[40] Homework 1. *Algorithms and Probability*

[10] In this problem assume that a binary trie is built from n strings. Using the external profile B_n^k (number of external nodes at level k) and the internal profile I_n^k (number of internal nodes at level k), define all other trie parameters that we discussed in class (e.g., depth, total path length, height, size, shortest path, fillup).

[10] Consider a code over a V-ary alphabet. Find the optimal average code length by solving the following optimization problem:

$$\begin{aligned} &\min_{l_1,\ldots,l_m} L = \textstyle\sum_i p_i l_i \\ &\textbf{subject to:} \quad \textstyle\sum_i V^{-l_i} \leq 1, \end{aligned}$$

where p_i is the probability of symbol i, and l_i are lengths (we shall ignore that l_i are integers), and V is the size of the underlying alphabet.

[10] For 5 points prove the following for a discrete nonnegative random variable X:

$$\mathbf{E}[X] = \sum_{m \geq 0} \Pr\{X > m\}.$$

For additional 5 points, prove the following general formula for the $(k+1)$st moment of a discrete nonnegative random variable X:

$$\mathbf{E}[X^{k+1}] = \sum_{m \geq 0} \Pr\{X > m\} \sum_{i=0}^{k} (m+1)^{k-i} m^i.$$

[10] Construct *your counterexample* to either

- $X_n \to X$ (pr.) implies $X_n \to X$ (a.s.)

or

- $X_n \to X$ (pr.) implies convergence in mean, that is, that $\mathbf{E}[X_n - X] \to 0$.