part of the Internet topology circa October/November 1999, and includes 284805, of which about 283685 can reach each other.

The second set was based on standard random graph topologies: A number of nodes \( N \) is picked and \( M \) of the \( N(N-1)/2 \) edges are picked randomly to form an undirected graph. This graph is transformed to a directed one by considering each link as bidirectional.

For each of the topologies tested, we picked randomly nodes as sources and created the global multicast tree as the tree of shortest (in number of hops) paths from the source to
all the network nodes. Once the global multicast tree is formed, the calculation of $F(k)$ is straightforward. However, the calculation of $R(n)$ requires further to simulate multicast tree formations for each $n$. Specifically, having fixed the source node and the corresponding global multicast tree, to compute $R(n)$ we perform the following simulations.

1. A number $n$ of destination hosts is picked randomly. The probability that each of the destination hosts is located at a given routing node is $1/N$ and independent of the location of the rest of the hosts.

2. The multicast tree is formed as the appropriate subtree of the global multicast tree.

3. The number of links, $l(n)$, of the multicast tree is computed.

4. The mean of number of links, $u$, in a unicast communication is computed.

5. The experiment in lines 1 to 4 is repeated 1000 times.

6. $L(n)$ is computed as the average of $l(n)$, and $U$ as the average of $u$, over the experiments performed.

7. We compute $R(n) = L(n)/U$.

We present below the results obtained for a real network with 284805 nodes and 860682 edges, as well as a random network with exactly the same number of nodes and edges. The random network contains one large connected component consisting of 268164 nodes. The graphs obtained for various randomly chosen sources belonging to the large connected components in the two networks, do not differ significantly. Also, the results remain qualitatively the same for other network sizes.

Figure 7 shows in a log-log scale the obtained $F(k)$, assuming that destination hosts can be located at any routing node. We observe that for the random topology, the slope is about $-1$. For the real topology, for values of $k$ between 5 and 100 the slope is about $-0.93$.

Figure 8 shows in log-log scale the obtained $R(n)$. The slopes of $R(n)$ for small $n$ (between 5 and 100) are now about $0.94$ for the real topology and $0.89$ for the random one.

Notice that as with the tree structures described in Section 3, from the graphs of Figure 8 one is tempted to say that the power law holds for both topologies, with $\theta = 1 - 0.94 = 0.06$ for the real topology and $\theta = 1 - 0.89 = 0.11$ for the random one. However, since in Figure 7 the slope of $F(k)$ for the random topology is $-1$ for small $k$, we must conclude that Theorem 3 a), rather than the power law holds in this case. In fact this behavior has been shown theoretically for another type of random graphs (complete graphs with exponentially distributed links weights) in [1]. Consider now the corresponding curves for the real network topology. Apart from a relative steep decline for $k \leq 5$, the slope of $F(k)$ in the log-log scale is about $-0.93$, while the slope of $R(n)$ is $0.94$. This is in agreement with the conditions of Theorem 3 b). Even though our confidence in the validity of the power law increases in this case, since the conditions are essentially asymptotic experimentation alone does not suffice to infer with absolute certainty the validity of the power law. A subject of further research is to identify classes of networks for which the conditions on $F(k)$ can be inferred based on even simpler network properties (e.g., distribution of node degree), or on the manner the networks are created.
Figure 7: Plots of $F(k)$ for a Real Internet Topology and a Randomly Generated Topology.

Figure 8: $R(n)$ for the Real and Random Topology
6 Conclusions

The question whether multicast communication follows the power law arose from the attempt to price multicast communication in [6]. Power laws related to Internet topology parameters were studied in [10]. As indicated in the latter reference, power laws of this nature may be valuable for affirming how realistic simulated topologies are. This is an important issue that received a lot of attention [3], [22].

In this paper we examined structural conditions on global multicast trees that give rise to the multicast power law. Regular $V$-ary trees do not exhibit the power law, while self-similar trees do. In fact, the power law rises under conditions weaker than tree self-similarity.

We provided a metric, $F(k)$, encompassing the structural properties of the global multicast tree, which is easy to measure and can be used to infer the validity of the power law. Simulation work demonstrated, based on observations of the graphs of $F(k)$, that randomly generated networks do not obey the power law, even though graphs of $R(n)$ may lead one to believe so. For the real topologies, similar measurements indicated that the power law is likely to hold. However, since the conditions on $F(k)$ are essentially asymptotic (for large $N$), further work is needed in order to ascertain the validity in the latter case. For example, can the behavior of $F(k)$ be deduced from even simpler network parameters? More generally, given the fact that several power laws related to various network parameters have been observed experimentally, see e.g. [10], the question arises as to why these laws appear and when one law implies the other.

A APPENDIX

In this appendix we establish our main theoretical results from Sections 2 and 4. As observed before, our results for self-similar trees cover as the special case ($\theta = 0$) the finding for regular $V$-ary trees. Thus in the next subsection we concentrate only on proving Theorem 2.

A.1 Proof of Theorem 2

We first need the following lemma.

Lemma 4 For fixed $a$, $0 < a = n/N < \infty$, and large $n$ the average number of links $L_\theta(n)$ in the self-similar tree attains the following asymptotic expansion.

$$L_\theta(n) = N \left( \frac{V^{\theta}}{V^{\theta} - 1} - c_1(a, \theta) \right) - \frac{N^{\theta} V^{\theta}}{V^{\theta} - 1} - \frac{1}{2} c_2(a, \theta) + O \left( \frac{1}{\ln n} \right), \quad (22)$$

where $\theta = 1 - \theta$ and

$$c_1(a, \theta) = \sum_{l=0}^{\infty} V^{-\theta l} \exp \left( -a V^{l} \right), \quad (23)$$

$$c_2(a, \theta) = \sum_{l=0}^{\infty} a V^{(1+\theta)l} \exp \left( -a V^{l} \right). \quad (24)$$
Proof. We saw that the average cost of the multicast self-similar tree is
\[ L_\theta(n) = \sum_{k=1}^{D} V^{(D-k)\theta} V^{-k} \left( 1 - \left( 1 - V^{-k} \right)^n \right). \]

Observe that since \( D = \log_V N \) and \( n = aN \), for \( k = D \) the last term of the above sum is approximately equal to \( V^D(1 - e^{-a}) \). This term is not small in general and thus we cannot extend the limit of the summation to infinity without introducing significant error. In order to provide an asymptotic analysis of \( L_\theta(n) \), we define \( \bar{\theta} = 1 - \theta \) and proceed as follows:
\[
L_\theta(n) = \sum_{k=1}^{D} V^{(D-k)\theta} V^{-k} \left( 1 - \left( 1 - V^{-k} \right)^n \right) = V^D \sum_{l=0}^{D-1} V^{-\bar{\theta}l} \left( 1 - \left( 1 - \frac{V^l}{V^D} \right)^n \right)
\]
\[
= V^D \left( \frac{1 - V^{-D\bar{\theta}}}{1 - V^{-\bar{\theta}}} \right) - V^D \sum_{l=0}^{D-1} V^{-\bar{\theta}l} \left( 1 - \frac{V^l}{V^D} \right)^n = N \left( \frac{1 - N^{-\theta}}{1 - V^{-\theta}} \right) - \bar{L}_\theta(n),
\]
where
\[
\bar{L}_\theta(n) = N \sum_{l=0}^{D-1} V^{-\bar{\theta}l} \left( 1 - \frac{aV^l}{n} \right)^n.
\]
We shall use the following expansion for \( x \leq \ln n \)
\[
\left( 1 - \frac{x}{n} \right)^n = \exp \left( -x + \frac{x^2}{2n} + \frac{x}{n} \left( \frac{\ln n}{3n} \right) \right).
\]
(25)

After setting
\[
A_n = \left\lfloor \frac{\ln \left( \frac{\ln n}{a} \right)}{\ln V} \right\rfloor,
\]
(hence \( \ln n < eaV^{A_n} \leq e \ln n \)) we obtain
\[
\bar{L}_\theta(n) = N \sum_{l=0}^{A_n} V^{-\bar{\theta}l} \left( 1 - \frac{aV^l}{n} \right)^n + N \sum_{l=A_n+1}^{D-1} V^{-\bar{\theta}l} \left( 1 - \frac{aV^l}{n} \right)^n.
\]
(26)

We analyze separately each of the terms on the right hand side of the previous equation. We first look at the term
\[
\sum_{l=0}^{A_n} V^{-\bar{\theta}l} \left( 1 - \frac{aV^l}{n} \right)^n.
\]
For \( l \leq A_n \) we have \( aV^l \leq \ln n \) and hence from (25),

\[
\sum_{l=0}^{A_n} V^{-\delta l} \left( 1 - \frac{aV^l}{n} \right)^n = \sum_{l=0}^{A_n} V^{-\delta l} \exp \left( -aV^l \right) \left( 1 - \frac{a^2V^{2l}}{2n} + \frac{aV^l}{n} O \left( \frac{(\ln n)^3}{n} \right) \right)
\]

\[
= \sum_{l=0}^{A_n} V^{-\delta l} \exp \left( -aV^l \right) - \frac{a}{2N} \sum_{l=0}^{A_n} V^{(1+\theta)l} \exp \left( -aV^l \right)
\]

\[+ \frac{1}{N^\theta} O \left( \frac{(\ln n)^{3+\theta}}{n^{1+\theta}} \right) + \left[ \theta = 0 \right] \ln N \frac{1}{N} O \left( \frac{(\ln n)^3}{n} \right),
\]

where \( \left[ \theta = 0 \right] \) is equal to 1 when \( \theta = 0 \) and zero otherwise. We now look separately at each of the above terms. We find

\[
\sum_{l=0}^{A_n} V^{-\delta l} \exp \left( -aV^l \right) = \sum_{l=0}^{\infty} V^{-\delta l} \exp \left( -aV^l \right) - \sum_{l=A_n+1}^{\infty} V^{-\delta l} \exp \left( -aV^l \right).
\]

But

\[
\sum_{l=A_n+1}^{\infty} V^{-\delta l} \exp \left( -aV^l \right) = \frac{1}{N^\theta} O \left( \frac{1}{n^\theta (\ln n)^\theta} \right),
\]

which finally yields

\[
\sum_{l=0}^{A_n} V^{-\delta l} \left( 1 - \frac{aV^l}{n} \right)^n = c_1(a, \theta) - \frac{1}{2N} c_2(a, \theta) + \frac{1}{N^\theta} O \left( \frac{1}{n^\theta (\ln n)^\theta} \right),
\]

where

\[
c_1(a, \theta) = \sum_{l=0}^{\infty} V^{-\delta l} \exp \left( -aV^l \right)
\]

\[
c_2(a, \theta) = \sum_{l=0}^{\infty} aV^{(1+\theta)l} \exp \left( -aV^l \right).
\]

To complete the proof of the lemma we must estimate the term

\[
\sum_{l=A_n+1}^{D-1} V^{-\delta l} \left( 1 - \frac{aV^l}{n} \right)^n.
\]

But for \( l > A_n \) we find

\[
\left( 1 - \frac{aV^l}{n} \right)^n < \left( 1 - \frac{\ln n}{n} \right)^n = O(1/n),
\]

and hence

\[
\sum_{l=A_n+1}^{D-1} V^{-\delta l} \left( 1 - \frac{aV^l}{n} \right)^n = V^{-\delta A_n} O \left( \frac{1}{n} \right) = \left( \frac{a}{\ln n} \right)^\theta O \left( \frac{1}{n} \right) = \frac{1}{N^\theta} O \left( \frac{1}{n^\theta (\ln n)^\theta} \right).
\]

Combining our previous estimates we finally prove lemma.
A.1.1 Derivation of Theorem 2

We now prove Theorem 2, that is, we find asymptotic expansions of $L_\theta(n)$ for $n$. Observe that we only need to analyze the quantities $c_1(a, \theta)$ and $c_2(a, \theta)$ defined in (23) and (24), respectively. The regimes: (i) $a \to 1$ and (ii) $a \to \infty$ are easy and are omitted due to lack of space.

Next we look at the regime $a \to 0$ which is the most interesting case, and the hardest. It turns out that this case can be handled by a special analytic tool, namely the Mellin transform. The Mellin transform found myriad of applications in the analysis of algorithms. The reader is referred to an excellent survey by Flajolet, Gourdon and Dumas [11] (cf. [14, 21]). For reader convenience, we collected the most important properties of the Mellin transform in Section A.3. In particular, the definition of Mellin transform is given in (52). Property (M2) defines the so called fundamental strip of the complex plane where the Mellin transform exists. The harmonic sum property (M3) and the mapping properties (M4) are crucial. We shall use them to derive asymptotics of $c_1(a, \theta)$ and $c_2(a, \theta)$ as $a \to 0$.

Let us first consider $c_1(a, \theta) = \sum_{i=0}^{\infty} V^{-\theta i} \exp(-a V^i)$. Observe that by (M3) the sum in $c_1(a, \theta) := c_1(a)$ is a harmonic sum with $\lambda_k = V^{-k}$ and $g(x) = e^{-x}$ with $\mu_k = V^k$. Thus the Mellin transform $c_1^*(s)$ with respect to $a$ of $c_1(a)$ is by (M3) (and the well known fact that the Mellin of $e^{-x}$ is the Euler gamma function $\Gamma(s)$ for $\Re(s) > 0$):

$$c_1^*(s) = \frac{\Gamma(s)}{1 - V^{-(s+1-\theta)}}.$$

We now use (M4) to find $c_1(a)$ as $a \to 0$, that is, we shall find the inverse to the Mellin transform which according to (M1)

$$c_1(a) = \frac{1}{2\pi i} \int_{\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} \frac{\Gamma(s)}{1 - V^{-(s+1-\theta)}} a^{-s} ds. \tag{33}$$

The goal is to apply the Cauchy residue theorem. But first we must consider a large rectangle left the the line, say from the line $(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$ to $(-M - i\infty, -M + i\infty)$ for some large $M > 0$. Due to the factor $a^{-s}$ the left line contributes $O(a^{-M})$ for any $M > 0$, which is negligible. The top and bottom lines of the big rectangle cancel out, thus the integral in (33) is equal to the residues inside the rectangle.

We now evaluate the residues. We note that the function $c_1^*(s)$ has poles at $s_k = -1 + \theta - (2\pi i k)/\ln V$, $k = \pm 1, \pm 2, \ldots$. All these poles are single. The pole at 0 has residue

$$c_{0,0} = \frac{V^\theta}{V^\theta - 1}.$$

The poles at $s_k$, $k \neq 1$ have residues

$$c_{0,s_k} = \frac{\Gamma(-1 + \theta - 2\pi i k/\ln V)}{\ln V}.$$  

Using now the Reverse Mapping Theorem (M4) and the property $\Gamma(-1+\theta) = \Gamma(\theta)/(-1+\theta)$, we see that for $a \to 0$ we have the expansion

$$c_1(a, \theta) = \frac{v^\theta}{v^\theta - 1} - \frac{\Gamma(\theta)}{\theta \ln v} a^\theta + a^\theta \psi_3(\ln a) + O(a^2), \tag{34}$$

21
where
\[
\psi_3(x) = \sum_{k=-\infty \atop k\neq 0}^{\infty} \frac{\Gamma(-1 + \theta - 2\pi ik/\ln V)}{\ln V} \exp\left(2\pi ik \frac{x}{\ln V}\right).
\]

Now we consider \(c_2(a,\theta) = \sum_{\ell=0}^{\infty} aV^{(1+\theta)\ell} \exp\left(-aV^\ell\right)\). It is again a harmonic sum, hence by (M2) we find its Mellin transform to be
\[
c_2^*(a) = \frac{s \Gamma(s)}{1 - (-s-\theta)}.
\]
But \(c_2^*(s)\) has a single poles at \(\theta\) with residue
\[
c_{0,0} = \frac{\theta \Gamma(\theta)}{\ln v},
\]
and single poles at \(s_k = \theta - (2\pi ik)/\ln V, k \neq 0\) with residues
\[
\frac{(\theta - i2\pi k)\Gamma(\theta - 2\pi ik/\ln V)}{\ln V}.
\]
Hence using again the Reverse Mapping Theorem (M4) we obtain
\[
c_2(a,\theta) = \frac{\theta \Gamma(\theta)}{\ln V}a^{-\theta} - a^{-\theta}\psi_4(\ln a) + O(a^2)
\]
where
\[
\psi_4(x) = \sum_{k=-\infty \atop k\neq 0}^{\infty} \frac{(\theta - 2\pi ik/\ln V)\Gamma(\theta - 2\pi ik/\ln V)}{\ln V} \exp\left(2\pi ik \frac{x}{\ln V}\right).
\]
Combining everything we finally prove Corollary 2.

A.2 Proof of Theorem 3

Let \(n \geq 2\). We will develop bounds on \(L(n)\), based on (20).

Using Taylor’s expansion of \((1-x)^n\) around \(x = k/N, 1 \leq k \leq N\), we have,
\[
\frac{n}{N} \left(1 - \frac{k}{N}\right)^{n-1} + \frac{n(n-1)}{2N^2} \left(1 - \frac{k}{N}\right)^{n-2} \leq \left(1 - \frac{k-1}{N}\right)^n - \left(1 - \frac{k}{N}\right)^n \leq \frac{n}{N} \left(1 - \frac{k}{N}\right)^{n-1} + \frac{n(n-1)}{2N^2} \left(1 - \frac{k-1}{N}\right)^{n-2}.
\]
Assume for simplicity in the notation that \(\beta N\) is integer. Using the last expansion, the fact that \(F(k)\) is decreasing, (20) and (21) we conclude that
\[
L(n) \geq A_N N^{\theta-1} n \sum_{k=1}^{\beta N} \left(\frac{k}{N}\right)^{\theta-1} \left(1 - \frac{k}{N}\right)^{n-1} \frac{1}{N} + A_N N^{\theta-1} n \sum_{k=1}^{\beta N} \left(\frac{k}{N}\right)^{\theta-1} \left(1 - \frac{k}{N}\right)^{n-2} \frac{1}{2N} \frac{n-1}{2N},
\]
\[(35)\]
and

\[ L(n) \leq AN^{\theta-1}n \sum_{k=1}^{\beta N} \left( \frac{k}{N} \right)^{\theta-1} \left( 1 - \frac{k}{N} \right)^{n-1} \frac{1}{N} \]

\[ + AN(\beta N)^{-1+\theta}n \sum_{k=\beta N+1}^{N} \left( 1 - \frac{k}{N} \right)^{n-1} \frac{1}{N} \]

\[ + AN^{\theta-1}n \left( \sum_{k=1}^{\beta N} \left( \frac{k}{N} \right)^{\theta-1} \left( 1 - \frac{k-1}{N} \right)^{n-2} \frac{1}{N} \right) \frac{n-1}{N} \]

\[ + AN(\beta N)^{-1+\theta}n \left( \sum_{k=\beta N+1}^{N} \left( 1 - \frac{k-1}{N} \right)^{n-2} \frac{1}{N} \right) \frac{n-1}{N} \] (36)

The second and fourth term of the last inequality are small. This can be seen as follows.

For the second term, taking into account that the function 
\( (1 - x)^{n-1}, n \geq 1, \) is decreasing for \( 0 \leq x \leq 1, \) we find,

\[ \sum_{k=\beta N+1}^{N} \left( 1 - \frac{k}{N} \right)^{n-1} \frac{1}{N} \leq \int_{1}^{1/\beta} (1 - x)^{n-1} \, dx \]

\[ = \frac{(1 - \beta)^n}{n} \] (37)

For the fourth term we have for \( N > 1/\beta \) (in a similar fashion as with the second term),

\[ \sum_{k=\beta N+1}^{N} \left( 1 - \frac{k-1}{N} \right)^{n-2} \frac{1}{N} \leq \sum_{k=\beta N}^{N} \left( 1 - \frac{k}{N} \right)^{n-2} \frac{1}{N} \]

\[ \leq \int_{\beta - 1/N}^{1} (1 - x)^{n-2} \, dx \]

\[ = \frac{(1 - \beta + 1/N)^{n-1}}{n-1}. \] (38)

We next treat the first and second term of (35), which are the same as the first and third term of (36) respectively.

Let us look at the first term of (35). Since \( \theta < 1, \) the function \( x^{\theta-1} (1 - x)^{n-1} \) is decreasing for \( 0 < x \leq 1. \) Hence we have

\[ \sum_{k=1}^{\beta N} \left( \frac{k}{N} \right)^{\theta-1} \left( 1 - \frac{k}{N} \right)^{n-1} \frac{1}{N} \geq \int_{1/N}^{1} x^{\theta-1} (1 - x)^{n-1} \, dx \] (39)

and

\[ \sum_{k=1}^{\beta N} \left( \frac{k}{N} \right)^{\theta-1} \left( 1 - \frac{k}{N} \right)^{n-1} \frac{1}{N} \leq \frac{1}{N^\theta} \left( 1 - \frac{1}{N} \right)^{n-1} + \int_{1/N}^{1} x^{\theta-1} (1 - x)^{n-1} \, dx. \]

\[ \leq \frac{1}{N^\theta} + \int_{1/N}^{1} x^{\theta-1} (1 - x)^{n-1} \, dx. \] (40)
Similarly, for the second term of (35) we have,

\[
\beta N \sum_{k=1}^{\beta N} \left( \frac{k}{N} \right)^{\theta-1} \left( 1 - \frac{k-1}{N} \right)^{n-2} \frac{1}{N} \geq \sum_{k=1}^{\beta N} \left( \frac{k}{N} \right)^{\theta-1} \left( 1 - \frac{k-1}{N} \right)^{n-2} \frac{1}{N}
\]

\[
\geq \int_{1/N}^{1} x^{\theta-1} (1-x)^{n-2} dx
\]

(41)

and

\[
\sum_{k=1}^{\beta N} \left( \frac{k}{N} \right)^{\theta-1} \left( 1 - \frac{k-1}{N} \right)^{n-2} \frac{1}{N} = \frac{1}{N^\theta} + \sum_{k=2}^{N} \left( \frac{k}{N} \right)^{\theta-1} \left( 1 - \frac{k-1}{N} \right)^{n-2} \frac{1}{N}
\]

\[
\leq \frac{1}{N^\theta} + \sum_{k=2}^{N} \left( \frac{k}{N} \right)^{\theta-1} \left( 1 - \frac{k-1}{N} \right)^{n-2} \frac{1}{N}
\]

\[
= \frac{1}{N^\theta} + \sum_{k=1}^{N-1} \left( \frac{k}{N} \right)^{\theta-1} \left( 1 - \frac{k}{N} \right)^{n-2} \frac{1}{N}
\]

\[
\leq \frac{1 + (1 - 1/N)^{n-2}}{N^\theta} + \int_{1/N}^{1} x^{\theta-1}(1-x)^{n-2} dx
\]

\[
\leq \frac{2}{N^\theta} + \int_{1/N}^{1} x^{\theta-1}(1-x)^{n-2} dx
\]

(42)

It remains to evaluate the asymptotics of the integral \( \int_{1/N}^{1} x^{\theta-1}(1-x)^{n-1} dx, n \geq 1 \). The form of these asymptotics depends on whether \( \theta = 0 \) or \( \theta > 0 \). We treat each case separately. 

**Case 1, \( \theta = 0 \):** Observe that,

\[
\int_{1/N}^{1} x^{-1} (1-x)^{n-1} dx = \int_{1/N}^{1} x^{-1} \left( \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^l x^l \right) dx
\]

\[
= \ln N + \sum_{l=1}^{n-1} \binom{n-1}{l} (-1)^l \frac{1 - N^{-l}}{l}
\]

\[
= \ln N + \sum_{l=1}^{n-1} \binom{n-1}{l} \frac{(-1)^l}{l}
\]

\[
+ N^{-1} \sum_{l=1}^{n-1} \binom{n-1}{l} (-1)^{l+1} \frac{N^{-l+1}}{l}
\]

(43)

Now, it is well known [12] that,

\[
\sum_{l=1}^{n} \binom{n}{l} \frac{(-1)^l}{l} = -\sum_{l=1}^{n} \frac{1}{l} = -\ln n - \gamma + \Theta(1/n)
\]

(44)

On the other hand it can be shown using standard calculus techniques that the function

\[
f(x) = \sum_{l=1}^{n-1} \binom{n-1}{l} (-1)^{l+1} \frac{x^{l+1}}{l},
\]
is increasing in \([1, \infty)\), that is

\[
f(1) = + \ln(n - 1) + \gamma + \Theta\left(\frac{1}{n}\right) \leq f(x) \leq \lim_{x \to \infty} f(x) = n - 1
\]

Hence,

\[
\frac{\ln(n - 1)}{N} + \Theta\left(\frac{1}{n}\right) \leq N^{-1} \sum_{l=1}^{n-1} \frac{(n - 1)(-1)^{l+1} N^{-l+1}}{l} \leq \frac{n}{N}
\]

(45)

From (43), (44), (45), and taking into account that \(\ln(n - 1) = \ln n + \Theta(1/n)\), we conclude that

\[
\ln N - \ln n - \gamma + \Theta\left(\frac{1}{n}\right) + \frac{N}{n} \leq f(x) \leq \lim_{x \to \infty} f(x) = n - 1
\]

(46)

Case 2: \(\theta > 0\). We have

\[
\int_{1/N}^{1} x^{\theta-1} (1-x)^{n-1} dx = \int_{0}^{1} x^{\theta-1} (1-x)^{n-1} dx - \int_{0}^{1/N} x^{\theta-1} (1-x)^{n-1} dx.
\]

Now,

\[
\frac{1}{N^{\theta}} \left(1 - \frac{1}{N}\right)^{n-1} \leq \int_{0}^{1/N} x^{\theta-1} (1-x)^{n-1} dx \leq \frac{1}{\theta N^{\theta}}
\]

and

\[
\int_{0}^{1} x^{\theta-1} (1-x)^{n-1} dx = \beta(\theta, n) = \frac{\Gamma(\theta)\Gamma(n)}{\Gamma(n+\theta)}
\]

where \(\beta(\theta, n)\) and \(\Gamma(x)\) are respectively the Beta and Gamma functions [21]. Using the approximation [21],

\[
\frac{\Gamma(n)}{\Gamma(n+\theta)} = n^{-\theta} (1 + \Theta(n^{-1})),
\]

we finally obtain,

\[
\frac{\Gamma(\theta)}{n^{\theta}} (1 + \Theta(n^{-1})) - \frac{1}{\theta N^{\theta}} \leq \int_{1/N}^{1} x^{\theta-1} (1-x)^{n-1} dx \leq \frac{\Gamma(\theta)}{n^{\theta}} (1 + \Theta(n^{-1})) - \frac{1}{N^{\theta}} \left(1 - \frac{1}{N}\right)^{n-1}
\]

(47)

Summarizing the development, from relations (35), (36), (37), (38), (39), (40), (41), (42), above we have the following bounds for \(L(n)\), \(n \geq 2\). When \(\theta = 0\), the previous relations and (46) imply that

\[
\ln N \frac{A_N}{N} n \left(1 - \frac{\ln n}{\ln N} - \frac{\gamma}{\ln N} + \Theta\left(\frac{1}{n \ln N}\right) + \Theta\left(\frac{n}{N}\right)\right) \leq L(n),
\]

(48a)

\[
L(n) \leq \ln N \frac{A_N}{N} n \left(1 - \frac{\ln n}{\ln N} - \frac{\gamma}{\ln N} + \Theta\left(\frac{1}{n \ln N}\right) + \Theta\left(\frac{n}{N}\right)\right).
\]

(48b)
When $\theta > 0$, the previous relations and (47) imply that

$$
\frac{a_N}{N^{1-\theta}} \Gamma(\theta)n^{1-\theta} \left( 1 + \Theta \left( \frac{1}{n^{1-\theta}} \right) + \Theta \left( \frac{n}{N} \right) \right) \leq L(n),
$$

(49a)

$$
L(n) \leq \frac{A_N}{N^{1-\theta}} \Gamma(\theta)n^{1-\theta} \left( 1 + \Theta \left( \frac{1}{n^{1-\theta}} \right) + \Theta \left( \frac{n}{N} \right) \right).
$$

(49b)

It remains to develop bounds for $U(n)$. From (20) we have $U(n) = \sum_{k=1}^{N} \frac{1}{N} F(k)$.

Following similar reasoning as before we have for $\theta = 0$,

$$
\frac{A_N}{N} \ln N \left( 1 + \Theta \left( \frac{1}{\ln N} \right) \right) \geq U(n) \geq \frac{a_N}{N} \ln N \left( 1 + \Theta \left( \frac{1}{\ln N} \right) \right),
$$

(50)

while for $\theta > 0$ we get,

$$
\frac{a_N}{N^{1-\theta}} \theta^{-1} \beta^\theta \geq U(n) \geq \frac{a_N}{N^{1-\theta}} \theta^{-1} \left( \beta^\theta - (1/N)^\theta \right)
$$

(51)

Part a) of the lemma follows from (48) and (50), and part b) follows from (49) and (51).

### A.3 Main Properties of Mellin Transform

For the reader convenience, we collected here the main properties of the Mellin transform. For details and proofs see [11, 21].

**M1 Direct and Inverse Mellin Transforms.** Let $c$ belong to the fundamental strip defined below. Then

$$
f^*(s) := \mathcal{M}(f(x); s) = \int_0^\infty f(x)x^{s-1}dx \iff f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f^*(s)x^{-s}ds.
$$

(52)

**M2 Fundamental Strip.** The Mellin transform of $f(x)$ exists in the fundamental strip $\Re(s) \in (-\alpha, -\beta)$, where

$$
f(x) = O(x^\alpha) \quad (x \to 0), \quad f(x) = O(x^\beta) \quad (x \to \infty)
$$

for $\beta < \alpha$.

**M3 Harmonic Sum Property.** By linearity and the scale rule $\mathcal{M}(f(ax); s) = a^{-s}\mathcal{M}(f(x); s)$,

$$
f(x) = \sum_{k \geq 0} \lambda_k g(\mu_k x) \iff f^*(s) = g^*(s) \sum_{k \geq 0} \lambda_k \mu_k^s s.
$$

(53)

**M4 Mapping Properties** (Asymptotic expansion of $f(x)$ and singularities of $f^*(s)$).

$$
f(x) = \sum_{(\xi, k) \in A} c_\xi, k x^\xi (\log x)^k + O(x^M) \iff f^*(s) \asymp \sum_{(\xi, k) \in A} c_\xi, k \frac{(-1)^k k!}{(s + \xi)^k + 1}.
$$

(54)

— (i) **Direct Mapping.** Assume that $f(x)$ admits as $x \to 0^+$ the asymptotic expansion (54) for some $-M < -\alpha$ and $k > 0$. Then for $\Re(s) \in (-M, -\beta)$, the transform $f^*(s)$ satisfies the singular expansion (54).

— (ii) **Reverse Mapping.** Assume that $f^*(s) = O(|s|^{-r})$ with $r > 1$, as $|s| \to \infty$ and that $f^*(s)$ admits the singular expansion (54) for $\Re(s) \in (-M, -\beta)$. Then $f(x)$ satisfies the asymptotic expansion (54) at $x = 0^+$. 

26
References


[13] Ramesh Govindan and Hongsuda Tangmunarunkit, "Heuristics for Internet Map Discovery", in Proc IEEE Infocom ’00, Tel Aviv, Israel, 2000.


27


