Solutions of Homework 8: *Basic Counting*

Q1. How many ordered pairs of integers (a, b) are needed to guarantee that there are two ordered pairs (a_1, b_1) and (a_2, b_2) such that

$$\begin{aligned} a_1 &= a_2 \mod 5, \\ b_1 &= b_2 \mod 5. \end{aligned}$$

Answer

Enumerating all possible ordered pairs of the form $(a \bmod 5, b \bmod 5)$; where a and b are any integers; results in the following set $S = \{(0,0), (0,1), \ldots, (0,4), (1,0), (1,1), \ldots, (1,4), \ldots\ldots, (4,0), \ldots, (4,4)\}$. Let the # of ordered pairs needed to guarantee a collision be n. Since $|S| = 25$, and using the pigeonhole principle, $\lceil \frac{n}{25} \rceil > 1 \rightarrow \mathbf{n = 26}$.

Q2. How many ways are there to seat 20 people around a circular table, where seating are considered to be the same if they can be obtained from each other by rotating the table? Justify your answer.

Answer

One way to solve this problem is to first assume that the people will be seated on a row ($P(20, 20) = 20!$ ways) and count those arrangements that are obtained from each other by simply rotating the table only once. Since the same arrangement of people is obtained 20 times by rotating the table, the total number of arrangements is

$$\frac{20!}{20} = \mathbf{19!}.$$

OR

Assuming that the first person sits in the northernmost seat. There are $P(19, 19) = 19!$ ways to seat the remaining people, since they form a permutation reading clockwise from the first person. Therefore the answer is **19!**.

Q3. How many subsets with more than two elements does a set with 15 elements have?

Answer

The total number of subsets $= 2^{15}$. The number of subsets with 0 element is 1, with 1 element is 15, and with 2 elements is $\mathrm{C}(15, 2) = (15 \times 14)/2! = 105$. Therefore, the number of subsets with more than two elements $= 2^{15} - (1 + 15 + 105) = 32,647$.

Q4. How many solutions are there to the inequality

$$x_1 + x_2 + x_3 \leq 18$$

where x_1, x_2, and x_3 are nonnegative integers?
Hint: Introduce an auxiliary variable x_4.

Answer

Introducing an auxiliary variable x_4 and then solving the equation $x_1+x_2+x_3+x_4 = 18$ gives a set of $\binom{18+4-1}{18} = \binom{21}{18} = \binom{21}{3} = \frac{21\cdot 20\cdot 19}{3\cdot 2\cdot 1} = 1330$ legitimate solutions (as shown in class). Note that since x_4 may take any value between 0 and 18, each solution of the new equation is also a solution of the old inequality. Hence, there are **1330** solutions to the given inequality.