Solution of Homework 2: *Language of Mathematics*

Q1. Prove

$$A \cap B = \overline{\bar{A} \cup \bar{B}}$$

without using the De Morgan's law and Venn's Diagram.

Answer

To prove that two sets are equal, we need to prove that each set is a subset of the other:

i) To prove that $\overline{\bar{A} \cup \bar{B}} \subseteq A \cap B$:

$$\begin{aligned} \forall x \in \overline{\bar{A} \cup \bar{B}} &\Rightarrow x \notin \bar{A} \cup \bar{B} \\ &\Rightarrow (x \notin \bar{A}) \wedge (x \notin \bar{B}) \\ &\Rightarrow (x \in A) \wedge (x \in B) \\ &\Rightarrow x \in A \cap B \end{aligned}$$

Thus, all elements of $\overline{\bar{A} \cup \bar{B}}$ are also elements of $A \cap B$. That is,

$$\overline{\bar{A} \cup \bar{B}} \subseteq A \cap B \cdots\cdots\cdots (1)$$

ii) Likewise, to prove that $A \cap B \subseteq \overline{\bar{A} \cup \bar{B}}$, we follow the exact same steps but in the reverse order thus proving:

$$A \cap B \subseteq \overline{\bar{A} \cup \bar{B}} \cdots\cdots\cdots (2)$$

From $(1), (2):\ A \cap B = \overline{\bar{A} \cup \bar{B}}$.

Q.2 What is the image of $f(\mathbf{R})$, where $\mathbf{R}$ is the set of all reals:

- $f(x) = x^4$
- $f(x) = x^2 - 4x$
- $f(x) = 2^{x^2}$
- $f(x) = \sin(x)$

Answer

If we plot the functions (or by substituting with different values for the independent variable x), the image of $f(\mathbf{R})$ will be:

1. $f(x) = x^4$
   $f(\mathbf{R}) = [0, \infty)$.

2. $f(x) = x^2 - 4x$
   $f(\mathbf{R}) = [-4, \infty)$.

3. $f(x) = 2^{x^2}$
   $f(\mathbf{R}) = [1, \infty)$.

4. $f(x) = \sin(x)$
   $f(\mathbf{R}) = [-1, 1]$.

Q.3 Is $f(x) = x^2 + 1$ a bijection of $\mathbf{R} \to \mathbf{R}$?
Compute also $f^{-1}(\{y : 0 \leq y \leq 1\})$, if exists, where $f^{-1}(Y)$ denotes an inverse image, that is, the set of all x such that $f(x) \in Y$.

Answer

The function $f(x)$ is not a bijection because it is not one-to-one. For example, $f(1) = f(-1) = 2$. Therefore, the inverse of $f(x)$ does not exist.

Q.4 What are the values of the following:

$$\sum_{i=5}^{99} 5 \cdot 2^{i-4},$$

$$\sum_{j=3}^{100} (2^{j+1} - 2^j),$$

$$\prod_{k=1}^{100} (-1)^k.$$

Answer

To solve this problem we will be using the following two formulas that we have learned in class:

$$1. \sum_{i=0}^{n} r^i = \frac{r^{n+1}-1}{r-1}$$

$$2. \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

The first formula is the sum of $n+1$ terms of the geometric progression and the second formula is the sum of the first n positive integers.

1.

$$\begin{aligned} \sum_{i=5}^{99} 5 \cdot 2^{i-4} &= \frac{5}{2^4} \sum_{i=5}^{99} 2^i \\ &= \frac{5 \cdot 2^5}{2^4} \sum_{i=0}^{94} 2^i \\ &= 10 \cdot \frac{2^{94+1}-1}{2-1} \\ &= 100(2^{95}-1) \end{aligned}$$

2. One way to solve this problem is:

$$\begin{aligned} \sum_{j=3}^{100} (2^{j+1} - 2^j) &= \sum_{j=3}^{100} 2^{j+1} - \sum_{j=3}^{100} 2^j \\ &= 2 \sum_{j=3}^{100} 2^j - \sum_{j=3}^{100} 2^j \\ &= \sum_{j=3}^{100} 2^j \\ &= 2^3 \sum_{j=0}^{97} 2^j \\ &= 8 \cdot \frac{2^{97+1}-1}{2-1} \\ &= 8(2^{98}-1) \end{aligned}$$

3.

$$
\begin{aligned}
\prod_{k=1}^{100}(-1)^k &= \Big(-1\Big)^{\sum_{k=1}^{100} k} \\
&= \Big(-1\Big)^{\frac{100(100+1)}{2}} \\
&= (-1)^{5050} \\
&= 1
\end{aligned}
$$