Module 4: Mathematical Induction

Theme 1: Principle of Mathematical Induction

Mathematical induction is used to prove statements about natural numbers. As students may remember, we can write such a statement as a predicate $P(n)$ where the universe of discourse for n is the set of natural numbers $\mathbf{N} = \{1, 2, \ldots\}$.

Example 1: Here are some examples of what we mean by $P(n)$:

$$
\begin{aligned}
P(n) &\equiv 1 + 2 + \cdots + n = \frac{n(n+1)}{2}, \quad \forall n \in \mathbf{N}, \\
P(n) &\equiv 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}, \quad \forall n \in \mathbf{N}, \\
P(n) &\equiv 1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2, \quad \forall n \in \mathbf{N}, \\
P(n) &\equiv \sum_{i=0}^{n} a^i = \frac{a^{n+1} - 1}{a - 1}, \qquad \forall n \in \mathbf{N}, \\
P(n) &\equiv n! > 2^n \quad \text{for} \quad n \geq 4, \\
P(n) &\equiv \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} < 1, \quad n \geq 1
\end{aligned}
$$

where $\equiv$ means "logically equivalent".

The first three expressions above provide closed-form formulas for the sum of n consecutive positive integers, the sum of squares of n consecutive positive integers, and the sum of cubes of n consecutive positive integers, respectively. The fourth expression is the sum of the first n terms in the geometric series and we studied it already in Module 2. The last two expressions are useful inequalities for factorial and the sum of negative powers of 2.

Every statement $P(n)$ above is about natural numbers or a subset of natural numbers (e.g., for $n \geq 4$). How can we prove such statements? Consider the first example above regarding the sum of the first n consecutive positive integers. We can easily verify that $P(n)$ is true for *some* selected n. Indeed,

$$
\begin{aligned}
P(1) &\quad \text{is true since} \quad 1 = \frac{1 \cdot 2}{2}, \\
P(2) &\quad \text{is true since} \quad 1 + 2 = \frac{2 \cdot 3}{2}, \\
P(3) &\quad \text{is true since} \quad 1 + 2 + 3 = \frac{3 \cdot 4}{2}.
\end{aligned}
$$

But how can we prove that $P(n)$ is true for **all** $n \in \mathbf{N}$?

The principle of mathematical induction (PMI) can be used to prove statements about natural numbers.

> **The principle of mathematical induction**: Let A be a set of natural numbers such that the following two properties hold:
>
> (1) $1 \in A$;
>
> (2) for every natural number n
>
> $$\text{if } \ n \in A \quad \text{then} \quad n+1 \in A. \tag{1}$$
>
> Then
>
> $$A = \mathbf{N} = \{1, 2, \ldots\},$$
>
> that is, A contains **all** natural numbers.

How is it related to proving statements like $P(n)$ above? Let us define

$$A = \{n : \ P(n) \text{ is true for } n\},$$

that is, A is the set of natural numbers for which P is true. The goal is to show that A is the same as the set of all natural numbers, that is, $A = \mathbf{N}$. Imagine that one verifies that $P(1)$ is true. Then we can set $A = \{1\}$. Let's now assume that one can prove step (2) of PMI (that we shall call the *inductive step*). Thus since we know that $1 \in A$, and we know the inductive step is valid, say for $n = 1$, we conclude that $2 \in A$. Therefore, $A = \{1, 2\}$, that is, $P(1)$ and $P(2)$ are true. But using again the inductive step, we conclude that $3 \in A$. Etc. Actually, PMI allows us to replace the imprecise "etc" by $A = \mathbf{N}$, that is, $P(n)$ is true for *all* natural numbers!

But why is PMI true, in the first place? We demonstrate its truth using the proof by contradiction. Suppose that (1) and (2) of PMI hold but A is not equal to $\mathbf{N}$. Hence, it must be at least one natural number is omitted from $\mathbf{N}$. Let n_0 be the *first* number (smallest) among $1, 2, \ldots$ omitted from $\mathbf{N}$. We know that n_0 cannot be 1 since we assumed that $1 \in A$ by (1) of PMI. But by our construction, $n_0 - 1 \in A$. Then by step (2) of PMI we must conclude that $n_0 \in A$, which is the desired contradiction. Therefore, $A = \mathbf{N}$.

Let us introduce some additional notation. The first step (1) of PMI is called the **basis step**, while the second step is known as the **inductive step**. It is usually trivial to verify the basis step, and most work has to be done to prove the inductive step. We shall illustrate it on the following example.

Example 2: Prove the first identity above about the sum of n consecutive natural numbers, that is,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad n \geq 1.$$

In this case, the property $P(n)$ is a predicate saying that the above is true for n, and

$$A = \{n : \text{ identity above is true for } n \in \mathbf{N}\}.$$

We prove $P(n)$ is true for all n (i.e., $A = \mathbf{N}$) using PMI. We need to go through the basis step and the inductive step.

Basis Step: We must prove $P(1)$ is true, but this was already established before.

Inductive Step: We now assume that $P(n)$ is true for a *fixed* but *arbitrary* n. The above assumption is called the **inductive hypothesis** and in our case it takes the following form

$$S_n := \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

for arbitrary n. (In the above symbol := means equal by definition.) The reader must understand that the statement immediately above and the statement that we want to prove are not the same, even if they look alike. In the statement above we assume that the identity to be proved (about *all* sums of the first n consecutive natural numbers) is true for one value of n (but an arbitrary one).

We now perform the inductive step. We must establish the inductive step, that is, to show that the formula for S_n above implies that

$$S_{n+1} = \sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$$

is true, too. Observe that above we replaced n by $n+1$ (on the left-hand side of the equation as well as on the right-hand side). Indeed, we have

$$\begin{aligned} S_{n+1} &= 1 + 2 + \cdots + n + (n+1) = \sum_{i=1}^{n} i + (n+1) \\ &= \frac{n(n+1)}{2} + (n+1) \\ &= (n+1)\left(\frac{n}{2} + 1\right) \\ &= \frac{(n+1)(n+2)}{2}, \end{aligned}$$

where in the second line above we invoked the induction hypothesis, in the third line we factored out the term $(n+1)$, and then added what is left. This is exactly what we need to prove the inductive step.

But, there is actually another, direct, proof originally proposed by the 18th century mathematician Carl Friedrich Gauss. Let, as before, $S_n = \sum_{i=1}^{n} i$. We write the sum S_n twice one starting the sum from 1 up to n, and the second time starting from n down to 1. Then, we add the individual elements vertically. Here is what comes out:

$$\begin{array}{ccccccccccc} S_n & = & 1 & + & 2 & + & \cdots & + & (n-1) & + & n \\ S_n & = & n & + & n-1 & + & \cdots & + & 2 & + & 1 \\ \hline 2S_n & = & (n+1) & + & (n+1) & + & \cdots & + & (n+1) & + & (n+1) \end{array}$$

Since there are n terms $(n+1)$ in the bottom line, we prove that

$$2S_n = n(n+1).$$

Again, we recover the same identity.

Exercise 4A: Using mathematical induction prove that

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Exercise 4B: Using mathematical induction prove that

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

Induction on a Subset of Natural Numbers

In the PMI discussed above in the first step we assumed that $1 \in A$, however, if we start the induction from another natural number, say k, then it holds for all $n \geq k$. This is shown in the next example.

Example 3: Prove that

$$n! > 2^n \quad \text{for} \quad n \geq 4.$$

We recall that $n! = 1 \cdot 2 \cdot 3 \cdots n = (n-1)!n$. We are asked to prove the above inequality only for $n \geq 4$. Thus let

$$P(n) = \{n! > 2^n \text{ is true for } n \geq 4\}.$$

We first check that $4! = 24 > 2^4 = 16$, thus $P(4)$ is true. (Observe that $P(3)$ is not true.) Now assume this statement is true for arbitrary $n \geq 4$. We must prove that

$$(n+1)! > 2^{n+1}$$

for $n \geq 4$. This is easy since

$$\begin{aligned} (n+1)! &= n!(n+1) \\ &> 2^n(n+1) \\ &> 2^{n+1}. \end{aligned}$$

The first inequality follows from the induction hypothesis $n! \geq 2^n$ while the second identity is a consequence of $(n+1) > 2$ for $n \geq 4$. This proves the desired inequality for all $n \geq 4$.

The next two examples require a little bit of work before the induction can be applied.

Example 4: *Bernoulli's inequality*. We shall prove the following result.

Theorem 1 *If n is a natural number and $1+x>0$, then*

$$(1+x)^n \geq 1+nx. \tag{2}$$

Proof. The proof is by induction. In the basis step, we assume $n=1$ and verify that $(1+x)^n \geq 1+nx$ is true for $1+x>0$. Now, we assume (inductive hypothesis) that $(1+x)^n \geq 1+nx$ is true for an arbitrary n, and we must prove that

$$(1+x)^{n+1} \geq 1+(n+1)x$$

for $1+x>0$. To prove this we proceed as follows:

$$\begin{array}{lcll}
(1+x)^{n+1} & = & (1+x)^n(1+x) & \\
& \stackrel{\text{by induction}}{\geq} & (1+nx)(1+x) & \text{since} \quad 1+x>0 \\
& = & 1+(n+1)x+nx^2 & \\
& \geq & 1+(n+1)x, &
\end{array}$$

where the first inequality is a consequence of the induction assumption (i.e., we **know** that $(1+x)^n \geq 1+nx$ so we can replace $(1+x)^n$ by $1+nx$ because $1+x>0$; observe that if $1+x<0$, then we had to reverse the inequality sign[1]). The next step is simple algebra, while the last step follows from the fact that nx^2 is nonnegative; it doesn't matter what the value of x, because $a+nx^2 \geq a$ for any a. This proves the theorem.

Example 5: Let us prove that

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}<1 \tag{3}$$

for $n \geq 1$. We prove it by induction. The first step for $n=1$ is easy to check, so we concentrate on the inductive step. We adopt the inductive hypothesis, which in this case is

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}<1,$$

and must prove that

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}+\frac{1}{2^{n+1}}<1.$$

A natural approach fails. If we invoke the induction hypothesis to the first n terms of the above, we will get

$$1+\frac{1}{2^{n+1}}$$

[1] Think of $3 \leq 5$ that after multiplying by -3 becomes $-9 \geq -15$.

which does *not* imply that it is less than or equal to 1 since $1/2^{n+1} > 0$. Here's how we proceed

$$\begin{array}{rcl} \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}+\frac{1}{2^{n+1}} & = & \frac{1}{2}+\frac{1}{2}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}\right) \\ & \overset{\text{by induction}}{<} & \frac{1}{2}+\frac{1}{2} \\ & \leq & 1, \end{array}$$

where in the first line on the right-hand side we factor $1/2$ and observe that what is left in the parenthesis must be smaller than 1 by the inductive hypothesis. The rest is simple algebra. This proves the inequality.

In some cases, we must use a **generalized** mathematical induction that we formulate in a little different form than before.

> *If a statement $P(n)$ is true for $n = 1$, and if for every $n > 1$, the truth of $P(n)$ for all natural numbers $< n$ implies the truth of $P(n)$ for n, then $P(n)$ is true for all natural numbers.*

The only difference between the basis PMI and the above is that in in the inductive step of the generalized mathematical induction we assume that the truth of $P(1), P(2), \ldots, P(n-1)$ implies the truth of $P(n)$. In other words, the second step of the generalized PMI can be written as

$$\{1, 2, \ldots, n-1\} \subseteq A \quad \text{then} \quad n \in A$$

where A is the set defined in the original PMI.

Recurrences[2]

We now apply mathematical induction to establish some facts about recurrences. We come back to recurrences in Theme 3.

We start with an example that illustrates an application of the generalized mathematical induction.

Example 6: Let us define $T(0) = 1$ and then

$$T(n) = 1 + \frac{2}{n}\sum_{i=0}^{n-1} T(i), \qquad n \geq 1. \tag{4}$$

This is an example of a **recurrence** that we shall study in some details later in this module. Observe that we can compute consecutive values $T(1), T(2)$ and so on from the recurrence itself. For example,

$$\begin{array}{rcl} T(1) & = & 1 + \frac{2}{1}T(0) = 3, \\ T(2) & + & 1 + \frac{2}{2}\left(T(0) + T(1)\right) = 5, \\ T(3) & = & 7. \end{array}$$

[2]This subsection can be omitted in the first reading.

But can we guess how $T(n)$ grows for arbitrary n. In the table below we computed some numerical values of $T(n)$ and compared them to the growth of n and n^2.

n	$T(n)$	n^2
1	3	1
3	7	9
6	13	36
9	19	81
12	25	144
15	31	225
18	37	324

From this table we should observe that $T(n)$ grows faster than n and much slower than n^2. Let us then conjecture that[3]

$$T(n) \leq 4n \log_2 n, \qquad n \geq 2. \tag{5}$$

We now use mathematical induction to prove this guess. Observe that $T(2) = 5 \leq 8 \log_2 2 = 8$ (since $\log_2 2 = 1$), but $T(1) = 3 > 4 \log_2 1 = 0$, therefore, we must start the induction from $n = 2$.

To carry out the inductive step we shall assume that for all $j \leq n - 1$ the above guess is is true. We now prove that this guess is also true for n. Indeed,

$$\begin{aligned}
T(n) \quad &= \quad 1 + \frac{2}{n} + \frac{6}{n} + \frac{2}{n} \sum_{j=2}^{n-1} T(j) \\
&\overset{\text{induction}}{=} \quad 1 + \frac{8}{n} + \frac{2}{n} \sum_{j=2}^{n-1} 4j \log_2 j \\
&\overset{\log j \leq \log n}{=} \quad 1 + \frac{8}{n} + \frac{8}{n} \log_2 n \sum_{j=2}^{n-1} j \\
&= \quad 1 + \frac{8}{n} + \frac{8}{n} \log_2 n \left(\frac{n(n-1)}{2} - 1 \right) \\
&= \quad 1 + \frac{8}{n} - \frac{8}{n} \log_2 n - 4 \log_2 n + 4n \log_2 n \\
&\leq \quad 1 - 4 \log_2 n + 4n \log_2 n \qquad \text{since} \quad \frac{8}{n} \leq \frac{8}{n} \log_2 n, \;\; n \geq 2, \\
&\leq \quad 4n \log_2 n, \qquad n \geq 2,
\end{aligned}$$

where (i) in the first step we use the recurrence and extract the first two terms from the sum; (ii) in the second line we use the induction assumption in its general form and bound every $T(j)$ by $4j \log_2 j$

[3]We recall that $y = \log_b x$ is a logarithm to base b of x, that is, it is the exponent to which b must be raised to obtain x as shown here $b^y = x$.

for $2 \leq j < n$; (iii) in the third line we observe that $\log_2 j \leq \log_2 n$ (since $j \leq n$) and factor the constant term $\log_2 n$ in front of the sum; (iv) in the fourth line we apply the formula for the sum of the first $n-1$ consecutive integers proved in Example 2; (v) the fifth line is simple algebra; (vi) in the sixth line we observe that

$$\frac{8}{n} \leq \frac{8}{n} \log_2 n$$

for $n \geq 2$ and therefore cancel out the terms $8/n$; finally the last inequality follows from the fact that $1 - 4\log_2 n \leq 0$ for $n \geq 2$. (As we said at the beginning of this subsection, if this derivation is too involved in the first reading, the student can move forward to the next section since it will not be used in the forthcoming discussion.)

Theme 2: Newton's Summation Formula

From high school we know that

$$\begin{aligned}
(a+b)^2 &= a^2+2ab+b^2,\\
(a+b)^3 &= a^3+3a^2b+3ab^2+b^3,\\
(a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4.
\end{aligned}$$

But what about a formula for

$$(a+b)^n$$

for arbitrary n. We shall derive it here, and it is called *Newton's summation formula*.

Before we handle the general case of $(a+b)^n$, we must introduce some new notation. In particular, **binomial coefficients** also known as *Newton's coefficients*. We define for natural k and $n \geq k$

$$C(n,k) := \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!},$$

where we remind that $n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$. By definition $0! = 1$. In literature the Newton coefficients $C(n,k)$ are also denoted as

$$C_n^r := \left(\begin{array}{c} n \\ k \end{array} \right).$$

From the definition we immediately find

$$\begin{aligned}
C(n,0) &= 1\\
C(n,1) &= n\\
C(n,k) &= C(n,n-k).
\end{aligned}$$

But we shall also need the following lemma.

Lemma 1 *For natural k and n*

$$C(n,k) = C(n-1,k) + C(n-1,k-1). \tag{6}$$

Proof. We give a direct proof. Observe that

$$\begin{aligned}
C(n-1,k-1)+C(n-1,k-1) &= \frac{(n-1)!}{k!(n-k-1)!} + \frac{(n-1)!}{(k-1)!(n-k)!}\\
&= \frac{(n-1)!(n-k)}{k!(n-k-1)!(n-k)} + \frac{(n-1)!k}{(k-1)!k(n-k)!}\\
&= \frac{(n-1)!}{k!(n-k)!}(n-k+k)\\
&= \frac{n!}{k!(n-k)!}\\
&= C(n,k)
\end{aligned}$$

where in the second line we multiply and divide the first term by $n-k$ and the second term by k. Then we factorize $\frac{(n-1)!}{k!(n-k)!}$ and after some simple algebra obtain the desired result.

Now, we are ready to formulate and prove the Newton summation formula.

Theorem 2 *For any natural* n

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}. \tag{7}$$

Proof.[4] The proof is by induction with respect to n. The basis step for $n = 1$ is easy to check since $C(1,0) = C(1,1) = 1$.

We now start the inductive step, and postulate that if (7) is true for arbitrary n, then

$$(a+b)^{n+1} = \sum_{k=0}^{n+1} C(n+1,k) a^k b^{n+1-k}$$

must be true. We proceed as follows

$$\begin{aligned}
(a+b)^{n+1} &= (a+b)^n(a+b) = \sum_{k=0}^{n} C(n,k)a^{k+1}b^{n-k} + \sum_{k=0}^{n} C(n,k)a^k b^{n+1-k} \\
&= C(n+1,0)b^{n+1} + ab^n\left[C(n,0) + C(n,1)\right] \\
&+ a^2b^{n-1}\left[C(n,1) + C(n,2)\right] + \cdots \\
&+ a^kb^{n-k+1}\left[C(n,k-1) + C(n,k)\right] + \cdots \\
&+ a^nb\left[C(n,n) + C(n,n-1)\right] + C(n+1,n+1)a^{n+1} \\
&\overset{\text{Lemma1}}{=} \sum_{k=0}^{n+1} C(n+1,k)a^k b^{n+1-k}.
\end{aligned}$$

In the first line above we use mathematical induction and then multiply out. In the next few lines we group terms with the same power, that is $a^i b^{n+1-i}$ for all i. Finally, we applied Lemma 1 (i,e., $C(n,k) + C(n,k-1) = C(n+1,k)$) to finish the derivation.

Exercise 4C: Apply Newton's formula to the following

$$(1+x^2)^4.$$

The above formula can lead to surprisingly interesting identities. Here are two of them

$$2^n = \sum_{i=0}^{n} \binom{n}{i} \tag{8}$$

$$0 = \sum_{i=0}^{n} \binom{n}{i} (-1)^i \tag{9}$$

[4]The proof can be omitted in the first reading.

The first identity follows immediately from the Newton formula applied to

$$(1+1)^n,$$

while the second follows from

$$(1-1)^n.$$

We shall re-derive these identities using combinatorial arguments in one of the next modules.

Theme 3: Recursion and Recurrences

Sometimes it is difficult to define an object explicitly. In such cases, it is better to define this object in terms of itself but of a smaller size. (Actually, we have seen this principle at work in mathematical induction.) This process is called **recursion** and often it is described mathematically by a **recurrence**.

Example 7: Define $a_0 = 1$ and for $n \geq 0$

$$a_{n+1} = 2a_n.$$

Let's see what we get. We first compute some sample values:

$$\begin{aligned} a_1 &= 2, \\ a_2 &= 2a_1 = 4, \\ a_3 &= 2a_2 = 2(2a_1) = 8 = 2^3, \\ a_4 &= 2a_3 = 2(2a_2) = 2(2(2a_1)) = 2^4. \end{aligned}$$

Based on this numerical evidence, we conjecture that $a_n = 2^n$. We can prove it using mathematical induction. But, in this it is easier to give a direct proof that is called **telescoping**. We proceed as follows:

$$a_{n+1} = 2a_n = 2 \cdot 2a_{n-1} = 2^2 \cdot 2a_{n-2} = \cdots 2^{i+1} a_{n-(n-i)} = \cdots = 2^n a_0 = 2^n.$$

In the above we successively used the recurrence $a_i = 2a_{i-1}$ until we reached the initial value a_0 and we know that $a_0 = 1$. Observe that without knowing a_0 we can neither start the recurrence nor finish it.

Exercise 4D: Derive an explicit formula for the following recurrence for $n \geq 1$

$$a_n = 3a_{n-1}$$

with $a_0 = 1$.

We can define some other functions recursively. For example, $F(n) = n!$ can be defined recursively as follows

$$\begin{aligned} F(0) &= 1, \\ F(n+1) &= (n+1)F(n), \qquad n \geq 0. \end{aligned}$$

Furthermore, let

$$S_n = \sum_{k=0}^{n} a_k$$

where $\{a_k\}_{k=0}^n$ is a given sequence. For a computer to understand such a sum, we must define it recursively. For example, we can do it this way

$$\begin{aligned} S_0 &= a_0, \\ S_{n+1} &= S_n + a_{n+1}, \quad n \geq 0. \end{aligned}$$

But, let us consider a more general recurrences. We underline that in order to start a recurrence we must define some initial values, and to provide a "method" how to compute the next value. Consider the following recurrence

$$\begin{aligned} a_0 &= 1, \\ a_n &= a_{n-1} + 2^n, \qquad n \geq 1. \end{aligned}$$

This recurrence starts with

$$1, 3, 7, 15, \ldots$$

but what is a general formula for a_n? Let us move the term a_{n-1} to the other side of the recurrence and write down all the values as follows

$$\begin{aligned} a_1 - a_0 &= 2 \\ a_2 - a_1 &= 2^2 \\ a_3 - a_2 &= 2^3 \\ &\ldots \\ a_i - a_{i-1} &= 2^i \\ &\ldots \\ a_{n-1} - a_{n-2} &= 2^{n-1} \\ a_n - a_{n-1} &= 2^n. \end{aligned}$$

Now, when we add all these equations together most of them will cancel out (we say that the sum *telescopes*) except a_n and a_0 giving us

$$a_n - a_0 = \sum_{i=1}^{n} 2^i,$$

which is the same as saying

$$a_n = \sum_{i=0}^{n} 2^i. \tag{10}$$

Is this better than the original recurrence? Not yet since we must compute the sum. Actually, in Module 2 we defined the geometric progression as follows

$$b_n = r^n, \quad n = 0, 1, \ldots$$

and we derived

$$S_n := \sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}, \qquad r \neq 1. \tag{11}$$

Actually, we shall re-prove this formula using mathematical induction. It is easy to check its truth for $n = 0$ (the basis step). Let us move to the inductive step. We first assume that the statement above is true for arbitrary n, and we try to prove that this would imply that

$$S_{n+1} = \sum_{i=0}^{n+1} r^i = \frac{1-r^{n+2}}{1-r}.$$

We proceed as follows

$$\begin{aligned} S_{n+1} &= \sum_{i=0}^{n+1} r^i \\ &= \sum_{i=0}^{n} r^i + r^{n+1} \\ &\overset{\text{induction}}{=} \frac{1-r^{n+1}}{1-r} + r^{n+1} \\ &= \frac{1-r^{n+1}+r^{n+1}-r^{n+2}}{1-r}, \end{aligned}$$

where in the second line we extract the last term from the sum and write it separately as r^{n+1}, then in the next line we apply to the first sum the inductive hypothesis, and finally after some algebra we prove the desired formula.

Now, we can return to (10) to conclude that

$$a_n = \sum_{i=0}^{n} = 2^{n+1} - 1.$$

Let us solve some more recurrences. This is the only way to learn how to handle them. Let $b_0 = 0$ and

$$b_n = b_{n-1} + n, \qquad n \geq 1.$$

We do the following telescoping

$$\begin{aligned} b_n &= b_{n-1} + n \\ &= (b_{n-2} + n - 1) + n \\ &= b_{n-3} + (n-2) + (n-1) + n \\ \ldots & \\ &= b_i + (i+1) + (i+2) + \cdots + n \\ \ldots & \end{aligned}$$

$$
\begin{aligned}
&= b_0 + 1 + 2 + 3 \cdots + (n-1) + n \\
&= \sum_{i=1}^{n} i \\
&= \frac{n(n+1)}{2}
\end{aligned}
$$

where in the second line we substitute $b_{n-1} = b - n - 2 + (n-1)$, in the third line we start observing a pattern in which b_i is followed by the sum of the first $i+1$ natural numbers. Then we apply the sum of n consecutive natural numbers derived in Example 2. In every step of the above derivation we used the recurrence itself to reduce it until we reach the value that we know, that is, b_0. We can do it since in step i we know that $b_i = b_{i-1} + i$.

Consider now a more complicated recurrence:[5]

$$
\begin{aligned}
c_0 &= 0, \\
c_n &= 2c_{n-1} + n.
\end{aligned}
$$

Let us start the telescoping process and try to find a general pattern. We have

$$
\begin{aligned}
c_n &= 2c_{n-1} + n \\
&= 2(2c_{n-2} + n - 1) + n = 2^2 c_{n-2} + 2(n-1) + n \\
&= 2^2(2c_{n-3} + n - 2) + 2(n-1) + n \\
&= 2^3 c_{n-3} + 2^2(n-2) + 2(n-1) + n \\
&= 2^3(2c_{n-4} + n - 3) + 2^2(n-2) + 2(n-1) + n \\
&= 2^4 c_{n-4} + 2^3(n-3) + 2^2(n-2) + 2(n-1) + n \\
\cdots \\
&= 2^i c_{n-i} + 2^i(n-i) + 2^{i-1} c_{n-i+1} + \cdots + 2^2(n-2) + 2(n-1) + n \\
\cdots \\
&= 2^n c_0 + 2^{n-1}[n-(n-1)] + 2^{n-2}[n-(n-2)] + \cdots + 2^2(n-2) + 2(n-1) + n \\
&= \sum_{k=0}^{n-1} 2^k (n-k).
\end{aligned}
$$

In the second line above, we substitute c_{n-1} by $2c_{n-2} + n - 1$ and observe that the "additive term" is now $n + 2(n-1)$ (the additive term is the one that does not involve c_i). After another substitution the additive term is enlarged to $2^2(n-2) + 2(n-1) + n$. Now you should be able to see the pattern which becomes $2^i(n-i) + 2^{i-1}c_{n-i+1} + \cdots + 2^2(n-2) + 2(n-1) + n$. After the last substitution the additive term is finally $2^{n-1}[n-(n-1)] + 2^{n-2}[n-(n-2)] + \cdots + 2^2(n-2) + 2(n-1) + n$.

[5]The next two examples can be omitted in the first reading.

Now to finish the recurrence we must find a formula for the following sum

$$c_n = \sum_{k=0}^{n-1} 2^k(n-k) = n\sum_{k=0}^{n-1} 2^k - \sum_{k=0}^{n-1} k2^k.$$

Observe that in the first sum we could factorize n since the summation is over k, thus n is fixed. After this observation, the first sum is easy to estimate. We just found above that it is equal to $2^n - 1$. But the second one is harder. To estimate it we first observe that

$$2^{k+1} - 2^k = 2^k(2-1) = 2^k.$$

Then

$$\begin{aligned}
S_n &= \sum_{k=0}^{n} k2^k = \sum_{k=1}^{n} k2^k \\
&= \sum_{k=1}^{n} k(2^{k+1} - 2^k) \\
&= \sum_{k=1}^{n} k2^{k+1} - \sum_{k=1}^{n} k2^k \\
&\overset{(A)}{=} \sum_{k=1}^{n} k2^{k+1} - \sum_{k=0}^{n-1} (k+1)2^{k+1} \\
&= \sum_{k=0}^{n} k2^{k+1} - \sum_{k=0}^{n-1} k2^{k+1} - \sum_{k=0}^{n} 2^{k+1} \\
&\overset{(B)}{=} \sum_{k=0}^{n-1} k2^{k+1} + n2^{n+1} - \sum_{k=0}^{n-1} k2^{k+1} - \sum_{k=0}^{n} 2^{k+1} \\
&\overset{(C)}{=} n2^{n+1} - 0 - (2^{n+1} - 2) \\
&= (n-1)2^{n+1} + 2.
\end{aligned}$$

In line (A) we change the index of summation from k to $k+1$, in line (B) we expand the first sum and observe that it cancels out the second sum, finally in line (C) we apply the geometric sum that we already discussed above.

Coming back to the recurrence, putting everything together we have

$$c_n = 2^{n+1} - n - 2$$

which is our final answer. Uffff . . . it was not that hard.

Finally, we solve one non-linear recurrence. Consider the following[6]

$$a_n = 3a_{n-1}^2, \quad n \geq 1$$

[6]The forthcoming analysis may be completely omitted, and come back only if a student is interested in a better understanding of non-linear recurrences.

where $a_0 = 1$. It is a non-linear recurrence since a_{n-1} is squared. Telescoping might be difficult for this recurrence. So we first simplify it. Define $b_n = \log_2 a_n$. Then we have

$$\begin{aligned} b_0 &= 1 \\ b_n &= 2b_{n-1} + \log_2 3 \end{aligned}$$

since $\log(xy) = \log x + \log y$. Now we are on familiar grounds. Using telescoping we find

$$b_n = (2^n - 1)\log_2 3,$$

which implies

$$a_n = 2^{(2^n-1)\log_2 3} = 3^{2^n-1}.$$

Finally, we should say it is always a good idea to verify numerically our solution by comparing its some initial values to the values computed from the recurrence itself.