Solution of Homework 9: *Discrete Probability*

Q1. Show that if A and B are independent events, then $\overline{A}$ and $\overline{B}$ are also independent events.

Answer

$$
\begin{aligned}
P(\overline{A})P(\overline{B}) &= \Big(1 - P(A)\Big)\Big(1 - P(B)\Big) \\
&= 1 - P(A) - P(B) + P(A)P(B) \\
&= 1 - \Big(P(A) + P(B) - P(A)P(B)\Big) \\
&= 1 - \Big(P(A) + P(B) - P(A \cap B)\Big) \qquad A, B \text{ are independent} \\
&= 1 - P(A \cup B) \\
&= P(\overline{A \cup B}) \\
&= P(\overline{A} \cap \overline{B}) \\
&\rightarrow \overline{A},\ \overline{B} \text{ are independent}
\end{aligned}
$$

Note that this was only one way to do it and there are lots of other ways to prove that the L.H.S. = R.H.S.

Q2. What is the probability of a five-card poker hand contains the ace of hearts?

Answer

Let E be the event that a five-card poker hand contains the ace of hearts. The number of elements in the event E is the number of ways to pick the other four cards, which is $C(51, 4)$.

$$\therefore P(E) = \frac{|E|}{|S|} = \frac{C(51,4)}{C(52,5)} = \frac{5}{52}.$$

Q3. What is the conditional probability that exactly four heads appear when a fair coin is flipped five times, given that the first flip came up tails?

Answer

Let A be the event that exactly four heads appear when a fair coin is flipped five times and B be the event that the first flip came up tails. We need to find

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{(0.5)^5}{0.5} = (0.5)^4 = 0.0625.$$

Q4. Let X_n be the random variable that counts the difference in the number of tails and the number of heads when n coins are flipped. Assume that the probability of throwing a head is $p = 0.3$. Compute $\mathbf{E}[X_n]$ and $\mathbf{Var}[X_n]$.

Answer

Let N_T be the number of tails and N_H be the number of heads obtained from tossing the n coins. Note that, both N_T and N_H have Binomial Distribution (with probabilities of success q equal to 0.7 and 0.3, respectively), mean $= nq$, and variance $= nq(1-q)$. Now that $X_n = N_T - N_H$, $\mathbf{E}[X_n] = \mathbf{E}[N_T] - \mathbf{E}[N_H] = 0.7n - 0.3n = 0.4n$.

Morever, since $n = N_T + N_H$, $X_n = 2N_T - n$ and, therefore, $\mathbf{Var}[X_n] = \mathbf{Var}[2N_T - n] = \mathbf{Var}[2N_T]$ (since N_T, n are independent and $\mathbf{Var}[n] = 0$). Hence, $\mathbf{Var}[X_n] = \mathbf{Var}[2N_T] = 4\mathbf{Var}[N_T] = 4n(0.7)(1-0.7) = 0.84n$.