Module 1: Basic Logic

Theme 1: Propositions

English sentences are either true or false or neither. Consider the following sentences:

1. Warsaw is the capital of Poland.

2. $2 + 5 = 3$.

3. How are you?

The first sentence is true, the second is false, while the last one is neither true nor false. A statement that is either **true** or **false** but not both is called a **proposition**. Propositional logic deals with such statements and **compound propositions** that combine together simple propositions (e.g., combining sentences (1) and (2) above we may say "Warsaw is the capital of Poland and $2 + 5 = 3$").

In order to build compound propositions we need rules on how to combine propositions. We denote propositions by lowercase letters p, q or r. Let us define:

- The **conjunction** of p and q, denoted as $p \wedge q$, is the proposition

$$p \ \textbf{and} \ q,$$

and it is **true** when both p and q are true and false otherwise.

- The **disjunction** of p and q, denoted as $p \vee q$, is the proposition

$$p \ \textbf{or} \ q,$$

and it is **false** when both p and q are false and true otherwise.

- The **negation** of p, denoted either as $\neg p$ or $\bar{p}$, is the proposition

  It is not **true** that p.

Example 1: Let $p =$"Hawks swoop" and $q =$"Gulls glide". Then $p \vee q$ is the same as "Hawks swoop or gulls glide". We also can translate back. For example, the English sentence "it is not true that hawks swoop" can be written as $\neg p$.

Exercise 1A: With the same notation as in the example above write the following propositions symbolically:

- It is not true that "Hawks swoop and gulls glide".

- "Hawks do not swoop or gulls do not glide".

Theme 2: Truth Tables

We can express compound propositions using a **truth table** that displays the relationships between the truth values of the simple propositions and the compound proposition. In the next three tables we show the truth tables for the negation, conjunction, and disjunction. Observe that any proposition p can take only two values, namely **true**, denoted T, or **false**, denoted F. Therefore, for a compound proposition consisting of two propositions (e.g., $p \wedge q$) we must consider only four possible assignments of T and F.

Table 1: The truth table for the negation.

p	$\neg p$
T	F
F	T

Table 2: The truth table for the conjunction.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Table 3: The truth table for the disjunction.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

In this module we will often use truth tables. To construct a truth table for a statement (e.g., $\neg p \vee q$) containing two propositions, say p and q, one first builds two columns with all possible vales of p and q (i.e., $(T,T),(T,F),(F,T),(F,F)$), and then follows already accepted rules of inference to determine the truth value of the compound statement (say $\neg p \vee q$).

Exercise 1B: Construct truth tables for the following statements:

- $p \wedge \neg p$;
- $p \vee \neg q$.

Theme 3: Implications

In mathematics we often deal with *conditional statements* like: “**if** $x = 2$, **then** $x^2 = 4$. The **if–then** statement is called **implication** and it is denoted as $p \to q$. It is false when p is true and q is false and true otherwise. The reader may inspect the truth table of $p \to q$ in Table 4 below.

Table 4: The truth table for the implication.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

It is important to emphasize that $p \to q$ is false only when p is **true** and q is **false**. In words, **truth** can*not* imply a **false** statement, but **false** can imply **truth**. For example, consider the following statement

$$\textbf{if} \;\; x = -2, \quad \textbf{then} \;\; x^2 = 4$$

which is true even if the first part of this compound statement is not true, say when $x = 2$.

In the implication $p \to q$, the proposition p is called **hypothesis** or **antecedent** and the proposition q is known as **conclusion** or **consequent**. The conclusion expresses a **necessary condition** for p, while the hypothesis expresses a **sufficient condition** for q to hold. Some other common ways of expressing the implication $p \to q$ are:

- if p, then q;
- p implies q;
- if p, q;
- p only if q;
- p is sufficient for q;
- q if p;
- q whenever p;
- q is necessary for p.

Exercise 1C: Make truth tables for the following statements:

1. $p \to \neg q$;
2. $(p \wedge \neg q) \to r$.

There are some important related implications following from $p \to q$, namely:

1. The proposition $q \to p$ is called the **converse**.
2. The **contrapositive** of $p \to q$ is $\neg q \to \neg p$;
3. The **inverse** is $\neg p \to \neg q$.

In Table 5 we compare the truth values of these propositions.

Table 5: The truth table for the implication, contrapositive, converse, and inverse.

p	q	$p \to q$	$\neg q \to \neg p$	$q \to p$	$\neg p \to \neg q$
T	T	T	T	T	T
T	F	F	F	T	T
F	T	T	T	F	F
F	F	T	T	T	T

We say that two compound propositions P and Q are **logically equivalent** if they have the same truth values. We shall write

$$P \equiv Q$$

or

$$P \Leftrightarrow Q.$$

It should be observed from Table 5 that the implication $p \to q$ has the same truth values as the contrapositive $\neg q \to \neg p$, but not as the converse and the inverse. Thus we can write

$$\begin{aligned} p \to q &\equiv \neg q \to \neg p, \\ p \to q &\not\equiv \neg p \to \neg q, \\ p \to q &\not\equiv q \to p. \end{aligned}$$

Example 2: Prove that

$$p \to q \equiv \neg p \vee q.$$

We use the truth table. Our computation is shown in Table 6. Comparing the second column with the last one, we see that the truth values are the same for $p \to q$ and $\neg p \vee q$, so the above two compound propositions are logically equivalent.

Table 6: The truth table for Example 2.

p	q	$p \to q$	$\neg p$	$\neg p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Exercise 1D: Using the truth table prove that the following propositions are logically equivalent:

$$p \vee (q \wedge r) \;\equiv\; (p \vee q) \wedge (p \vee r).$$

In Exercise 1D the reader was asked to prove logical equivalence that is known under the name **distributive law**. This is an example of many other logical equivalences that we list in Table 7 and prove in the sequel.

Table 7: Logical Equivalences

Equivalence	*Name*
$p \wedge T \;\equiv\; p$ $p \vee F \;\equiv\; p$	Identity laws
$p \vee T \;\equiv\; T$ $p \wedge F \;\equiv\; F$	Domination laws
$p \vee p \;\equiv\; p$ $p \wedge p \;\equiv\; p$	Idempotent laws
$\neg(\neg p) \;\equiv\; p$	Double negation law
$p \vee q \;\equiv\; q \vee p$ $p \wedge q \;\equiv\; q \wedge p$	Commutative laws
$p \vee (q \vee r) \;\equiv\; (p \vee q) \vee r$ $p \wedge (q \wedge r) \;\equiv\; (p \wedge q) \wedge r$	Associative laws
$p \vee (q \wedge r) \;\equiv\; (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \;\equiv\; (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \;\equiv\; \neg p \vee \neg q$ $\neg(p \vee q) \;\equiv\; \neg p \wedge \neg q$	De Morgan's laws

All laws listed above can be easily proved using the truth table. The reader is encouraged to try to work out all the truth tables. Having such laws under our belt, we can prove many new logical equivalences *without* using the truth table.

Example 3: Prove that

$$\neg(p \vee (\neg p \wedge q)) \;\equiv \neg p \wedge \neg q \;\equiv\; \neg(p \vee q).$$

We proceed as follows

$$\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{De Morgan's law}\\
&\equiv \neg p \wedge (\neg(\neg p) \vee \neg q) && \text{De Morgan's law}\\
&\equiv \neg p \wedge (p \vee \neg q) && \text{double negation law}\\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{distributive law}
\end{aligned}$$

$$
\begin{aligned}
&\equiv F \vee (\neg p \wedge \neg q) && \text{since } \neg p \wedge p \equiv F \\
&\equiv (\neg p \wedge \neg q) \vee F && \text{commutative law} \\
&\equiv (\neg p \wedge \neg q) && \text{identity law} \\
&\equiv \neg(p \vee q) && \text{De Morgan's law.}
\end{aligned}
$$

Thus the above logical equivalence is proved. The above is largely self-explanatory, but a few words of additional information follows: In the first statement above we, naturally, apply De Morgan's law $\neg(P \vee Q) = \neg P \wedge \neg Q$. In our case, Q is a compound statement $Q = \neg p \wedge q$, thus another application of De Morgan's law implies $\neg Q = p \vee \neg q$. Then we "multiply out", that is, $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$. The rest is simple.

A compound proposition is called a **tautology** if it is always true, no matter what the truth values of the propositions (e.g., $p \vee \neg p \equiv T$ no matter what is the value of p. Why?).

A compound proposition is called a **contradiction** if it is always false, no matter what the truth values of the propositions (e.g., $p \wedge \neg p \equiv T$ no matter what is the value of p. Why?).

Finally, a proposition that is neither a tautology nor a contradiction is called a **contingency**.

Theme 4: Predicates and Quantifiers

In mathematics we often have to deal with sentences like

$$p: \; x^2 - 2x + 1 = 0 \quad \text{or} \quad q: \; n \; \text{ is a prime number},$$

which are **not** propositions since their values are neither true nor false since the values of the variables x and n are not specified. We shall denote such statements as $P(x)$ or $Q(n)$ and call *propositional functions* or *predicates* of x or n.

More formally, let P be a statement involving the variable x that belongs to the set $\mathcal{D}$. Then P is called a **propositional function** or **predicate** with respect to $\mathcal{D}$ if for each $x \in \mathcal{D}$ the sentence $P(x)$ is a proposition. The domain $\mathcal{D}$ is often called the **universe of discourse** of P.

Example 4: The statement above

$$P(x): \; x^2 - 2x + 1 = 0$$

is true when $x = 1$ and is false for any $x \neq 1$. The statement $Q(3)$ is true, where $Q(n)$: "n is a prime number".

Predicates are very important in mathematics and computer science since they allow us to justify logical inferences or *syllogisms*. Consider the following famous syllogism:

All men are mortal.

Fermat is a man.

Therefore, Fermat is mortal.

This conclusion seems to be perfectly correct, but we do not have rules of inference for propositional logic to justify it. We shall come back in Module 3 to such logical inferences when we discuss mathematical proofs.

We saw above how to change a propositional function into a proposition: by assigning truth values to the variable x. There is another way of changing a predicate $P(x)$ into a proposition: either by saying that $P(x)$ is true for *all* values of x belonging to $\mathcal{D}$ or that $P(x)$ is true for *some* value of x in $\mathcal{D}$. The former is called the *universal quantification* while the latter the *existential quantification*.

Universal quantification

The **universal quantification** $P(x)$ is the proposition

$P(x)$ is true for all values of x in the universe of discourse $\mathcal{D}$.

We shall denote is as

$$\forall x \; P(x).$$

We can also read it as "for all x $P(x)$" or "for every x $P(x)$". The symbol $\forall$ (notice that it is an upside down A) is called a universal quantifier.

Example 5: The statement

$$\forall x \; x^2 \geq 0$$

is a universally quantified statement that is true. But

$$\forall x \; x^2 > 0$$

is a universally quantified statement that is false since for $x = 0$ we have $x^2 = 0$. We have just learned how to prove that a universal quantification is false. We must show *at least one value* of x for which $P(x)$ is not true. Such a value of x is called a **counterexample** for $\forall x, \; P(x)$.

Finally, observe that if the universe of discourse consists of a finite number of elements, say $x_1, x_2, \ldots, x_n$, then

$$\forall x P(x) \;\equiv\; P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n)$$

since this conjunction is true if and only if $P(x_1), P(x_2), \ldots, P(x_n)$ are all true.

Existential quantification

The **existential quantification** $P(x)$ is the proposition

> $P(x)$ is true for some value(s) of x in the universe of discourse $\mathcal{D}$.

We shall denote it as

$$\exists x \; P(x).$$

We can also read it as "for some x $P(x)$" or "there is an x such that $P(x)$" or "there is at least one x such that $P(x)$". The symbol $\exists$ (notice that it is mirror image of E) is called an existential quantifier.

Example 6: Let $Q(x)$ denote the statement: $4x^2 = 1$. What is the truth value of the quantification

$$\exists x \; Q(x)$$

when the universe of discourse for x is the set of real numbers? Since $Q(1/2)$ and $Q(-1/2)$ are true propositions, we conclude that $\exists x \; Q(x)$ is true in the defined universe of discourse. But if we demand that the universe of discourse for x is the set of integers, then $\exists x \; Q(x)$ is false since there is no integer satisfying $4x^2 = 1$. Here, we observe that in order to prove that an existentially qualified statement $P(x)$ is false, one must show that for **all** x in the universe of discourse the predicate $P(x)$ is false.

Finally, observe that if the universe of discourse consists of a finite number of elements, say $x_1, x_2, \ldots, x_n$, then

$$\exists x P(x) \;\equiv\; P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n)$$

since this disjunction is true if and only if at least one of $P(x_1), P(x_2), \ldots, P(x_n)$ is true.

We now generalize De Morgan's laws to quantifications. We claim that

$$\neg\forall x P(x) \equiv \exists x \neg P(x), \tag{1}$$

$$\neg\exists x P(x) \equiv \forall x \neg P(x) \tag{2}$$

Let us try to *prove* the first statement. Suppose that $\neg\forall x P(x)$ is true. Hence, $\forall x P(x)$ is false. But, as we seen before such a statement is false if there exists at least one x for which $P(x)$ is false. This implies that for such x the statement $\neg P(x)$ is true, form which we infer that $\exists x \neg P(x)$ is true. We have shown that if $\neg\forall x P(x)$ is true, then $\exists x \neg P(x)$ is true. In a similar manner, we conclude that if $\neg\forall x P(x)$ is false, then $\exists x \neg P(x)$ is false. In conclusion, the pair of propositions $\neg\forall x P(x)$ and $\exists x \neg P(x)$ have the same truth values, so they must be logically equivalent.