Solutions of Homework 5: *Big* O, Ω

<u>Important Note</u>: $\log n \leq \sqrt{n} \leq n$.

Q1.

1. $100n + \log n \leq 100n + n = 101n = O(n) \Rightarrow c = 101, n_0 = 1$.
2. $(5n+1)^3 + 1000n^2 \leq (5n+n)^3 + 1000n^3 = (6n)^3 + 1000n^3 = 1216n^3 = O(n^3) \Rightarrow c = 1216, n_0 = 1$.
3. $n\sqrt{n^3} + \log^5 n \leq n^{5/2} + (\sqrt{n})^5 = 2n^{5/2} = O(n^{5/2}) \Rightarrow c = 2, n_0 = 1$.
4. $n^3 + n + \sqrt{n} + \sqrt{\log n} \leq n^3 + n^3 + n^3 + n^3 = 4n^3 = O(n^3) \Rightarrow c = 4, n_0 = 1$.

Q.2

i.

$$\begin{aligned} 6n^2 \log n - 2n &\leq 6n^2 \log n = O(n^2 \log n) && (1) \\ 6n^2 \log n - 2n &\geq 6n^2 \log n - 2n^2 \log n = 4n^2 \log n = \Omega(n^2 \log n) && (2) \end{aligned}$$

Form (1) and (2): $6n^2 \log n - 2n = \Theta(n^2 \log n)$.

ii.

$$\begin{aligned} \frac{6n^2+n}{n\log^6 n + 1} &\leq \frac{6n^2+n^2}{n\log^6 n} = \frac{7n^2}{n\log^6 n} = \frac{7n}{\log^6 n} = O\Big(\frac{n}{\log^6 n}\Big) && (3) \\ \frac{6n^2+n}{n\log^6 n + 1} &\geq \frac{6n^2}{n\log^6 n + n\log^6 n} = \frac{6n^2}{2n\log^6 n} = \frac{3n}{\log^6 n} = \Omega\Big(\frac{n}{\log^6 n}\Big) && (4) \end{aligned}$$

Form (3) and (4): $\frac{6n^2+n}{n\log^6 n+1} = \Theta(\frac{n}{\log^6 n})$.

iii.

$$\begin{aligned} \sum_{i=1}^{n} i^2 &\leq \sum_{i=1}^{n} n^2 = n^3 = O(n^3) && (5) \\ \sum_{i=1}^{n} i^2 &\geq \sum_{i=n/2}^{n} i^2 \geq \sum_{i=n/2}^{n} \Big(\frac{n}{2}\Big)^2 = \Big(\frac{n}{2}\Big)\Big(\frac{n}{2}\Big)^2 = \Big(\frac{1}{8}\Big)n^3 = \Omega(n^3) && (6) \end{aligned}$$

Form (5) and (6): $\sum_{i=1}^{n} i^2 = \Theta(n^3)$.

Q.3 Proof by contradiction: Assume that

$$\begin{aligned} & (\log n)^3 = O(\log n^3) \\ \rightarrow\ & (\log n)^3 \leq c \log n^3 = k \log n \text{ (note that } \log n^3 = 3\log n) \\ \rightarrow\ & (\log n)^2 \leq k. \end{aligned}$$

However, it is impossible to find such a constant k which is always greater than $(\log n)^2$ for all possible values of n (taking into account that it is a monotonically increasing function). Therefore, $(\log n)^3 \neq O(\log n^3)$.

Q.4 Note that:

- A constant function (such as 10^8) does not grow with n as opposed to any other function and, therefore, it is upper bounded by any of these functions regardless the value of this constant.
- By sketching the graphs for $\log^4 \sqrt{n}$ and n, you can conclude that $\log^4 \sqrt{n} \leq n$.
- The exponential function a^n always dominates polynomial and logarithmic functions.
- $2^{\log_2 n} = n$
- $2^{3 \log_2 n} = 2^{\log_2 n^3} = n^3$
- $n \leq \sqrt{n^3} \log^2 n \leq n^{3/2} \cdot n \leq n^3$
- $\left(\frac{5}{3}\right)^{2n} = \left(\frac{25}{9}\right)^n \geq 2^n$

Hence, the required ascending order is:

$$10^8 \prec \log^4 \sqrt{n} \prec 2^{\log_2 n} \prec \sqrt{n^3} \log^2 n \prec 2^{3 \log_2 n} \prec 2^n \prec \left(\frac{5}{3}\right)^{2n}.$$