ON SOME NON-LINEAR RECURRENCES THAT ARISE IN COMPUTER SCIENCE

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We survey recent work on non-linear recurrence equations that arise in computer science or combinatorics. We consider the examples of the height of digital trees, the QUICKSORT algorithm, height of binary search trees, and enumeration problems concerning random binary trees characterized by nodes and path lengths. In each case a singular perturbation analysis of the recurrence yields insights into the asymptotic behavior, such as limiting distributions and tail probabilities.

1. Introduction

Many problems that arise in theoretical computer science lead naturally to solving non-linear recurrence or difference equations. These may sometimes be solved exactly but more frequently this is not the case, and an approximate analysis is necessary. Often the problems have a natural large parameter suggesting that an asymptotic analysis is appropriate. This large parameter could be the number of strings that are to be stored in a digital tree, or the number of items that are to be sorted by some algorithm. Some of the problems that arise are basic combinatorial enumeration problems.

Recent books that describe and analyze these types of problems are by Flajolet and Sedgewick [1] and Szpankowski [2].

In recent years we have analyzed some basic problems in computer science by using asymptotic methods of applied mathematics. These include methods for asymptotically evaluating sums and integrals, as well as perturbation methods such as matched asymptotics and WKB-type expansions.

The latter are especially useful for difficult non-linear problems that cannot be solved explicitly.

In this note we survey some of the problems and the asymptotic solutions that we obtained. We consider digital trees, sorting algorithms, binary search trees, and enumeration problems that arise in studying numbers of nodes and paths in random binary trees.

2. Digital trees

Suppose that we have a set $\mathcal{S}$ of n strings, say $\mathcal{S} = \{s_1, s_2, \ldots, s_n\}$ with each s_j being a finite or infinite sequence of 0's and 1's. Thus, e.g., $s_1 = 1\,0\,0\,1\,0\,1\,1 \ldots$. We assume a probabilistic model, namely that in a given position a 0 or 1 occurs with equal probability $= 1/2$, and that the positions are independent of one another. This is called a "symmetric Bernoulli model". We store the n strings in a tree. Different trees are obtained by using different rules for this storage. In a "trie" we store the string to the left or right of the root if $|\mathcal{S}| = 1$, according to whether the first symbol is a 1 or a 0. If $|\mathcal{S}| > 1$ we split the set of strings into two subsets according to whether the first symbol is a 1 or a 0. The trie is then built recursively and in Fig. 1 we illustrate this for an example with 4 strings. A second type of digital tree is a "PATRICIA trie", which can be obtained from a trie by eliminating nodes that have only one branch (see Fig. 1). A "digital search tree (DST)" is a further refinement that stores the strings in the internal nodes of the tree. To measure how efficiently a digital tree will store a large number n of strings we consider the "height" of the tree, which is defined as the largest path in the tree, and is related to the maximum search time. In the examples in Fig. 1 the heights are 3, 2, 2 for the respective cases of the trie, PATRICIA and DST.

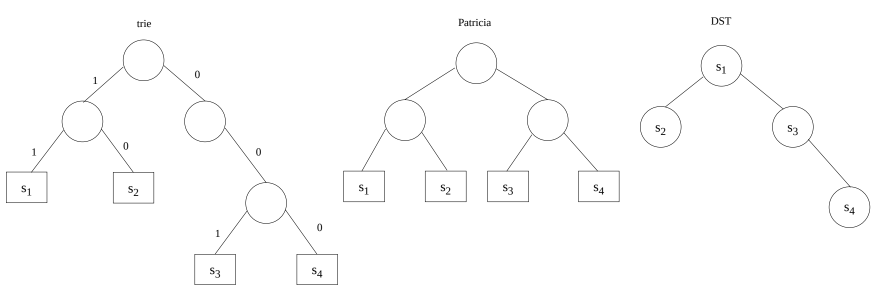

Fig. 1. A sketch of the three types of digital trees, as they each store the 4 strings $s_1 = 1\,1\,1\,0\,0 \ldots$, $s_2 = 1\,0\,1\,1\,1 \ldots$, $s_2 = 0\,0\,1\,1\,0 \ldots$ and $s_4 = 0\,0\,0\,0\,1\,1 \ldots$.

The height is a random variable whose probability distribution we denote by

$$h^{\mathrm{T}}(k,n) = \mathrm{Prob}\left[\mathcal{H}_n^{\mathrm{T}} \leqslant k\right] \tag{1}$$

where T is used to denote trie. Similarly we define the heights of PATRICIA and DST by $\mathcal{H}_n^{\mathrm{PAT}}$ and $\mathcal{H}_n^{\mathrm{DST}}$. The distribution functions satisfy the respective recurrences

$$h^{\mathrm{T}}(k+1,n) = 2^{-n}\sum_{i=0}^{n}\binom{n}{i}h^{\mathrm{T}}(k,i)h^{\mathrm{T}}(k,n-i),\ k \geqslant 0; \tag{2}$$
$$h^{\mathrm{T}}(0,0) = h^{\mathrm{T}}(0,1) = 1;\ h^{\mathrm{T}}(0,n) = 0,\ n \geqslant 2;$$

$$\begin{aligned} h^{\mathrm{PAT}}(k+1,n) &= 2^{1-n}h^{\mathrm{PAT}}(k+1,n) \\ &+ 2^{-n}\sum_{i=0}^{n-1}\binom{n}{i}h^{\mathrm{PAT}}(k,i)h^{\mathrm{PAT}}(k,n-i),\ n \geqslant 2,\ k \geqslant 0; \end{aligned} \tag{3}$$

$$h^{\mathrm{PAT}}(0,0) = h^{\mathrm{PAT}}(0,1) = 1;\ h^{\mathrm{PAT}}(0,n) = 0, n \geqslant 2;$$

$$h^{\mathrm{DST}}(k+1,n+1) = 2^{-n}\sum_{i=0}^{n}h^{\mathrm{DST}}(k,i)h^{\mathrm{DST}}(k,n-i),\ k \geqslant 0; \tag{4}$$
$$h^{\mathrm{DST}}(0,0) = h^{\mathrm{DST}}(0,1) = 1;\ h^{\mathrm{DST}}(0,n) = 0,\ n \geqslant 2.$$

These three recurrences look very similar but their solutions turn out to be much different. They can be recast as functional equations by introducing exponential generating functions, with $H_k(z) = \sum_{n=0}^{\infty} h(k,n)z^n/n!$. For example, for DST we obtain $H'_{k+1}(z) = [H_k(z/2)]^2$. The case of tries can be explicitly solved and the mean height satisfies

$$E\left[\mathcal{H}_n^{\mathrm{T}}\right] \sim 2\log_2(n),\ n \to \infty. \tag{5}$$

The distribution follows a double exponential or extreme value law as $n \to \infty$ [3,4].

For the case of PATRICIA we analyzed (3) asymptotically as $n \to \infty$ using perturbation methods and found that $h^{\mathrm{PAT}}(k,n)$ behaves differently for the three scales (i) $k = n - O(1)$, (ii) $k = \log_2 n + O(1)$ and (iii) $2^k - n = O(1)$. Below we give some of our asymptotic formulas:

(i) $k = n - j,\ j = O(1),\ n \to \infty$

$$1 - h^{\mathrm{PAT}}(k,n) = \mathrm{Prob}\left[\mathcal{H}_n^{\mathrm{PAT}} > k\right] \sim 2^{-n^2/2}2^{(j-3/2)n}n!\rho_0\mathrm{K}_j,$$

$$\mathrm{K}_j = 2^{-j^{-2}/2} 2^{3j/2} \frac{1}{4\pi i} \oint z^{1-j} e^z \prod_{m=0}^{\infty} \left[\frac{1-\exp\left(-z2^{-m-1}\right)}{z2^{-m-1}} \right] dz$$

$$\rho_0 = \prod_{\ell=2}^{\infty} (1-2^{-\ell})^{-1} = 1.731\ldots.$$

(ii) $k, n \to \infty;\ \xi = n2^{-k}(1), 0 < \xi < 1$

$$h^{\mathrm{PAT}}(k,n) \sim \sqrt{1 + 2\xi\Phi'(\xi) + \xi^2\Phi''(\xi)}\, e^{-n\Phi(\xi)}$$

where $\Phi(\xi) > 0$ was determined numerically. Also, as $\xi \to 0^+$

$$\Phi(\xi) \sim \frac{1}{2}\rho_0 e^{\phi(\log_2 \xi)} \xi^{3/2} \exp\left(-\frac{\log^2 \xi}{2\log 2}\right),$$

$$\phi(x) = \frac{\log 2}{2} x(x+1) + \sum_{\ell=0}^{\infty} \log\left[\frac{1-\exp\left(-2^{x-\ell}\right)}{2^{x-\ell}}\right] + \sum_{\ell=1}^{\infty} \log\left[1-\exp\left(-2^{x+\ell}\right)\right],$$

and as $\xi \to 1^-$

$$\Phi(\xi) \sim D_1 + (1-\xi)\left[\log(1-\xi) - 1 - \log D_2\right],$$

$$D_1 = 1 + \log\left(\mathrm{K}_0^*\right), \quad D_2 = \mathrm{K}_1^* \mathrm{K}_0^*/e$$
$$\mathrm{K}_0^* = .6832\ldots, \quad \mathrm{K}_1^* = 1.259\ldots.$$

(iii) $k, n \to \infty$; $2^k - n = M = O(1)$

$$h_n^{\mathrm{PAT}}(k,n) \sim \frac{\sqrt{2\pi}}{M\,!} D_2^M n^{M+1/2} e^{-\mathrm{D}_1 n}.$$

We can view (i) as the right tail and (iii) as the left tail of the distribution. Most of the probability mass is concentrated in that range of k where h^{PAT} changes from being ≈ 0 to being ≈ 1. We can show that this occurs in the asymptotic matching region between cases (i) and (ii). By using the behavior of $\Phi(\xi)$ as $\xi \to 0^+$ (which is in the matching region) we conclude that most probability mass occurs at the single point

$$k_1 = k_1(n) = 1 + \left\lfloor \log_2 n + \sqrt{2\log_2 n} - 3/2 \right\rfloor.$$

Thus $h^{\mathrm{PAT}}(k_1(n)-1, n) \approx 0$ while $h_n^{\mathrm{PAT}}(k_1(n), n) \approx 1$. This is true outside of very special subsequences of n, which lead to mass at exactly two points and which we precisely characterized in [5]. The mean thus satisfies

$$\mathrm{E}\left[\mathcal{H}_n^{\mathrm{PAT}}\right] = \log_2 n + \sqrt{2\log_2 n} + O(1), \tag{6}$$

which is smaller than the mean for tries (cf. (5)) by a factor of 1/2. Results similar to (6) were also obtained in [6,7] by probabilistic methods.

In Fig. 2 we illustrate the height distribution occurring at one or two points.

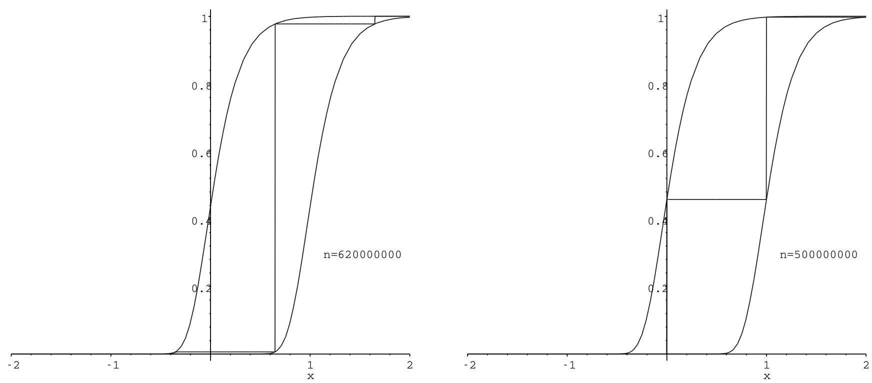

Fig. 2. Asymptotic distributions for the height and their corresponding lower and upper bounds for PATRICIA tries with $n = 6.2 \cdot 10^8$ (one point distribution) and $n = 5 \cdot 10^8$ (two point distribution).

An analogous analysis of (4) for the DST model showed [8] that there are now four ranges of (k, n) that must be analyzed, which includes the three cases for PATRICIA and also the new scale where $k, n \to \infty$ with $k/n \in (0, 1)$. We give below only our results for the mean

$$\mathrm{E}\left[\mathcal{H}_n^{\mathrm{DST}}\right] = \log_2 n + \sqrt{2\log_2 n} - \log_2\left(\sqrt{2\log_2 n}\right) + O(1), \qquad (7)$$

which shows that the DST height is typically smaller than the PATRICIA height by the $O(\log\log n)$ term. The first two terms in (7) were previously obtained in [9] by probability arguments, but his is not enough to distinguish DST from PATRICIA.

3. QUICKSORT algorithm

A popular sorting algorithm, that is taught in even elementary computer science classes, is QUICKSORT. This gives an efficient method of sorting n items. We assume that all possible $n!$ orderings of the items are equally likely and let $\mathcal{L}_n$ be the number of comparisons needed to sort completely the list.

In the algorithm we choose randomly one of the n items and then compare its rank to the remaining $(n-1)$ items. This divides the remaining items into two sublists, and these sublists are then sorted recursively by this method. It is well known [10] that as $n \to \infty$ the mean $\mathrm{E}[\mathcal{L}_n] \sim 2n \log n = O(n \log n)$, while the best case performance is $\sim n \log_2 n$, and the worst case is $\sim n^2/2$. Thus the mean is of the same order as the best case, with a larger constant since $2 > 1/\log 2$. Higher order moments are also readily computable but the full distribution of $\mathcal{L}_n$, $\Pr[\mathcal{L}_n = k]$, seems much harder. Its generating function

$$\mathrm{L}_n(u) = \sum_{k=0}^{\infty} \Pr\left[\mathcal{L}_n = k\right] u^k \tag{8}$$

satisfies the non-linear recurrence

$$\mathrm{L}_{n+1}(u) = \frac{u^n}{n+1} \sum_{i=0}^{n} \mathrm{L}_i(u)\mathrm{L}_{n-i}(u),\ \ \mathrm{L}_0(u) = 1. \tag{9}$$

We wish to study this equation for $n \to \infty$. We found that several different ranges of u lead to different expansions, but focus here only on one specific range of u, which will correspond to that range of k where most of the probability mass accumulates.

If we set $u = 1 + w/n$ (thus $u - 1 = O(n^{-1})$) with

$$\mathrm{L}_n(u) = e^{\mathrm{A}_n(u-1)}\mathrm{G}(n(u-1); n) = e^{\mathrm{A}_n w/n}\mathrm{G}(w; n), \tag{10}$$

$$\mathrm{A}_n = \mathrm{E}[\mathcal{L}_n] = 2(n+1)\left[\sum_{i=1}^{n} \frac{1}{i}\right] - 2n$$

then $\mathrm{G}(w; n) \to \mathrm{G}_0(w)$ as $w \to \infty$ where G_0 satisfies the non-linear integral equation

$$e^{-w}\mathrm{G}_0(x) = \int_0^1 e^{2\phi(x)w}\mathrm{G}_0(wx)\mathrm{G}_0(w - wx)\, dx, \tag{11}$$

$$\phi(x) = x \log x + (1-x)\log(1-x),\ \ \mathrm{G}_0(0) = 1,\ \ \mathrm{G}_0'(0) = 0.$$

Furthermore, $\Pr[\mathcal{L}_n - \mathrm{E}[\mathcal{L}_n] = ny] \sim n^{-1}P(y)$ where $P(y)$ satisfies the double integral equation

$$P(y+1) = \int_0^1 \int_{-\infty}^{\infty} P\left(xt + \frac{y - 2\phi(x)}{2(1-x)}\right) \times P\left(-(1-x)t + \frac{y - 2\phi(x)}{2(1-x)}\right)\, dt\, dx, \tag{12}$$

$$\int_{-\infty}^{\infty} P(y)\,dy = 1,\ \int_{-\infty}^{\infty} yP(y)\,dy = 0.$$

The functions G_0 and P are closely related, in fact $\mathrm{G}_0(w) = \int_{-\infty}^{\infty} e^{wy}P(y)\,dy$ is just the moment generating function of the continuous probability density $P(y)$.

An asymptotic analysis of (11) and (12) yielded [11] the following results for $\mathrm{G}(w)$ as $w \to \pm\infty$:

$$\mathrm{G}_0(w) \sim \frac{2\sqrt{2}}{\sqrt{\pi \log 2}}\sqrt{-w}\exp\left[\left(\frac{1}{\log 2} - 2\right) w\log(-w) + \beta_0 w\right], \quad w \to -\infty \tag{13}$$

$$\mathrm{G}_0(w) \sim \frac{C_*}{w} e^{-w^2} e^{(1-2\gamma-2\log 2)w} \exp\left[\int_1^w \frac{2e^u}{u}\,du\right],\ w \to +\infty. \tag{14}$$

Here γ is the Euler constant, and β_0 and C_* are constants that must be evaluated numerically. The corresponding results for $P(y)$ as $y \to \pm\infty$ (the tails of the limiting QUICKSORT density) are

$$P(y) \sim \frac{1}{\pi e}\sqrt{\frac{2}{a}}\exp\left[\frac{\beta_0 - y}{a} - \frac{a}{e}\exp\left(\frac{\beta_0 - y}{a}\right)\right],\ y \to -\infty, \quad a = 2 - \frac{1}{\log 2}, \tag{15}$$

$$P(y) \sim \frac{C_*}{\sqrt{8\pi}}\sqrt{\frac{y}{1 - 1/w_*}} e^{-w_*^2} e^{-(2\gamma + 2\log 2)w_*} \times \exp\left[-yw_* + \int_1^{w_*} \frac{2e^u}{u}\,du\right],\ y \to +\infty \tag{16}$$

where $w_* = w_*(y)$ is defined implicitly from

$$y = \frac{2}{w_*}e^{w_*};\ w_* \sim \log\left(\frac{y}{2}\right), y \to \infty. \tag{17}$$

It follows that the left tail of $P(y)$ is very thin (decaying as a double exponential) while the right tail is slightly thinner than an exponential (with $\log[P(y)] \sim -y\log y$ as $y \to +\infty$). Our results are much sharper than previous estimates in [12,13,14].

4. Binary search trees

Binary search trees are the most fundamental data structure used for searching. The height distribution $\mathrm{L}_n^k = \mathrm{Prob}\left[\mathcal{H}_h^{\mathrm{BST}} \leqslant k\right]$ satisfies the recurrence

$$\mathrm{L}_{n+1}^{k+1} = \frac{1}{n+1} \sum_{\ell=0}^{n} \mathrm{L}_\ell^k L_{n-\ell}^k, \; k \geqslant 0, \tag{18}$$

with $\mathrm{L}_0^0 = 1$ and $\mathrm{L}_n^0 = 0$ for $n \geqslant 1$. We contrast (18) to the digital tree recurrences in (2)–(4). Some previous analyses of this model for $n \to \infty$ are given in [15,16,17].

In [18] we analyzed (18) for $k, n \to \infty$, identifying the following five asymptotic ranges: (i) $n - k = O(1)$, (ii) $0 < k/n < 1$, (iii) $k/\log n = \nu$ fixed and $\nu > A = 4.311\ldots$, (iv) $k = A \log n + B \log\log n + O(1)$, $B = -\frac{3}{2}\frac{A}{A-1}$, (v) $n2^{-k} = \omega$ fixed and $0 < \omega < 1$, and (vi) $2^k - n = O(1)$. The most interesting case is (iv) where we have $\mathrm{L}_n^k \sim f(\zeta)$, $\zeta = k - A\log n - B \log\log n$ and f satisfies the non-linear integral equation

$$f(\zeta+1) = \int_0^1 f(\zeta - A\log t) f\left(\zeta - A\log(1-t)\right)\, dt, \tag{19}$$

for $\zeta \in \mathbb{R}$. While not being able to solve (19) exactly, matched asymptotics can be used to argue the tail behaviors

$$1 - f(\zeta) \sim c_1 \zeta \exp\left[-\left(1 - A^{-1}\right)\zeta\right], \; \zeta \to +\infty \tag{20}$$

$$f(\zeta) \sim 2\sqrt{\frac{2c_0}{\pi}}\sqrt{\frac{A\log 2}{A\log 2 - 1}}\, e^{-\beta_*\zeta} \exp\left[-c_0 e^{-\beta_*\zeta}\right], \; \zeta \to -\infty \tag{21}$$

where c_0, c_1 are constants that were found numerically, $\beta_* = \log 2/(A\log 2 - 1) = .3486\ldots$ and $A > 1$ satisfies $A\log A - A - A\log 2 + 1 = 0$ (thus $A = 4.311\ldots$).

We therefore see that the right tail is exponential with an additional algebraic factor of ζ, and the left tail is again a very thin double exponential.

5. Binary trees

In Fig. 3 we sketch a typical binary tree with $n = 5$ nodes. Each node has an associated left and right path length. In going from the root of the tree to a given node we take a number of steps to the left and a number of steps to the right. The sum of these over all nodes is the right (resp. left) path length $\mathcal{R}$ (resp. $\mathcal{L}$) for the tree. The (total) path length is $\mathcal{P} = \mathcal{R} + \mathcal{L}$, which

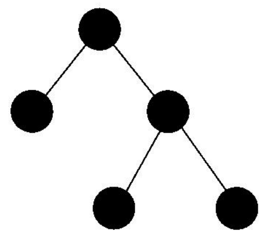

Fig. 3. A sketch of a binary tree with 5 nodes, total path length $\mathcal{P} = 6$, right path length $\mathcal{R} = 4$, and left path length $\mathcal{L} = 2$ (thus $\mathcal{J} = 2$).

is the sum of the depths of the nodes in the tree. We then let $\mathcal{J} = \mathcal{R} - \mathcal{L}$ be the difference between the right and left path lengths.

We let $b(n,p)$ be the number of binary trees with n nodes and total path length p. The generating function $B_n(w) = \sum_{p=0}^{\infty} b(n,p) w^p$ satisfies

$$B_{n+1}(w) = w^n \sum_{k=0}^{n} B_k(w) B_{n-k}(w), \;\; n \geqslant 0 \tag{22}$$

with $B_0(w) = 1$. It was previously established that $B_n(1)$, the total number of trees with n nodes, is the Catalan number $\mathcal{C}_n = \frac{1}{n+1}\binom{2n}{n}$. Also, the fraction of trees with n nodes that have path length p, $b(n,p)/\sum_{p=0}^{\infty} b(n,p)$, follows an Airy distribution [19] with the scaling $p = O(n^{3/2})$. A more difficult problem is to fix the path length p and then study the distribution of the number of nodes, i.e., $b(n,p)/\sum_{n=0}^{\infty} b(n,p)$, or to study the double sequence $b(n,p)$. In [20] we analyzed $b(n,p)$ for the following scales: (i) $p, n \to \infty$ with $p = \binom{n}{2} - O(1)$, (ii) $p = O(n^2)$, (iii) $p = O(n^{3/2})$ (leading to leading order to the Airy distribution), (iv) $p = O(n^{4/3})$, and (v) $p = n \log_2 n + O(n)$. Having a thorough understanding of $b(n,p)$ in all of the different ranges allowed us to first estimate the number of trees (regardless of the number of nodes) that have path length p. Its exponential growth rate takes the form

$$\log\left[\sum_{n=0}^{\infty} b(n,p)\right] = \frac{2p\log^2 2}{\log p}\left[1 - C_0(\log p)^{-2/3} + O\left((\log p)^{-1}\right)\right] \tag{23}$$

where $p \to \infty$ and $C_0 = (2\log 2)^{1/3}|r_0|$, where $r_0 = \max\{z\colon Ai(z) = 0\} = -2.3381\ldots$ is the maximal root of the Airy function. The leading term in (23) was also obtained using combinatorial arguments by Seroussi [21]. To understand the fraction of trees that have n nodes for a fixed path length p we must analyze the asymptotic matching region between cases (iv) and (v) above. Examining carefully the scale $p = n\log_2 n + O[n(\log n)^{1/3}]$ we obtained the Gaussian limit law

$$\frac{b(n,p)}{\sum_{n=0}^{\infty} b(n,p)} \approx \frac{1}{\sqrt{2\pi\mathcal{V}(p)}} \exp\left[-\frac{(n-\mathcal{N}(p))^2}{2\mathcal{V}(p)}\right] \tag{24}$$

where the mean and variance are

$$\mathcal{N}(p) = \frac{p\log 2}{\log p}\left[1 - \frac{2^{4/3}}{3}(\log 2)^{1/3}\frac{|r_0|}{(\log p)^{2/3}} + O\left((\log p)^{-1}\right)\right] \tag{25}$$

$$\mathcal{V}(p) \sim \frac{p}{(\log p)^{5/3}}\frac{2^{1/3}}{9}(\log 2)^{1/3}|r_0|.$$

We next examine binary trees, but now distinguish between the left and right paths. We let $b(n,r,\ell)$ be the number of such trees with right (resp., left) path $= r$ (resp., ℓ). Its double generating function $G_n(w,v) = \sum_{r=0}^{\infty}\sum_{\ell=0}^{\infty} b(n,r,\ell)w^r v^\ell$ satisfies

$$G_{n+1}(w,v) = \sum_{k=0}^{\infty} w^k v^{n-k} G_k(w,v) G_{n-k}(w,v) \tag{26}$$

with $G_0(w,v) = 1$, and we note that (18) and (26) are related by $G_n(w,w) = B_n(w)$. In [22] we analyzed the joint left–right path distribution for $n \to \infty$ with the scaling $\mathcal{P}(n^{3/2})$ and $\mathcal{J}(n^{5/4})$. Here we only discuss the path length difference, whose generating function is $G_n(w,w^{-1})$. The fraction of trees whose path length difference is $\mathcal{J} = n^{5/4}\beta = O(n^{5/4})$ satisfies the limit law

$$\frac{1}{\mathcal{C}_n}\sum_{\ell} b(n,\ell+\mathcal{J},\ell) \sim n^{-5/4} p_-(\beta) \tag{27}$$

as $n \to \infty$, where $p_-(\beta)$ is a continuous density that satisfies $p_-(\beta) = p_-(-\beta)$, $\int_{-\infty}^{\infty} p_-(\beta)\,d\beta = 1$ and has the properties

$$p_-(\beta) \sim \sqrt{\frac{5}{6}}(5\beta)^{1/3}\widetilde{C}\exp\left[-\frac{3}{4}5^{1/3}\beta^{4/3}\right],\ \beta \to \infty,$$

$$\widetilde{C} = .5513\ldots,$$

$$p_-(0) = .4572\ldots,\ p_-''(0) = -.7146\ldots.$$

Also, $p_-(\beta)$ has a unique inflection point for $\beta > 0$, at $\beta = .7589\ldots$ (see Fig. 4). Other recent investigations into the path length difference and its asymptotic properties appear in [23,24]. We also show in [22] that the moment generations function of p_- is

$$\int_{-\infty}^{\infty} e^{\beta\theta} p_-(\beta)\,d\beta = 1 + \sqrt{\pi}\,\overline{\mathrm{H}}(\theta)$$

where $\overline{\mathrm{H}}(\theta) = \theta^{6/5}\Delta(\theta^{4/5}) = B^{3/2}\Delta(B)$ $(B = \theta^{4/5})$ and $\Delta(B)$ satisfies the non-linear integral equation

$$\int_0^B \Delta(\xi)\Delta(B-\xi)\,d\xi + 2B^2\Delta(B) + 2\sqrt{\frac{B}{\pi}} = \frac{4}{\sqrt{\pi}}\int_0^B \frac{\Delta'(\xi)}{\sqrt{B-\xi}}\,d\xi. \quad (28)$$

This characterizes the moment generating function for θ real and positive. For θ purely imaginary we let $\theta = ix$, $y = x^{4/5}$, $\overline{\mathrm{H}}(ix) = -y^{3/2}\Lambda(y)$ and $\mathrm{U}(\phi) = \int_0^\infty e^{-y\theta}\Lambda(y)\,dy$. Then $\mathrm{U}(\phi)$ satisfies the non-linear ODE

$$2\mathrm{U}''(\phi) + \mathrm{U}^2(\phi) + 4\sqrt{\phi}\,\mathrm{U}(\phi) = \phi^{-3/2}, \quad (29)$$

which is closely related to the first Painlevè transcendent.

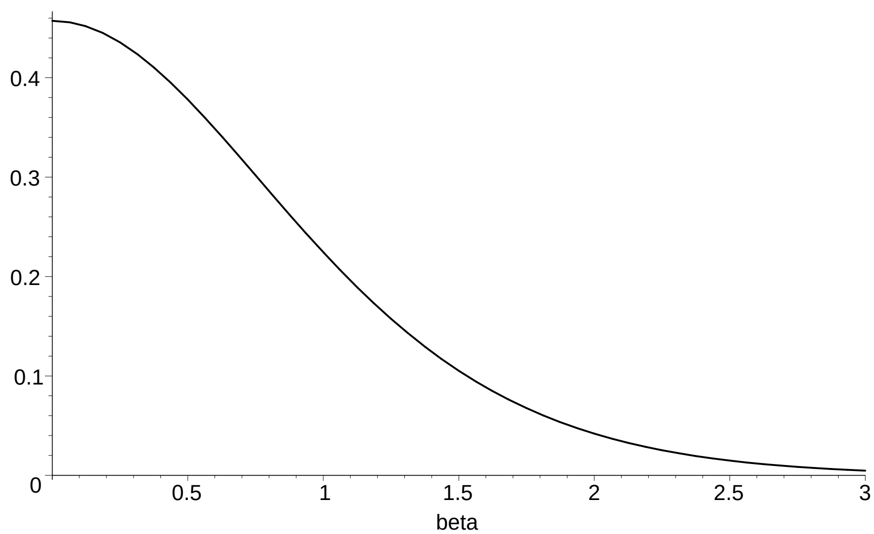

Fig. 4. The density $p_-(\beta)$ for $\beta \in [0, 3]$.

6. Summary

We have shown that many classic problems in theoretical computer science and combinatorics involve solving non-linear recurrences, where the non-linearity takes the form a convolution sum, as in (3), (4), (9), (18), (22) or (26). In each case an exact solution seems out of the question, but a singular perturbation analysis is fruitful and yields valuable asymptotic results. Thus applied math techniques which are now fairly standard tools, such as WKB expansions and matched asymptotics, have proved useful in a new area of applications.

Acknowledgements

The work of C.K. was partially supported by NSF grant DMS-0503745; the work of W.S. by NSF grants DMS-0503742, DMS-0800568, and CCF-0513636, NSA grant H 98230-08-1-0092, NIH grant R01 GM068959-01, and AFOSR grant FA8655-07-1-3071.

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