Multicast Tree Structure and the Power Law

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Abstract— In this paper we investigate structural properties of multicast trees that give rise to the so called multicast *power law*. The law asserts that the ratio $R(n)$ of the average number of links in a multicast tree connecting the source to n destinations to the average number of links in a unicast path, satisfies asymptotically $R(n) \approx cn^{\phi}$, $0 < \phi < 1$. In order to obtain a better insight we first analyze some simple multicast tree topologies, which under appropriately chosen parameters give rise to the multicast power law. The asymptotic analysis of $R(n)$ in this case, indicates that it is very difficult to infer the validity of power law by observing graphs of $R(n)$ alone. Next we introduce a new metric, "reachability degre", which is easy to measure and applicable to general networks where multicast trees are constructed as subtrees of a given spanning tree which we call Global Multicast Tree. The reachability degree is indicative of the structure of the Global Multicast Tree. We show that this metric provides a more reliable means for inferring the validity of the power law. Finally, we perform experiments on real and simulated networks to demonstrate the use of the new metric.

Index Terms— Multicasting, Power Law, Asymptotic Analysis, Analysis of Algorithms.

I. INTRODUCTION

Multicast communication in the Internet was proposed more than a decade ago in [1], [5] (cf. also [6]), and the experimental MBone network has been operational since 1992. In multicast communication senders transmit to a logical address and receivers join a logical group. Multicast routing ensures that only a single copy of a packet destined to multiple destinations traverses each link, so that the overall traffic load is reduced. Also, multicast alleviates the overhead on senders who can reach an entire group by the transmission of a single packet. The trade-off is that multicast requires extra control and routing overhead at the routing nodes.

In this paper we concentrate on the quantification of the main advantage of multicast routing. That is, we address the question of what is the expected traffic load reduction due to multicast, when compared to unicast communication. One of the first works to address this issue is [3]. Motivated by the problem of pricing multicast communications, Chuang and Sirbu [3] performed experiments on a number of real and generated network topologies. For a network of size N, they measured the average number of links, $L_N(n)$, in the multicast tree needed to reach n randomly selected destination hosts from a given source; $L_N(n)$ represents the average cost of

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multicast tree per unit of bandwidth (it is assumed that all destinations require the same bandwidth). If unicast communication is employed, then the communication cost per unit of bandwidth is U_N where U_N is the average number of links in a unicast path. The efficiency gain of multicast versus unicast is reflected in how far $L_N(n)$ deviates from the (unicast) linear growth. Chuang and Sirbu [3], after extensive simulations, concluded experimentally that for small n, $L_N(n)/U_N = \Theta(n^{0.8})$, i.e., the ratio increases as a fractional power of n. This naturally raises the following questions. Is this behavior to be expected, or is it just an experimental approximation? Is the behavior specific to the chosen topologies, or should it be expected of other topologies as well? Can one identify conditions under which the power law relation holds? This paper attempts to provide theoretical answers to some of these questions.

The basic experimental result in the Chuang and Sirbu paper is reproduced here in Figure 1. The authors considered a network with N routing nodes. Multiple destination hosts may be connected to the network through a routing node (e.g., each routing node may have a number of dial-in ports, or may have a LAN connected to one of its ports [3]). A source and a number n of multicast destination hosts, is picked randomly. A multicast tree consisting of shortest paths from the source to each of the destinations is constructed. Denote by $L_N(n)$ the average number of links of the multicast trees created by the above procedure. Note that the number of destination hosts, n, can be either smaller or larger than the number of routing nodes N. Therefore, the ratio $a = n/N$ can vary from zero to infinity. Figure 1 shows the Chuang and Sirbu findings concerning the ratio $R_N(n) = L_N(n)/U_N$. For n small relative to N, i.e., when $a \ll 1$, the power law $R_N(n) = L_N(n)/U_N \approx n^{0.8}$ seems to be exhibited. Deviation from the power law and a phase transition appears around $a = 1$, and saturation occurs for $a \gg 1$.

The first attempt to obtain an analytical insight to the empirical multicast power law of Chuang and Sirbu was undertaken by Phillips, Shenker and Tangmunarunkit in [17]. The authors provided an approximate analysis of $R_N(n)$ for a regular V-ary tree topology. Their analysis indicated that $R_N(n)$ grows according to $n(A - B\ln n)$ where A and B may depend on N but are independent of n (cf. (17) and (18) of [17]). While this growth does not obey the power law, it turns out that on a log-log scale it looks very much like it for the network sizes considered and for n small relative to N. Another work that explores analytically the possibility of the appearance of the power law related to multicast communication is the work of Mieghem, Hooghiemstra and Hofstad [14]. In the latter work, it is assumed that behind each routing node there can be a single destination host. When n is small compared to N, the

case of interest in this paper, it is unlikely that more than one of the n destination hosts will be located behind each routing node. Hence for $n/N \ll 1$, the asymptotic results remain the same whether it is assumed that multiple or single destinations hosts can be located behind each routing node - see also the discussion in [17, Section 3]. The conclusion in [14], Corollary 6, is that for networks for which it holds $U_N/N^{0.2} \to 0$, the multicast power law *cannot* hold if $n = aN$ with $0 < a \le 1$ and large N. However, this conclusion does not exclude the possibility that the power law still holds for realistic Internet topologies when $a = n/N \to 0$ and $N \to \infty$, $n \to \infty$. This limiting regime is of significant interest since the Internet is growing fast and numerous multicast groups that are small relative to N, are likely to exist. In this paper we explore the possibility of appearance of the power law under these latter conditions.

As in [3], [17], [14], we assume that multicast is performed on a spanning tree joining a given source to all the network nodes, which we call *Global Multicast Tree*. Nodes of a multicast group are connected to the source using the appropriate subtree of the Global Multicast Tree. Multicast trees of this type usually are shortest-path trees and are, or can be used by several Internet multicast protocols such as DVMPRP [18], MOSPF [15], PIM-DM [7].

We address the following question. Can one determine general conditions on Global Multicast Tree structures, based on which it can be inferred that the multicast power law holds or not? In order to get a better insight, we first concentrate on generalizations of the simple tree structures considered in [17]. For these structures, detailed asymptotic results can be obtained. While we do not claim that these structures provide good models of multicast trees encountered in practice, as will be seen (see Section III) they provide insight and a means for understanding more clearly the problem and the difficulties involved.

First, to make sure the approximations of [17] did not bias the final result, we provide (cf. Theorem 1) an asymptotic analysis of the full $V-$ary Global Multicast Tree topology considered in [17] (i.e., a tree for which all the nodes except the leaf nodes have outdegree V, and all leaf nodes are at depth D), which confirms that their result indeed gives a good approximation for the leading term for $R_N(n)$ when a is small. Interestingly enough, for small a we discover some small oscillations of the coefficient in front of n. Next, we consider a more general Global Multicast Tree topology. Specifically, we start with a full $V-$ary multicast tree; between two successive branchings of the multicast tree we add a (possibly random) number of concatenated relay - otherwise called unary - nodes, i.e., nodes at which no branching occurs. The average number of these concatenated nodes decreases exponentially as the distance (in number of branchings) from the source increases. More precisely, a node in such a tree at level k, $k \ge 1$, is replicated on average $V^{(D-k)\theta}$ times, where D is the depth of the tree and $0 \le \theta \le 1$. A tree with such a property is shown in Figure 2. Note that for such a tree, each of the subtrees that have as root one of the nodes at level 1 has $1/V$ number of nodes as the original tree and is of the same form at the original tree. For this reason, we appropriately
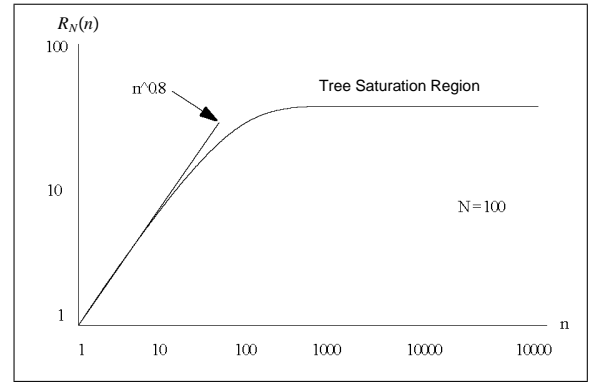

Fig. 1. Figure 7 from Chunag and Sirbu [3] showing the phase transition of the ratio of the number of links traverse in multicast and the average path length in unicast versus the number of destinations n.
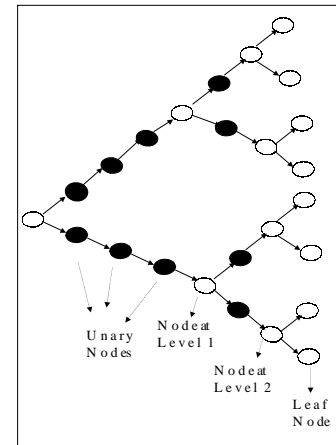

Fig. 2. A Self-Similar Tree with $D = 3$ and $\theta = 1$.

call such a tree *self-similar*. Notice that this definition of self-similar tree differs from other ones in the literature [16], [18]. When $\theta = 0$, we have the full $V-$ary tree. When $1 > \theta > 0$, we show that the ratio $R_N(n)$ exhibits the power law. More precisely, we show that for $a = n/N \to 0$, and $N \to \infty$, $n \to \infty$, it holds $R_N(n) \sim (c_N + \psi(n))n^{1-\theta}$ where c_N is an explicitly computable function of N and $\psi(n)$ is an oscillating function of n of rather small amplitude for small values of V. (cf. Theorem 2).

While self-similar trees provide concrete examples where the power law appears, they also show that it is rather difficult to confirm experimentally, by simply measuring and plotting multicast tree costs, whether the power law holds. This is so, since as will be seen in Section III, even with the asymptotic forms we obtain, it is very easy to confuse the multicast tree cost of a regular $V-$ary tree with that of a "corresponding" self-similar tree. Next we look for general structural properties of the Global Multicast Tree that may provide us with a more reliable means of deciding whether the multicast power law holds or not. It turns out (see Section IV) that such structural properties do exist and are easy to obtain for general networks. We define a metric $F(k)$ (see Section III for the exact definition) that incorporates these structural properties of the Global Multicast Tree and show that its behavior

determines the existence or non-existence of the power law. Finally, in Section V we perform experiments on random and real network topologies that demonstrate the use of the metric we propose.

II. SPECIFIC GLOBAL MULTICAST TREE TOPOLOGIES

In this section we present results concerning regular and self-similar Global Multicast Tree topologies.

A. Regular Trees

As in [17], we consider a $V-$ary Global Multicast Tree where the source is located at the root of the tree and all the potential destination hosts of the multicast tree are connected to the network through the leaf nodes of the tree. In Section II-C we consider the possibility that destination hosts may be connected to the network through other tree nodes, not just through the leaf nodes. We assume that behind each leaf node there may be multiple destination hosts.

Let D be the depth of the tree, i.e., its longest (in terms of hops) path. We assume that the $V-$ ary tree is full (all nodes but the leaves have outdegree V and all the leaves are at depth D). If N is the number of leaf nodes then clearly,

$$N = V^D. \tag{1}$$

Let the multicast group consist of n hosts and

$$a = \frac{n}{N} > 0. \tag{2}$$

Note that since more than one destination hosts may be behind each node, it is possible that $a > 1$. However, the possibility for the power law to appear arrises only when $a \ll 1$, which is the most interesting case from an analytical point of view, and very likely to occur in practice. We assume that the probability of a destination host being connected to the network through a given leaf node is uniform and independent of the way the rest of the hosts are connected.

Following Chuang and Sirbu [3], to quantify the reduction of traffic load in multicast over unicast, we shall analyze the average number of links $L_N(n)$ in the multicast tree that connects n randomly selected hosts. Then $U_N = L_N(1)$ denotes the average number of links in the path between the source and a host in unicast transmission. The reduction ratio $R_N(n)$ is defined as,

$$R_N(n) = \frac{L_N(n)}{U_N}. \tag{3}$$

Observe that for the full V-ary tree we have $U_N = D$.

To estimate the average number of links in the multicast tree connecting n nodes, we observe that there are V^l links connecting a node at level $l-1$ to a node at level l of the tree, $1 \leq l \leq D$; the probability that such a link is in the multicast tree when n destination hosts have been selected is,

$$1 - \left(1 - 1/V^l\right)^n.$$

Thus the average number of links in the multicast tree is

$$L_N(n) = \sum_{l=1}^{D=\log_V N} V^l \left(1 - \left(1 - V^{-l}\right)^n\right). \tag{4}$$

Our goal is to provide estimates of $L_N(n)$ that are valid for $a \to 0$, $N \to \infty$, $n \to \infty$. The analysis for other values of a, e.g. $a \to 1$ or $a \to \infty$ is in fact simpler but less interesting from our point of view. In the Appendix we prove the following result.

Theorem 1: The quantity $L_N(n)$ attains the following asymptotics,

$$\begin{aligned} L_N(n) = &n\left(D + \frac{1}{\ln V} - \frac{\ln n}{\ln V} + \left(\frac{1}{2} - \frac{\gamma}{\ln V}\right)\right.\\ &\left. - \psi_1(\ln a) + O\left(a^{M-1}\right)\right)\\ &- \frac{V}{V-1} + \frac{1}{2\ln V} - \frac{1}{2}\psi_2(\ln a)\\ &+ O\left(a^M\right) + O\left(\frac{1}{\ln n}\right) + D \times O\left(\frac{\ln n^3}{n}\right), \end{aligned} \tag{5}$$

where $M \geq 2$, $\gamma = 0.57721\ldots$ is the Euler constant and $\psi_1(x)$, $\psi_2(x)$, are oscillating periodic functions of small amplitude for small V that can be expressed as,

$$\psi_1(x) = \sum_{\substack{k=-\infty\\k\neq 0}}^{\infty} \frac{\Gamma(-1 - 2\pi ik/\ln V)}{\ln V} \exp\left(2\pi ik\frac{x}{\ln V}\right), \tag{6}$$

$$\psi_2(x) = \sum_{\substack{k=-\infty\\k\neq 0}}^{\infty} \frac{\Gamma(1 - 2\pi ik/\ln V)}{\ln V} \exp\left(2\pi ik\frac{x}{\ln V}\right). \tag{7}$$

For example, it can be computed, $|\psi_1(x)| < 0.0000001725$, 0.00041227, $0.0085, 0.068$, 0.153 for $V = 2, 3, 5, 100, 1000$, respectively.

The constants in the $O(x)$ notation above are independent of N, n, D, but may depend on the parameters V, M.

We can now compare the theoretical results of Theorem 1 to the experimental results of Chuang and Sirbu presented in Figure 1. In particular, taking into account that $\ln N = D \ln V$ we see from (5) that the ratio $R_N(n)$ for $a \to 0$ can be approximated by

$$R_N(n) \approx n\left(1 - \frac{\ln n}{\ln N} + \frac{1-\gamma}{\ln N} + \frac{\ln V}{2\ln N}\right), \tag{8}$$

which is not of the power law form.

B. Self-Similar Trees

According to the discussion in the previous section, the multicast power law does not hold for $V-$ary Global Multicast Trees. In this section we show that if the Global Multicast Tree has a "self-similar" structure in the sense to be discussed below, then we indeed have the power law behavior for $a \to 0$, $N \to \infty$, $n \to \infty$.

As in the previous section, consider a $V-$ary tree where all possible hosts are located at the leaves of the tree. However, we assume now that the link connecting a node at level $l-1$ and a node at level l consists of a concatenation of a random number of links. Let ℓ_l be the average number of these links. We postulate that ℓ_l is a fraction of ℓ_{l-1}, that is, for some A we have $\ell_1 = A$ and

$$\ell_l = \phi\ell_{l-1}, \ 0 \leq \phi \leq 1, \ l \geq 2.$$

Therefore, $\ell_l = \phi^{l-1}A$. Setting $\phi = V^{-\theta}$ we find,

$$\ell_l = V^{-\theta(l-1)}A = \ell_D V^{(D-l)\theta}, \qquad \theta > 0.$$

In the rest of the paper, we assume for simplicity and without loss of generality that $\ell_D = 1$.

We call a tree with the above structure, a self-similar $V-$ary tree with similarity factor θ. Note that when $\theta = 0$, we have the regular $V-$ ary tree. In the following we assume that $0 \le \theta < 1$.

We analyze now $L_{N,\theta}(n)$ for self-similar trees. In particular, as before, we derive

$$L_{N,\theta}(n) = \sum_{l=1}^{D} V^{(D-l)\theta} V^l \left(1 - \left(1 - V^{-l}\right)^n \right),$$

and for the average path length in a unicast connection we find

$$U_{N,\theta} = \sum_{l=1}^{D} V^{(D-l)\theta} = \frac{N^\theta - 1}{V^\theta - 1}. \tag{9}$$

In the Appendix we prove the following asymptotic expansions for $L_\theta(n)$.

Theorem 2: The quantity $L_N(n)$ attains the following asymptotics.

$$L_{N,\theta}(n) = N^\theta \left(n^{\bar\theta} \left(\frac{\Gamma(\theta)}{\theta \ln V} - \psi_3(\ln a) + O\left(a^{M-\bar\theta}\right) \right) \right.$$

$$\left. -\frac{V^{\bar\theta}}{V^{\bar\theta} - 1} + \frac{1}{2} \frac{\theta \Gamma(\theta)}{n^\theta \ln V} - \frac{1}{2n^\theta} \psi_4(\ln a) + \frac{1}{N^\theta} O\left(a^M\right) \right)$$

$$+ N^\theta O\left(\frac{1}{n^\theta (\ln n)^{\bar\theta}} \right) + \frac{N^\theta - 1}{V^\theta - 1} O\left(\frac{(\ln n)^3}{n} \right), \tag{10}$$

where $M \ge 2$, $\bar\theta = 1 - \theta$, $\Gamma(\theta)$ is the Gamma function and $\psi_3(a)$ and $\psi_4(a)$ are oscillating periodic functions of small amplitude for small V that can be expressed as

$$\psi_3(x) = \sum_{\substack{k=-\infty \\ k \ne 0}}^{\infty} \frac{\Gamma(-1 + \theta - 2\pi i k/\ln V)}{\ln V} \exp\left(2\pi i k \frac{x}{\ln V} \right),$$

$$\tag{11}$$

$$\psi_4(x) = \sum_{\substack{k=-\infty \\ k \ne 0}}^{\infty} \frac{\Gamma(1 + \theta - 2\pi i k/\ln V)}{\ln V} \exp\left(2\pi i k \frac{x}{\ln V} \right).$$

$$\tag{12}$$

Observe that for $a \to 0$, $N \to \infty$, $n \to \infty$, Theorem 2 and (9) suggest the following approximation.

$$R_{N,\theta}(n) = \frac{L_{N,\theta}(n)}{U_{N,\theta}} \approx n^{1-\theta} \left(V^\theta - 1 \right) \left(\frac{\Gamma(\theta)}{(1 - \theta) \ln V} \right). \tag{13}$$

Thus, we obtain the power law with exponent of n equal to $1 - \theta$.

C. Destination Hosts Located at Non-relay Nodes

In the previous sections we assumed that destination hosts are located at the leaves of the Global Multicast Tree. If destination hosts can also be located at any of the non-relay nodes of the Global Multicast Tree, then in order to find the average cost of multicast we argue as in Section II and find

$$L_{N,\theta}(n) = \sum_{l=1}^{D} \left(V^{(D-k)\theta} V^l \left(1 - \left(1 - \frac{V^{D-l+1} - 1}{V^{D+1} - V}\right)^n \right) \right)$$

$$= V^{D\bar\theta} \frac{V^{\bar\theta(D+1)} - V^{\bar\theta}}{V^{\bar\theta} - 1}$$

$$- V^D \sum_{l=0}^{D-1} V^{-\bar\theta l} \left(1 - \frac{a \frac{V^{l+1}-1}{V-1}}{n} \right)^n.$$

The term of $L_{N,\theta}(n)$ that needs to be analyzed asymptotically is

$$\widehat{L}_{N,\theta}(n) = \sum_{l=0}^{D-1} V^{-\bar\theta l} \left(1 - \frac{a \frac{V^{l+1}-1}{V-1}}{n} \right)^n.$$

This term has the same form as the one analyzed before and hence its asymptotic expansion has the same form. Therefore, the results are qualitatively the same as in the case where destination hosts are located at nodes at the leaves of the tree. We do not dwell into more details since, as explained in the introduction, our main motivation for studying regular and self-similar trees is to get a concrete example where the multicast power law appears and complete analysis is provided, in order to get a better insight.

III. COMPARISON BETWEEN REGULAR AND SELF-SIMILAR TREES.

From the discussion in the previous sections we have the following two asymptotic formulas for N and n large and as $n/N \to 0$. For self similar trees,

$$R_{N,\theta}(n) \approx n^{1-\theta} \left(V^\theta - 1 \right) \frac{\Gamma(\theta)}{(1 - \theta) \ln V},$$

while for regular $V-$ary trees,

$$R_N(n) \approx n \left(1 + \frac{1-\gamma}{\ln N} + \frac{\ln V}{2 \ln N} - \frac{\ln n}{\ln N} \right).$$

Clearly, plotting the above approximation for $R_{N,\theta}(n)$ in a log-log scale provides a linear curve with slope $1 - \theta$. Let us turn our attention now to $R_N(n)$. In the graph in Figure 3 we plot the asymptotic form of $R_N(n)$ in a log-log scale for $V = 2$, $N = 4096 (= 2^{12})$, 40960, 409600. As observed in [17] and [14], in this scale, the graphs look linear. The slopes, s_N, for $N = 4096$, 40960, 409600 are, 0.832, 0.879, .905, respectively. Therefore, one is tempted from the plots in Figure 3 to infer that the power law holds with $\theta = 1 - s_N$, i.e., respectively, 1-.832=.168, 1-.879=.121, 1-.905= 0.095.

From the discussion above we see that it is very difficult to decide experimentally by observing the slopes of multicast gain alone whether for the graphs under consideration the multicast power law holds. The question therefore arises whether there are metrics, applicable to general networks, based on which the inference can be made easier and more
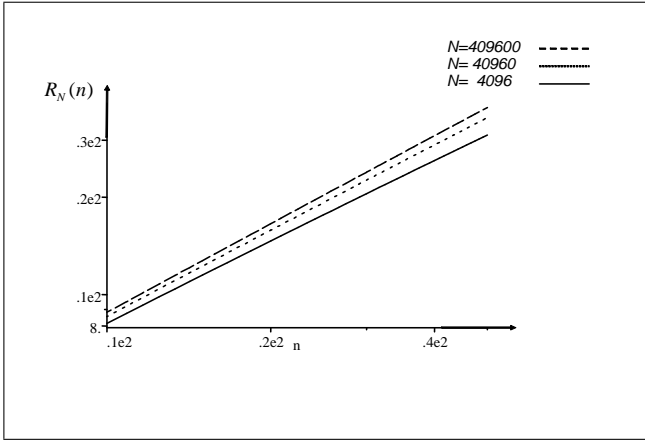
Fig. 3. Plots of $R_N(n) = R(n, 0)$ for $V = 2$ and $N = 4096, 40960, 409600$.

Fig. 4. Plots of $F(k)$ for Regular and Self-similar Trees.

robust. Next we examine the possibility of introducing such a general and robust metric for Global Multicast Trees. Let t be a node on the Global Multicast Tree and consider the subtree (of the Global Multicast Tree) $T(t)$ that has as root node t. Let $r(t)$ be the number of routing nodes (behind which destination hosts may exist) that are located on the subtree $T(t)$. Equivalently, $r(t)$ is the number of routing nodes that are reachable from the source through paths of the Global Multicast Tree traversing node t. For example, in the self-similar tree in Figure 2, if destination hosts are located only at leaf nodes, then we have $r(t) = 1$ if t is a leaf node. If t is located at level 1 or if t is a relay (unary) node between levels 0 and 1 we have $r(t) = 4 = 2^{3-1}$. We call $r(t)$ the "reachability degree of t". Intuitively the reachability degree of a node should play an important role in determining this node's participation in multicast tree construction. If $r(t)$ is large, it is likely that node t will participate in the multicast tree to be formed. As we will see in the next section, $r(t)$ also plays an important role in determining analytically the multicast tree performance in general networks.

Next, we need some definitions that are based on node reachability degree, $r(t)$. Let $Q(k)$ be the number of nodes, t, *not including the root node*, with reachability degree $r(t) = k$. The reason for the exclusion of the root node in the definition of $Q(k)$ is technical and will become clear from the analysis in Section IV. If destination nodes are located at the leaves of the Global Multicast Tree, it is easy to see that for the regular V-ary tree, we have $Q(k) = V^D k^{-1}$ if $k = V^l$, $0 \le l \le D - 1$, and $Q(k) = 0$ for all other values of k. In general, taking into account that there are $V^{\theta(D-l)} - 1$ relay nodes between levels $l - 1$ and l, where $0 \le \theta \le 1$, we find that for the self-similar V-ary tree (including the regular V-ary tree) we have $Q(k) = k^{-1+\theta} V^D$, if $k = V^l$, $0 \le l \le D - 1$, and $Q(k) = 0$ for all other values of k.

Now, let us define by $F(k)$ the number of tree nodes other than the root node, with reachability degree at least k, $1 \le k \le N$. That is,

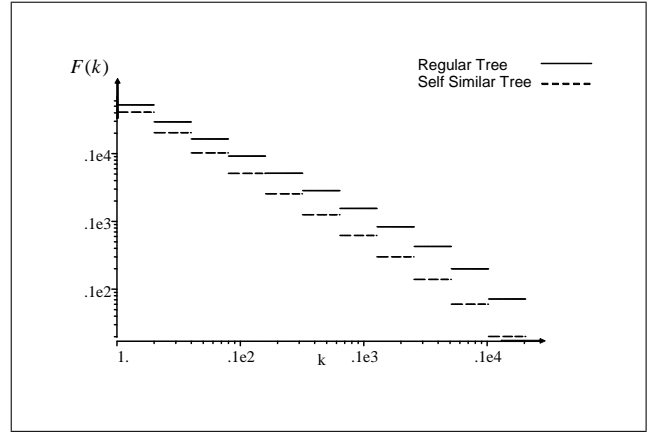$$F(k) = \sum_{l=k}^{N} Q(k).$$

From the previous discussion about self-similar trees, we have that $F(k) = 0$ for $k > V^{D-1}$ and for $1 \le k \le V^{D-1}$,

$$F(k) = \sum_{l=\lceil \log_V k \rceil}^{D-1} Q(V^l)$$

$$= N \sum_{l=\lceil \log_V k \rceil}^{D-1} V^{l(-1+\theta)}$$

$$= N V^{\lceil \log_V k \rceil(-1+\theta)} \frac{1 - V^{(D-\lceil \log_V k \rceil)(-1+\theta)}}{1 - V^{-1+\theta}}.$$

where $\lceil x \rceil$ is defined as the smallest integer larger than or equal to x. In Figure 4 we plot in log-log scale $F(k)$ for a regular tree with $N = 2^{12} = 4096$, $V = 2$, as well as its "corresponding" self-similar tree, i.e., the self-similar tree with $\theta = .168$. We observe that the two curves are now clearly distinguishable even for small k: the slopes we get by linear interpolation are clearly different. The linear interpolation slope for the regular tree is about -1, while that of the corresponding self-similar tree is about -.835. In fact, decaying (with respect to k) bounds for $F(k)$, for a wide range of k, can be easily developed as follows.

Using the fact that for $\theta \le 1$ it holds,

$$(-1+\theta)\log_V k + (-1+\theta) \le \lceil \log_V k \rceil (-1+\theta)$$
$$\le (-1+\theta)\log_V k,$$

we have,

$$F(k) \le \frac{N}{1 - V^{-1+\theta}} k^{-1+\theta}, \ 1 \le k \le N, \qquad (14)$$

and

$$F(k) \ge NV^{(-1+\theta)} k^{-1+\theta} \frac{1 - V^{(D-\lceil \log_V k \rceil)(-1+\theta)}}{1 - V^{-1+\theta}}.$$

Moreover, since for $k \le V^{D-1} = N/V$ it holds,

$$\frac{1 - V^{(D-\lceil \log_V k \rceil)(-1+\theta)}}{1 - V^{-1+\theta}} \ge 1,$$

we can write

$$F(k) \geq NV^{-1+\theta}k^{-1+\theta}, \ 1 \leq k \leq N/V. \qquad (15)$$

Note that given a general Global Multicast Tree, $F(k)$ is easy to measure, and therefore it is easy to obtain experimental results based on $F(k)$. It is not needed to experiment with, or simulate further, the multicast groups themselves. In addition, it turns out that under the assumption that destination host nodes are selected with uniform probability, any Global Multicast Tree that has polynomially decaying bounds of $F(k)$ as in (14), (15) gives rise to the multicast power law. This issue is undertaken in the next section.

IV. GENERAL GLOBAL MULTICAST TREE TOPOLOGIES

Let us assume that n destination hosts are forming a multicast group. We assume that destination hosts can be located behind any of the routing nodes N. If destination hosts can be located only at a subset S of the routing nodes, the results that follow hold by replacing N with the size of the set S. As in Section II we assume that the probability that a destination host is located behind a given node is $1/N$, independent of where the rest of the destination hosts are located. Under these assumptions, given $F(k)$, $L_N(n)$ can be derived as follows. Consider a node t on the Global Multicast Tree T, *other than the source node s*, with reachability degree $r(t)$. Let $\mathbf{I}_t$ be 1 if the link entering this node belongs to the created multicast tree and 0 otherwise. If $\mathbf{L}$ is the random variable representing the number of links in the multicast tree T, then it holds,

$$\mathbf{L} = \sum_{t \in T-\{s\}} \mathbf{I}_t,$$

and hence,

$$L_N(n) = \sum_{t \in T-\{s\}} E[\mathbf{I}_t].$$

Now, observe that $\mathbf{I}_t = 1$ if and only if at least one of the $r(t)$ nodes that are located on the subtree $T(t)$ is selected by the n destination hosts. The probability that one of the destination hosts picks some of the $r(t)$ nodes is $r(t)/N$ and the probability than none of the n destination hosts picks some of the $r(t)$ nodes is $(1 - r(t)/N)^n$. Hence,

$$E[\mathbf{I}_t] = 1 - (1 - r(t)/N)^n,$$

and, recalling that $Q(k)$ is the number of nodes (not including the source node) for which $r(t) = k$,

$$L_N(n) = \sum_{t \in T-\{s\}} (1 - (1 - r(t)/N)^n)$$
$$= \sum_{k=1}^{N} Q(k)\left(1 - \left(1 - \frac{k}{N}\right)^n\right).$$

Note that this formula holds for any Global Multicast Tree. We next transform this formula in order to make depend on $F(k)$.

Using Abel's partial summation formula (cf. [19]) we have that for two sequences v_k and u_k it holds,

$$\sum_{k=1}^{N}(u_k - u_{k+1})v_k = u_1 v_1 - u_{N+1}v_N + \sum_{k=2}^{N} u_k(v_k - v_{k-1}).$$

Using this and taking into account that $F(N+1) = 0$, we proceed as follows:

$$L_N(n) = \sum_{k=1}^{N} Q(k)\left(1 - \left(1 - \frac{k}{N}\right)^n\right)$$
$$= \sum_{k=1}^{N}(F(k) - F(k+1))\left(1 - \left(1 - \frac{k}{N}\right)^n\right)$$
$$= \sum_{k=2}^{N} F(k)\left(\left(1 - \frac{k-1}{N}\right)^n - \left(1 - \frac{k}{N}\right)^n\right)$$
$$\quad + F(1)\left(1 - \left(1 - \frac{1}{N}\right)^n\right)$$
$$= \sum_{k=1}^{N} F(k)\left(\left(1 - \frac{k-1}{N}\right)^n - \left(1 - \frac{k}{N}\right)^n\right). \quad (16)$$

Notice that if we consider randomly generated graphs, in which case the Global Multicast Tree will also be random, then (16) still holds with $F(k)$ replaced by $\overline{F}(k) = E[F(k)]$. Based on (16), we can now show that in general, a decay of $F(k)$ (or $\overline{F}(k)$) according to (14), (15) with $\theta > 0$, gives rise to the multicast power law, while a decay with $\theta = 0$ does not give rise to such law.

Theorem 3: Assume that for the Global Multicast Tree and $0 \leq \theta < 1$ it holds for large N and for $0 < \beta \leq 1$,

$$F(k) \leq A_N k^{-1+\theta}, \ 1 \leq k \leq N, \qquad (17)$$
$$F(k) \geq a_N k^{-1+\theta}, \ 1 \leq k \leq \beta N. \qquad (18)$$

a) If $\theta = 0$, then

$$R_N(n) \geq \frac{a_N}{A_N}n\left(1 - \frac{\ln n}{\ln N} - O\left(\frac{1}{\ln N}\right) - O\left(\frac{n}{N}\right)\right),$$
$$R_N(n) \leq \frac{A_N}{a_N}n\left(1 - \frac{\ln n}{\ln N} + O\left(\frac{1}{\ln N}\right) + O\left(\frac{n}{N}\right)\right).$$

b) If $\theta > 0$, then,

$$R_N(n) \geq \frac{a_N}{A_N}\Gamma(\theta)\theta n^{1-\theta}\left(1 - O\left(\frac{1}{n}\right) - O\left(\left(\frac{n}{N}\right)^\theta\right)\right),$$
$$R_N(n) \leq \frac{A_N}{a_N}\Gamma(\theta)\theta n^{1-\theta}\left(1 + O\left(\frac{1}{n}\right) + O\left(\left(\frac{n}{N}\right)^\theta\right)\right).$$

Although not explicitly denoted, A_N and a_N may depend on θ. The proof of Theorem 3 is given in the Appendix. We see that under the generalized assumptions of Theorem 3 the power law may appear again, albeit in a weaker form than in self-similar trees. If $\theta = 0$, i.e. $F(k)$ decays as $1/k$, then for large N and small ratios n/N, $R_N(n)$ behaves as $B_N n(1 - \ln n/N)$ and the power law does not hold. If $\theta > 0$, then the power law for $R_N(n)$ appears. Hence measuring $F(k)$ can provide a means to infer the validity of power law for multicast trees. For regular and self-similar trees, we see from (14) and (15) that $A_N = N/(1 - V^{-1+\theta})$, $a_N = N/V^{-1+\theta}$ and hence $A_N/a_N = V^{2(1-\theta)}/(V^{1-\theta} - 1)$. The parameter β

is introduced in order to include decays of the form (14), (15) as well as decays observed in experimental data presented in the next section. Refinements of the theorem can be made, but we leave it in this simple form since it contains the essential points of the discussion.

V. EXPERIMENTS

In this section we examine the behavior of $R_N(n)$ and $F(k)$ on real and simulated networks and compare the obtained graphs with the theoretical results obtained in Section IV and the corresponding behavior of self-similar trees in Section III.

We performed two sets of experiment. The first set was based on data from real internet topologies, obtained by the Scan project of ISI, with their software "mercator" implementing the ideas described [11]. We used the "Scan+Lucent" map which also merged data from the Internet Mapping project [2]. The graph is an approximation of part of the Internet topology circa October/November 1999, and includes 284805 nodes, of which about 283685 can reach each other. For the experiment, we considered the fully connected network consisting of the 283685 nodes.

The second set was based on random graph topologies: Specifically, a number of nodes N is picked and M of the $N(N-1)/2$ edges are picked randomly to form an undirected graph. This graph is transformed to a directed one by considering each link as bidirectional.

For each of the topologies tested, we picked randomly nodes as sources and created the Global Multicast Tree as the tree of shortest (in number of hops) paths from the source to all the network nodes. Once the Global Multicast Tree is formed, the calculation of $F(k)$ is straightforward. However, the calculation of $R_N(n)$ requires further to simulate multicast tree formations for each n. Specifically, having fixed the source node and the corresponding Global Multicast Tree, to compute $R_N(n)$ we perform the following simulations.

1) A number n of destination hosts is picked randomly. The probability that each of the destination hosts is located at a given routing node is $1/N$ and independent of the location of the rest of the hosts.
2) The multicast tree is formed as the appropriate subtree of the Global Multicast Tree.
3) The number of links, $l(n)$, of the multicast tree is computed.
4) The average of number of links, u, in a unicast communication is computed
5) The experiment in lines 1 to 4 is repeated 1000 times.
6) $L_N(n)$ is computed as the average of $l(n)$, and U_N as the average of u, over the experiments performed.
7) We compute $R_N(n) = L_N(n)/U_N$.

We present below the results obtained for the fully connected real network with 283685 nodes and 860682 edges, as well as a random network. We generated the random network as described above with exactly the same number of nodes and edges. The resulting random network contained one large connected component consisting of 268164 nodes. We performed the experiments using this connected component. The graphs obtained for various randomly chosen sources belonging to
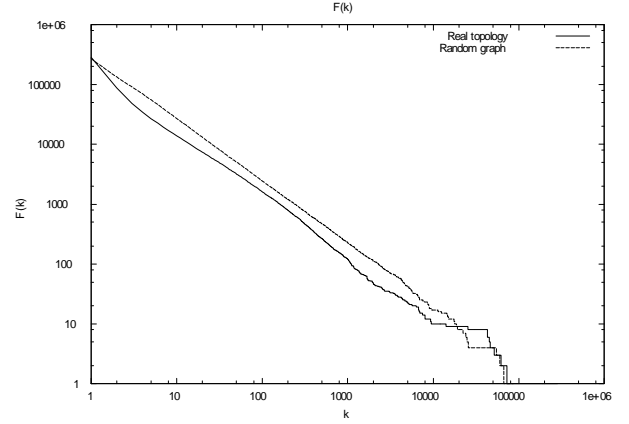

Fig. 5. Plots of $F(k)$ for a Real Internet Topology and a Randomly Generated Topology.

the two networks, do not differ significantly. Also, the results remain qualitatively the same for other network sizes that were tested.

Figure 5 shows in a log-log scale the obtained $F(k)$. We observe that for the random topology, the slope is about -1. For the real topology, for values of k between 5 and 100 the slope is about -.93.

Figure 6 shows in log-log scale the obtained $R_N(n)$. The slopes of $R_N(n)$ for small n (between 5 and 100) are now about .94 for the real topology and .89 for the random one.

Notice that as with the tree structures described in Section III, from the graphs of Figure 6 one is tempted to say that the power law holds for both topologies, with $\theta = 1 - .94 = .06$ for the real topology and $\theta = 1 - .89 = .11$ for the random one. However, in Figure 5 the slope of $F(k)$ in the log-log scale for the random topology is -1 for small k, i.e., $F(k)$ decays as $1/k$. Testing random graphs of various network sizes shows the same behavior, which leads us to conclude that Theorem 3 a), rather than the power law holds in this case. In fact this behavior has been shown theoretically for another type of random graphs (complete graphs with exponentially distributed links weights) in [?]. Consider now the corresponding curves for the real network topology. Apart from a relative steep decline for $k \leq 5$, the slope of $F(k)$ in the log-log scale is about $-.93$, while the slope of $R_N(n)$ is .94. This is in agreement with the conditions of Theorem 3 b). Tests with other real network sizes show the same qualitative behavior -although the numerical values vary. Of course in this case, one should be careful in drawing definite conclusions because it is difficult to say whether the networks obtained for different node numbers are of the same "type". A subject of further research is to identify classes of networks for which the conditions on $F(k)$ can be inferred based on even simpler network properties (e.g., distribution of node degree), or on the manner the networks are generated and evolve.

VI. CONCLUSIONS

The question whether multicast communication follows the power law arose from the attempt to price multicast communication in [3]. Power laws related to Internet topology
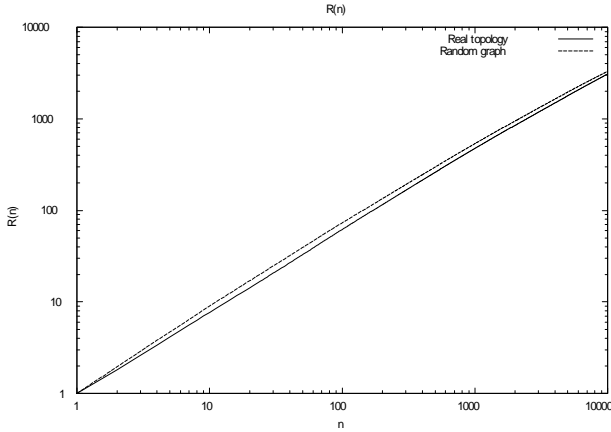
Fig. 6. $R(n)$ for the Real and Random Topology

parameters were studied in [8]. As indicated in the latter reference, power laws of this nature may be valuable for affirming how realistic simulated topologies are. This is an important issue that received a lot of attention [4], [20].

In this paper we examined structural conditions on Global Multicast Trees that give rise to the multicast power law. Regular $V-$ary trees do not exhibit the power law, while self-similar trees do. In fact, the power law rises under conditions weaker than tree self-similarity. We provided a metric, $F(k)$, encompassing the structural properties of the Global Multicast Tree, which is easy to measure and can be used to infer the validity of the power law. Simulation work demonstrated, based on observations of the graphs of $F(k)$, that randomly generated networks do not obey the power law, even though graphs of $R_N(n)$ may lead one to believe so. For the real topologies, similar measurements indicated that the power law is likely to hold. However, since the conditions on $F(k)$ are essentially asymptotic (for large N), further work is needed in order to ascertain the validity in the latter case. For example, can the behavior of $F(k)$ be deduced from even simpler network parameters? More generally, given the fact that several power laws related to various network parameters have been observed experimentally, see e.g. [8], the question arises as to why these laws appear and when one law implies the other.

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APPENDIX

In this appendix we establish our main theoretical results from Sections II and IV. In the next subsection we concentrate on the proof of Theorem 2 which refers to self-similar trees with $\theta > 0$. Theorem 1 for regular V-ary trees, i.e., the case $\theta = 0$ is proved in a similar fashion.

A. Proof of Theorem 2

We first need the following lemma.

Lemma 4: For $a = n/N$, $0 < \theta < 1$ the average number of links $L_{N,\theta}(n)$ in the self-similar tree attains the following asymptotic expansion.

$$
L_{N,\theta}(n) = N\left(\frac{V^{\bar{\theta}}}{V^{\bar{\theta}}-1} - c_1(a,\theta)\right) - \frac{N^\theta V^{\bar{\theta}}}{V^{\bar{\theta}}-1} + \frac{1}{2}c_2(a,\theta)
$$
$$
+ N^\theta O\left(\frac{1}{n^\theta (\ln n)^{\bar{\theta}}}\right) + \frac{N^\theta-1}{V^\theta-1}O\left(\frac{(\ln n)^3}{n}\right),
\tag{19}
$$

where $\bar{\theta} = 1 - \theta$ and

$$
c_1(a,\theta) = \sum_{l=0}^{\infty} V^{-\bar{\theta}l}\exp\left(-aV^l\right),
\tag{20}
$$

$$
c_2(a,\theta) = \sum_{l=0}^{\infty} aV^{(1+\theta)l}\exp\left(-aV^l\right).
\tag{21}
$$

For $\theta = 0$ the same expansion holds, with the provision that $\left(N^\theta - 1\right)/\left(V^\theta - 1\right)$ is replaced with its limiting value as $\theta \to 0$, i.e., $\ln N / \ln V = D$.

Proof. We saw that the average cost of the multicast self-similar tree is

$$L_{N,\theta}(n) = \sum_{k=1}^{D} V^{(D-k)\theta} V^k \left(1 - \left(1 - V^{-k}\right)^n\right).$$

Observe that since $D = \log_V N$ and $n = aN$, for $k = D$ the last term of the above sum is approximately equal to $V^D(1 - e^{-a})$. This term is not small in general and thus we cannot extend the limit of the summation to infinity without introducing significant error. In order to provide an asymptotic analysis of $L_{N,\theta}(n)$, we define $\bar{\theta} = 1 - \theta$ and proceed as follows:

$$L_{N,\theta}(n) = \sum_{k=1}^{D} V^{(D-k)\theta} V^k \left(1 - \left(1 - V^{-k}\right)^n\right)$$

$$= V^D \sum_{l=0}^{D-1} V^{-\bar{\theta}l} \left(1 - \left(1 - \frac{V^l}{V^D}\right)^n\right)$$

$$= V^D \left(\frac{1 - V^{-D\bar{\theta}}}{1 - V^{-\bar{\theta}}}\right) - V^D \sum_{l=0}^{D-1} V^{-\bar{\theta}l} \left(1 - \frac{V^l}{V^D}\right)^n$$

$$= N \left(\frac{1 - N^{-\bar{\theta}}}{1 - V^{-\bar{\theta}}}\right) - \bar{L}_{N,\theta}(n)$$

$$= N \frac{V^{\bar{\theta}}}{V^{\bar{\theta}} - 1} - \frac{N^\theta V^{\bar{\theta}}}{V^{\bar{\theta}} - 1} - \bar{L}_{N,\theta}(n), \tag{22}$$

where

$$\bar{L}_{N,\theta}(n) = N \sum_{l=0}^{D-1} V^{-\bar{\theta}l} \left(1 - \frac{aV^l}{n}\right)^n.$$

We concentrate on the analysis of $\bar{L}_{N,\theta}(n)$. First note that for $x/n < 1 - \delta$, $\delta > 0$, we have,

$$\ln\left(1 - \frac{x}{n}\right) = -\sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{x}{n}\right)^k \Rightarrow$$

$$\left(1 - \frac{x}{n}\right)^n = \exp\left(-n \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{x}{n}\right)^k\right)$$

$$= \exp(-x) \exp\left(-n \sum_{k=2}^{\infty} \frac{1}{k}\left(\frac{x}{n}\right)^k\right)$$

$$= \exp(-x) \exp\left(-\frac{x^2}{2n} - n \sum_{k=3}^{\infty} \frac{1}{k}\left(\frac{x}{n}\right)^k\right)$$

$$= \exp(-x) \exp\left(-\frac{x^2}{2n} - O\left(\frac{x^3}{n^2}\right)\right). \tag{23}$$

Using the inequalities $1 - y \leq \exp(-y) \leq 1 - y + y^2/2$, $y \leq 1$, and assuming further that $x \leq \ln n$ (hence $x/n < 0.5$) we have from (23),

$$\left(1 - \frac{x}{n}\right)^n = \exp(-x)\left(1 - \frac{x^2}{2n} + O\left(\frac{x^4}{n^2}\right)\right)$$

$$= \exp(-x)\left(1 - \frac{x^2}{2n} + \frac{x}{n}O\left(\frac{(\ln n)^3}{n}\right)\right). \tag{24}$$

Now, set $A_n = \left\lceil \frac{\ln\left(\frac{\ln n}{a}\right)}{\ln V} \right\rceil$. Hence

$$aV^{A_n-1} < \ln n \leq aV^{A_n}. \tag{25}$$

Since $aV^{A_n-1} < \ln n < n = aV^D$, we conclude that

$$A_n \leq D. \tag{26}$$

Therefore we can write,

$$\bar{L}_{N,\theta}(n) = N \sum_{l=0}^{A_n-1} V^{-\bar{\theta}l} \left(1 - \frac{aV^l}{n}\right)^n$$

$$+ N \sum_{l=A_n}^{D-1} V^{-\bar{\theta}l} \left(1 - \frac{aV^l}{n}\right)^n, \tag{27}$$

with the understanding that $\sum_{l=a}^{b} c_n = 0$ if $a > b$. We analyze separately each of the terms on the right hand side of the last equation.

A. The term $\sum_{l=0}^{A_n} V^{-\bar{\theta}l} \left(1 - \frac{aV^l}{n}\right)^n$.

For $l \leq A_n - 1$ we have from (25) that $aV^l \leq \ln n$, and taking into account (24),

$$\sum_{l=0}^{A_n-1} V^{-\bar{\theta}l} \left(1 - \frac{aV^l}{n}\right)^n =$$

$$\sum_{l=0}^{A_n-1} V^{-\bar{\theta}l} \exp\left(-aV^l\right) \left(1 - \frac{a^2 V^{2l}}{2n} + \frac{aV^l}{n}O\left(\frac{(\ln n)^3}{n}\right)\right) =$$

$$\sum_{l=0}^{A_n-1} V^{-\bar{\theta}l} \exp\left(-aV^l\right) - \frac{a}{2N} \sum_{l=0}^{A_n-1} V^{(1+\theta)l} \exp\left(-aV^l\right)$$

$$+ \left(\sum_{l=0}^{A_n-1} V^{\theta l} \exp\left(-aV^l\right)\right) \frac{1}{N}O\left(\frac{(\ln n)^3}{n}\right). \tag{28}$$

We look at each of the three terms of the last equality in (28) separately.

A.1 The term $\sum_{l=0}^{A_n-1} V^{-\bar{\theta}l} \exp\left(-aV^l\right)$.

We write,

$$\sum_{l=0}^{A_n-1} V^{-\bar{\theta}l} \exp\left(-aV^l\right) = \sum_{l=0}^{\infty} V^{-\bar{\theta}l} \exp\left(-aV^l\right)$$

$$- \sum_{l=A_n}^{\infty} V^{-\bar{\theta}l} \exp\left(-aV^l\right). \tag{29}$$

For $l \geq A_n$ we have from (25) that $\exp\left(-aV^l\right) \leq 1/n$ and hence

$$\sum_{l=A_n}^{\infty} V^{-\bar{\theta}l} \exp\left(-aV^l\right) \leq \frac{1}{n} \sum_{l=A_n}^{\infty} V^{-\bar{\theta}l}$$

$$\leq \frac{1}{1 - V^{-\bar{\theta}}} \frac{1}{N^{\bar{\theta}}} \frac{1}{n^\theta (\ln n)^{\bar{\theta}}}$$

$$= \frac{1}{N^{\bar{\theta}}}O\left(\frac{1}{n^\theta (\ln n)^{\bar{\theta}}}\right). \tag{30}$$

A.2 The term $\frac{a}{2N} \sum_{l=0}^{A_n-1} V^{(1+\theta)l} \exp\left(-aV^l\right)$.

We have,

$$\frac{a}{2N}\sum_{l=0}^{A_n-1} V^{(1+\theta)l}\exp\left(-aV^l\right) = \frac{a}{2N}\sum_{l=0}^{\infty} V^{(1+\theta)l}\exp\left(-aV^l\right)$$
$$- \frac{a}{2N}\sum_{l=A_n}^{\infty} V^{(1+\theta)l}\exp\left(-aV^l\right). \tag{31}$$

Using the fact that $x^2\exp(-x) < .6$ for $x \geq 0$ we have for the second term of the last equality,

$$\sum_{l=A_n}^{\infty}\frac{aV^{(1+\theta)l}}{2N}\exp\left(-aV^l\right) = \sum_{l=A_n}^{\infty}\frac{V^{\theta l}\left(aV^l\right)^2}{2NaV^l}\exp\left(-aV^l\right)$$
$$\leq \frac{1}{Na}\sum_{l=A_n}^{\infty} V^{-\bar\theta l}$$
$$= \frac{1}{Na^\theta}\left(aV^{A_n}\right)^{-\bar\theta}\frac{1}{1-V^{-\bar\theta}}$$
$$\leq \frac{1}{N^{\bar\theta}n^\theta\left(\ln n\right)^{\bar\theta}}\frac{1}{1-V^{-\bar\theta}}$$
$$= \frac{1}{N^{\bar\theta}}O\left(\frac{1}{n^\theta\left(\ln n\right)^{\bar\theta}}\right). \tag{32}$$

A.3 The term $\sum_{l=0}^{A_n-1} V^{\theta l}\exp\left(-aV^l\right)$.

We observe,

$$\sum_{l=0}^{A_n-1} V^{\theta l}\exp\left(-aV^l\right) \leq \sum_{l=0}^{A_n-1} V^{\theta l}$$
$$= \frac{V^{\theta A_n}-1}{V^\theta-1}$$
$$\leq \frac{N^\theta-1}{V^\theta-1} \quad\text{by (26)}. \tag{33}$$

Combining (28), (29), (30), (31), (32), (33) we finally get

$$\sum_{l=0}^{A_n} V^{-\bar\theta l}\left(1-\frac{aV^l}{n}\right)^n = c_1(a,\theta) - \frac{1}{2N}c_2(a,\theta)$$
$$+ \frac{1}{N^{\bar\theta}}O\left(\frac{1}{n^\theta\left(\ln n\right)^{\bar\theta}}\right)$$
$$+ \frac{N^\theta-1}{V^\theta-1}\frac{1}{N}O\left(\frac{\left(\ln n\right)^3}{n}\right), \tag{34}$$

where

$$c_1(a,\theta) = \sum_{l=0}^{\infty} V^{-\bar\theta l}\exp\left(-aV^l\right), \tag{35}$$

$$c_2(a,\theta) = \sum_{l=0}^{\infty} aV^{(1+\theta)l}\exp\left(-aV^l\right). \tag{36}$$

B. The term $\sum_{l=A_n}^{D-1} V^{-\bar\theta l}\left(1-\frac{aV^l}{n}\right)^n$.

For $l > A_n$, taking into account (25) we find,

$$\left(1-\frac{aV^l}{n}\right)^n \leq \left(1-\frac{\ln n}{n}\right)^n$$
$$= \left(\left(1-\frac{\ln n}{n}\right)^{\frac{n}{\ln n}}\right)^{\ln n}$$
$$\leq \frac{1}{n}, \quad\text{since } (1-1/x)^x \leq e^{-1} \text{ for } x \geq 1,$$

and hence

$$\sum_{l=A_n}^{D-1} V^{-\bar\theta l}\left(1-\frac{aV^l}{n}\right)^n \leq V^{-\bar\theta A_n}\frac{1}{n}$$
$$\leq \left(\frac{a}{\ln n}\right)^{\bar\theta}\frac{1}{n}$$
$$= \frac{1}{N^{\bar\theta}}\frac{1}{n^\theta\left(\ln n\right)^{\bar\theta}}. \tag{37}$$

From (27), (34) and (37) we finally conclude,

$$\bar L_{N,\theta}(n) = Nc_1(a,\theta) - \frac{1}{2}c_2(a,\theta) + N^\theta O\left(\frac{1}{n^\theta\left(\ln n\right)^{\bar\theta}}\right)$$
$$+ \frac{N^\theta-1}{V^\theta-1}O\left(\frac{\left(\ln n\right)^3}{n}\right).$$

Which together with (22) proves the lemma.

1) Derivation of Theorem 2: We now prove Theorem 2, that is, we find asymptotic expansions of $L_{N,\theta}(n)$ for $n \to \infty$. Observe that we only need to analyze the quantities $c_1(a,\theta)$ and $c_2(a,\theta)$ defined in (20) and (21), respectively.

We concentrate on the regime $a \to 0$ which is the most interesting case, and the hardest. It turns out that this case can be handled by a special analytic tool, namely the Mellin transform. The Mellin transform found myriad of applications in the analysis of algorithms. An excellent survey is given by Flajolet, Gourdon and Dumas in [9] (cf. [12], [19]). For reader convenience, we collected the most important properties of the Mellin transform in Appendix **??**. In particular, the definition of Mellin transform is given in (65). Property (M1) defines the so called *fundamental strip* of the complex plane where the Mellin transform exists. The *harmonic sum property* (M3) and the *mapping properties* (M4) are crucial. We shall use them to derive asymptotics of $c_1(a,\theta)$ and $c_2(a,\theta)$ as $a \to 0$.

Let us first consider $c_1(a,\theta) = \sum_{l=0}^{\infty} V^{-\bar\theta l}\exp\left(-aV^l\right)$. Observe that by (M3) the sum in $c_1(a,\theta) := c_1(a)$ is a harmonic sum with $\lambda_k = V^{-\bar\theta k}$ and $g(x) = e^{-x}$ with $\mu_k = V^k$. Thus the Mellin transform $c_1^*(s)$ with respect to a of $c_1(a)$ is by (M3) (and the well known fact that the Mellin of e^{-x} is the Euler gamma function $\Gamma(s)$ for $\Re(s) > 0$):

$$c_1^*(s) = \frac{\Gamma(s)}{1-V^{-(1+s-\theta)}}.$$

We now use (M4) to find $c_1(a)$ as $a \to 0$. If $\mathrm{Re}(s) \in (-M,\delta)$, $\infty > M \geq 1$, $\infty > \delta > 0$, from the properites of the Gamma function [13], it can be seen that $\Gamma(s) = O(|s|^{-r})$, $r > 1$ as $|s| \to \infty$. Also, for $\mathrm{Re}(s) \in (-M,\delta)$, the function $c_1^*(s)$ has poles at $\bar s = 0$ and $s_k = -1 + \theta - (2\pi i k)/\ln V$, $k =$

$0, \pm 1, \pm 2, \ldots$. All these poles are single. The pole at $\overline{s} = 0$ has residue

$$c_{\overline{s},0} = \frac{V^{\overline{\theta}}}{V^{\overline{\theta}} - 1}.$$

The poles at s_k have residues

$$c_{s_k,0} = \frac{\Gamma\left(-1 + \theta - 2\pi i k / \ln V\right)}{\ln V}.$$

Using now the Reverse Mapping Theorem (M4) and the property $\Gamma(-1 + \theta) = \Gamma(\theta)/(-1 + \theta)$, we see that for $a \to 0$ we have the expansion for any $M > 0$,

$$c_1(a, \theta) = \frac{V^{\overline{\theta}}}{V^{\overline{\theta}} - 1} - \frac{\Gamma(\theta)}{\theta \ln V} a^{\overline{\theta}} + a^{\overline{\theta}} \psi_3(\ln a) + O(a^M), \quad (38)$$

where

$$\psi_3(x) = \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{\Gamma(-1 + \theta - 2\pi i k / \ln V)}{\ln V} \exp\left(2\pi i k \frac{x}{\ln V}\right).$$

Now we consider $c_2(a, \theta) = \sum_{l=0}^{\infty} a V^{(1+\theta)l} \exp\left(-a V^l\right)$. It is again a harmonic sum, hence by (M3) we find its Mellin transform to be

$$c_2^*(a) = \frac{s\Gamma(s)}{1 - V^{-(s-\theta)}}.$$

But $c_2^*(s)$ has a single poles at θ with residue

$$c_{\theta,0} = \frac{\theta\Gamma(\theta)}{\ln V},$$

and single poles at $s_k = \theta - (2\pi i k)/\ln V$, $k \neq \pm 1, \pm 2, \ldots$ with residues

$$\frac{(\theta - i2\pi k)\Gamma(\theta - 2\pi i k / \ln V)}{\ln V}.$$

Hence using again the Reverse Mapping Theorem (M4) we obtain

$$c_2(a, \theta) = \frac{\theta\Gamma(\theta)}{\ln V} a^{-\theta} - a^{-\theta} \psi_4(\ln a) + O(a^M), \quad (39)$$

where

$$\psi_4(x) = \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{(\theta - 2i\pi k / \ln V)\Gamma(\theta - 2\pi i k / \ln V)}{\ln V} \exp\left(2\pi i k \frac{x}{\ln V}\right).$$

From Lemma 4, (38) and (39) Theorem 2 follows.

B. Proof of Theorem 3

We will develop bounds on $L_N(n)$ based on (16), (17) and (18). Assume $n \geq 2$.

Using Taylor's expansion of $(1 - x)^n$ around $x = k/N$, $1 \leq k \leq N$, we have,

$$\left(1 - \frac{k-1}{N}\right)^n - \left(1 - \frac{k}{N}\right)^n \leq \frac{n}{N}\left(1 - \frac{k}{N}\right)^{n-1}$$
$$+ \frac{n(n-1)}{2N^2}\left(1 - \frac{k-1}{N}\right)^{n-2} \quad (40)$$

$$\left(1 - \frac{k-1}{N}\right)^n - \left(1 - \frac{k}{N}\right)^n \geq \frac{n}{N}\left(1 - \frac{k}{N}\right)^{n-1}$$
$$+ \frac{n(n-1)}{2N^2}\left(1 - \frac{k}{N}\right)^{n-2}. \quad (41)$$

Using (40), (16) and (17) we conclude that,

$$L_N(n) \leq A_N N^{\theta-1} n \sum_{k=1}^{N}\left(\frac{k}{N}\right)^{\theta-1}\left(1 - \frac{k}{N}\right)^{n-1} \frac{1}{N}$$

$$+ A_N N^{\theta-1} n \left(\sum_{k=1}^{N}\left(\frac{k}{N}\right)^{\theta-1}\left(1 - \frac{k-1}{N}\right)^{n-2} \frac{1}{N}\right) \frac{n-1}{2N}. \quad (42)$$

Let us look at the summation in the first term of the right hand side in (42). Since $\theta < 1$, the function $x^{\theta-1}(1-x)^{n-1}$ is decreasing for $0 < x \leq 1$. Hence we have,

$$\sum_{k=1}^{N}\left(\frac{k}{N}\right)^{\theta-1}\left(1 - \frac{k}{N}\right)^{n-1}\frac{1}{N} \leq \frac{1}{N^{\theta}}\left(1 - \frac{1}{N}\right)^{n-1}$$
$$+ \int_{1/N}^{1} x^{\theta-1}(1-x)^{n-1}dx$$
$$\leq \frac{1}{N^{\theta}} + \int_{1/N}^{1} x^{\theta-1}(1-x)^{n-1}dx. \quad (43)$$

Regarding the summation in the second term of the right hand side in (42) we calculate,

$$\sum_{k=1}^{N}\left(\frac{k}{N}\right)^{\theta-1}\left(1 - \frac{k-1}{N}\right)^{n-2}\frac{1}{N} =$$
$$\frac{1}{N^{\theta}} + \sum_{k=2}^{N}\left(\frac{k}{N}\right)^{\theta-1}\left(1 - \frac{k-1}{N}\right)^{n-2}\frac{1}{N}$$
$$\leq \frac{1}{N^{\theta}} + \sum_{k=2}^{N}\left(\frac{k-1}{N}\right)^{\theta-1}\left(1 - \frac{k-1}{N}\right)^{n-2}\frac{1}{N}$$
$$= \frac{1}{N^{\theta}} + \sum_{k=1}^{N-1}\left(\frac{k}{N}\right)^{\theta-1}\left(1 - \frac{k}{N}\right)^{n-2}\frac{1}{N}$$
$$\leq \frac{1 + \left(1 - \frac{1}{N}\right)^{n-2}}{N^{\theta}} + \int_{1/N}^{1} x^{\theta-1}(1-x)^{n-2}dx$$
$$\leq \frac{2}{N^{\theta}} + \int_{1/N}^{1} x^{\theta-1}(1-x)^{n-2}dx. \quad (44)$$

Combining (42), (43) and (44) we obtain,

$$\frac{L_N(n)}{A_N N^{\theta-1} n} \leq \int_{1/N}^{1} x^{\theta-1}(1-x)^{n-1} dx$$
$$+ \frac{n-1}{2N} \int_{1/N}^{1} x^{\theta-1}(1-x)^{n-2} dx$$
$$+ \frac{1}{N^\theta}$$
$$+ \frac{n-1}{N^{1+\theta}}. \tag{45}$$

To develop a lower bound we proceed as follows. Assume for simplicity in the notation that βN is integer. From (16), (18), (41) and the fact that the function $x^{\theta-1}(1-x)^{n-1}$ is decreasing for $0 < x \leq 1$, $n \geq 1$, $\theta < 1$, we have,

$$\frac{L_N(n)}{a_N N^{\theta-1} n} \geq \sum_{k=1}^{\beta N} \left(\frac{k}{N}\right)^{\theta-1} \left(1 - \frac{k}{N}\right)^{n-1} \frac{1}{N}$$
$$+ \frac{n-1}{2N} \left(\sum_{k=1}^{\beta N} \left(\frac{k}{N}\right)^{\theta-1} \left(1 - \frac{k}{N}\right)^{n-2} \frac{1}{N} \right)$$
$$\geq \int_{1/N}^{\beta} x^{\theta-1}(1-x)^{n-1} dx$$
$$+ \frac{n-1}{2N} \left(\int_{1/N}^{\beta} x^{\theta-1}(1-x)^{n-2} dx \right),$$

or

$$\frac{L_N(n)}{a_N N^{\theta-1} n} \geq \int_{1/N}^{1} x^{\theta-1}(1-x)^{n-1} dx$$
$$+ \frac{n-1}{2N} \left(\int_{1/N}^{1} x^{\theta-1}(1-x)^{n-2} dx \right)$$
$$- \int_{\beta}^{1} x^{\theta-1}(1-x)^{n-1} dx$$
$$- \frac{n-1}{2N} \left(\int_{\beta}^{1} x^{\theta-1}(1-x)^{n-2} dx \right). \tag{46}$$

For unicast transmission we have,

$$U_N = L(1) = N^{-1} \sum_{i=1}^{N} F(k),$$

and in a similar fashion we derive

$$\frac{U_N}{A_N N^{\theta-1}} \leq \int_{1/N}^{1} x^{\theta-1} dx + \frac{1}{N^\theta}, \tag{47a}$$

$$\frac{U_N}{a_N N^{\theta-1}} \geq \int_{1/N}^{\beta} x^{\theta-1} dx. \tag{47b}$$

From (45) and (46) we see that we need to evaluate the asymptotics of the integral $\int_{1/N}^{1} x^{\theta-1}(1-x)^{n-1} dx$, $n \geq 1$. The form of these asymptotics depends on whether $\theta = 0$ or $\theta > 0$. We treat each case separately.

Case 1: $\theta = 0$: Observe that,

$$\int_{1/N}^{1} x^{-1}(1-x)^{n-1} dx =$$
$$\int_{1/N}^{1} x^{-1} \left(\sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^l x^l \right) dx$$
$$= \ln N + \sum_{l=1}^{n-1} \binom{n-1}{l} (-1)^l \frac{1 - N^{-l}}{l}$$
$$= \ln N + \sum_{l=1}^{n-1} \binom{n-1}{l} \frac{(-1)^l}{l}$$
$$+ N^{-1} \sum_{l=1}^{n-1} \binom{n-1}{l} (-1)^{l+1} \frac{N^{-l+1}}{l}. \tag{48}$$

It is well known [10] that,

$$\sum_{l=1}^{n} \binom{n}{l} \frac{(-1)^l}{l} = -\sum_{l=1}^{n} \frac{1}{l} = -\ln n - \gamma + \Theta(1/n). \tag{49}$$

On the other hand it can be shown using standard calculus techniques that the function

$$f(x) = \sum_{l=1}^{n-1} \binom{n-1}{l} (-1)^{l+1} \frac{x^{-l+1}}{l},$$

is increasing in $[1, \infty)$, that is,

$$f(1) = \ln(n-1) + \gamma + \Theta\left(\frac{1}{n}\right) \leq f(x) \leq \lim_{x \to \infty} f(x) = n-1.$$

Hence,

$$\frac{\ln(n-1)}{N} + \Theta\left(\frac{1}{nN}\right) \leq N^{-1} \sum_{l=1}^{n-1} \binom{n-1}{l} (-1)^{l+1} \frac{N^{-l+1}}{l}$$
$$\leq \frac{n}{N}. \tag{50}$$

From (48), (49), (50), and taking into account that $\ln(n-1) = \ln n + \Theta(1/n)$, we conclude that,

$$\int_{1/N}^{1} x^{-1}(1-x)^{n-1} dx \leq \ln N - \ln n - \gamma + \Theta\left(\frac{1}{n}\right) + \frac{n}{N} \tag{51}$$

$$\int_{1/N}^{1} x^{-1}(1-x)^{n-1} dx \geq \ln N - \ln n - \gamma + \frac{\ln n}{N} + \Theta\left(\frac{1}{n}\right). \tag{52}$$

To develop the lower bound from (46) we also need the following simple inequality,

$$\int_{\beta}^{1} x^{-1}(1-x)^{n-1} dx \leq -(1-\beta)^{n-1} \ln \beta. \tag{53}$$

Summarizing, we have from (45) and (51),

$$\frac{L_N(n)}{n A_N N^{-1}} \leq \ln N \left(1 - \frac{\ln n}{\ln N} + O\left(\frac{1}{\ln N}\right) + O\left(\frac{n}{N}\right) \right), \tag{54}$$

while from (46), (52) and (53),

$$\frac{L_N(n)}{n a_N N^{-1}} \geq \ln N \left(1 - \frac{\ln n}{\ln N} - O\left(\frac{1}{\ln N}\right) - O\left(\frac{n}{N}\right) \right). \tag{55}$$

For unicast transmission we have from (47),

$$\frac{U_N}{A_N N^{-1}} \le \ln N + \frac{1}{N} \tag{56a}$$

$$\frac{U_N}{a_N N^{-1}} \ge \ln N + \ln \beta. \tag{56b}$$

Part a) of the theorem is derived from (54), (55) and (56).
Case 2: $\theta > 0$. We have,

$$\int_{1/N}^{1} x^{\theta-1} (1-x)^{n-1} \, dx = \int_{0}^{1} x^{\theta-1} (1-x)^{n-1} \, dx$$
$$- \int_{0}^{1/N} x^{\theta-1} (1-x)^{n-1} \, dx. \tag{57}$$

Now,

$$\frac{1}{\theta N^\theta} \left(1 - \frac{1}{N}\right)^{n-1} \le \int_{0}^{1/N} x^{\theta-1}(1-x)^{n-1} \, dx \tag{58a}$$

$$\le \frac{1}{\theta N^\theta}, \tag{58b}$$

and

$$\int_{0}^{1} x^{\theta-1}(1-x)^{n-1} \, dx = \beta(\theta, n) = \frac{\Gamma(\theta)\Gamma(n)}{\Gamma(n+\theta)}, \tag{59}$$

where $\beta(\theta, n)$ and $\Gamma(x)$ are respectively the Beta and Gamma functions [19]. Using the approximation [19],

$$\frac{\Gamma(n)}{\Gamma(n+\theta)} = n^{-\theta} \left(1 + \Theta(n^{-1})\right),$$

we obtain from (57), (58), (59),

$$\int_{1/N}^{1} x^{\theta-1}(1-x)^{n-1} \, dx \le$$
$$\Gamma(\theta) n^{-\theta} \left(1 - \frac{n^\theta}{\Gamma(\theta)\theta N^\theta} \left(1 - \frac{1}{N}\right)^{n-1} + O\left(\frac{1}{n}\right)\right) \tag{60}$$

$$\int_{1/N}^{1} x^{\theta-1}(1-x)^{n-1} \, dx \ge$$
$$\Gamma(\theta) n^{-\theta} \left(1 - \frac{n^\theta}{\Gamma(\theta)\theta N^\theta} + O\left(\frac{1}{n}\right) + O\left(\frac{n}{N}\right)\right). \tag{61}$$

Summarizing, we have from (45) and (60),

$$\frac{L_N(n)}{A_N N^{\theta-1} n} \le \Gamma(\theta) n^{-\theta} \left(1 - \frac{n^\theta}{\Gamma(\theta)\theta N^\theta} \left(1 - \frac{1}{N}\right)^{n-1}\right.$$
$$\left. + \frac{n^\theta}{\Gamma(\theta) N^\theta} + O\left(\frac{1}{n}\right) + O\left(\frac{n}{N}\right)\right), \tag{62}$$

while from (46), (61),

$$\frac{L_N(n)}{a_N N^{\theta-1} n} \ge \Gamma(\theta) n^{-\theta} \left(1 - \frac{n^\theta}{\Gamma(\theta)\theta N^\theta} + O\left(\frac{1}{n}\right)\right.$$
$$\left. + O\left(\frac{n}{N}\right) - \frac{1-\beta^\theta}{\Gamma(\theta)\theta} n^\theta (1-\beta)^{n-1}\right). \tag{63}$$

From (47) we obtain,

$$\frac{U_N}{A_N N^{\theta-1}} \le \theta^{-1} \left(1 - \frac{1}{N^\theta} + \frac{\theta}{N^\theta}\right) \tag{64a}$$

$$\frac{U_N}{a_N N^{\theta-1}} \ge \theta^{-1} \left(1 - \frac{1}{N^\theta} - (1-\beta^\theta)\right). \tag{64b}$$

Part b) of the theorem is derived from (62), (63) and (64).

C. Main Properties of Mellin Transform

For the reader convenience, we collected here the main properties of the Mellin transform. For details and proofs see [9], [19].

Let $f(x)$ be a complex-valued continous function in $(0, \infty)$. The Mellin transform of f is defined as

$$f^*(s) := \mathcal{M}(f(x); s) = \int_{0}^{\infty} f(x) x^{s-1} dx. \tag{65}$$

(M1) FUNDAMENTAL STRIP. The Mellin transform of $f(x)$ exists in the *fundamental strip* $\text{Re}(s) \in (-\alpha, -\beta)$, where

$$f(x) = O(x^\alpha) \quad (x \to 0), \qquad f(x) = O(x^\beta) \quad (x \to \infty)$$

for $\beta < \alpha$.

(M2) DIRECT AND INVERSE MELLIN TRANSFORMS. Let c belong to the fundamental strip. Then

$$f^*(s) := \mathcal{M}(f(x); s) = \int_{0}^{\infty} f(x) x^{s-1} dx \iff$$

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f^*(s) x^{-s} ds. \tag{66}$$

(M3) HARMONIC SUM PROPERTY. It holds,

$$f(x) = \sum_{k \ge 0} \lambda_k g(\mu_k x) \iff f^*(s) = g^*(s) \sum_{k \ge 0} \lambda_k \mu_k^{-s}. \tag{67}$$

(M4) MAPPING PROPERTIES (Asymptotic expansion of $f(x)$ and singularities of $f^*(s)$).

$$f(x) = \sum_{(\xi, m) \in A} c_{\xi, m} x^\xi (\log x)^m + O(x^M) \iff$$

$$f^*(s) \asymp \sum_{(\xi, m) \in A} c_{\xi, k} \frac{(-1)^m m!}{(s+\xi)^{m+1}}. \tag{68}$$

— (i) *Direct Mapping.* Assume that $f(x)$ admits as $x \to 0^+$ the asymptotic expansion (68) for some $-M < -\alpha$. Then for $\text{Re}(s) \in (-M, -\beta)$, the transform $f^*(s)$ satisfies the singular expansion (68).

— (ii) *Reverse Mapping.* Assume that $f^*(s) = O(|s|^{-r})$ with $r > 1$, as $|s| \to \infty$ and that $f^*(s)$ admits the singular expansion (68) for $\text{Re}(s) \in (-M, -\beta)$. Then $f(x)$ satisfies the asymptotic expansion (68) at $x = 0^+$.