[60] Homework 2. *The Moment Methods*

[10] Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables distributed as the standard normal distribution $N(0,1)$, that is, the density $f(x)$ of X_1 is

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

for $-\infty < x < \infty$. Find a_n such that

$$\max\{X_1, \ldots, X_n\} \to a_n \quad \text{(pr.)}.$$

Hint. It is known that for $x \to \infty$

$$\Pr\{X > x\} \sim \frac{1}{\sqrt{2\pi}x} e^{-x^2/2}$$

when X is distributed as $N(0,1)$.

[10] Prove the following alternative formulation of the second moment method. *Let* $\{A_i\}_{i=1}^n$ *be a set of identically distributed events such that* $n\Pr\{A_i\} \to \infty$. *If*

$$\sum_{i \neq j} \frac{\Pr\{A_i | A_j\}}{n\Pr\{A_i\}} \to 1$$

as $n \to \infty$, *then* $\Pr\{\bigcup_{i=1}^n A_i\} \to 1$.

[20] Consider the following "comparison" of two random strings. Let X_1^n and Y_1^m of length n and $m < n$, respectively, be generated by a binary memoryless source with probabilities p and $q = 1 - p$ of the two symbols. Define C_i to be the number matches between X_i^{i+m-1} and Y; that is,

$$C_i = \sum_{j=1}^{m} equal(X_{i+j-1}, Y_j),$$

where $equal(x, y)$ is one when $x = y$ and zero otherwise. Define $M_{mn} = \max_{1 \le i \le n-m+1}\{C_i\}$. Prove that *if* $\log n = o(m)$ *then*

$$\lim_{n\to\infty} \frac{M_{m,n}}{m} = P_2 \qquad \text{(a.s.)},$$

where $P_2 = p^2 + q^2$.

[20] Consider the fill-up level F_n (the maximum level in a tree that is full) in a trie built over n binary strings generated by an independent memoryless source. Prove the following result that was first established by Pittel in 1985.

Theorem 1 (Pittel, 1985) *Consider a trie built over* n *binary strings generated by* n *independent memoryless sources with* p *being the probability of generating a "0", and* $q = 1 - p$. *Then*

$$\lim_{n\to\infty} \frac{F_n}{\log n} = \frac{1}{\log p_{\min}^{-1}} \qquad \text{(pr.)},$$

where $p_{\min} = \min\{p, 1-p\}$.