

[60] **Homework 2. The Moment Methods**

- [10] Let X_1, X_2, \dots, X_n be i.i.d. random variables distributed as the standard normal distribution $N(0, 1)$, that is, the density $f(x)$ of X_1 is

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

for $-\infty < x < \infty$. Find a_n such that

$$\max\{X_1, \dots, X_n\} \rightarrow a_n \quad (\text{pr.}).$$

Hint. It is known that for $x \rightarrow \infty$

$$\Pr\{X > x\} \sim \frac{1}{\sqrt{2\pi}x} e^{-x^2/2}$$

when X is distributed as $N(0, 1)$.

- [10] Prove the following alternative formulation of the second moment method. Let $\{A_i\}_{i=1}^n$ be a set of identically distributed events such that $n\Pr\{A_i\} \rightarrow \infty$. If

$$\sum_{i \neq j} \frac{\Pr\{A_i A_j\}}{n\Pr\{A_i\}} \rightarrow 1$$

as $n \rightarrow \infty$, then $\Pr\{\bigcup_{i=1}^n A_i\} \rightarrow 1$.

- [20] Consider the following “comparison” of two random strings. Let X_1^n and Y_1^m of length n and $m < n$, respectively, be generated by a binary memoryless source with probabilities p and $q = 1 - p$ of the two symbols. Define C_i to be the number matches between X_i^{i+m-1} and Y ; that is,

$$C_i = \sum_{j=1}^m \text{equal}(X_{i+j-1}, Y_j),$$

where $\text{equal}(x, y)$ is one when $x = y$ and zero otherwise. Define $M_{mn} = \max_{1 \leq i \leq n-m+1} \{C_i\}$. Prove that if $\log n = o(m)$ then

$$\lim_{n \rightarrow \infty} \frac{M_{m,n}}{m} = P_2 \quad (\text{a.s.}),$$

where $P_2 = p^2 + q^2$.

- [20] Consider the fill-up level F_n (the maximum level in a tree that is full) in a trie built over n binary strings generated by an independent memoryless source. Prove the following result that was first established by Pittel in 1985.

Theorem 1 (Pittel, 1985) Consider a trie built over n binary strings generated by n independent memoryless sources with p being the probability of generating a “0”, and $q = 1 - p$. Then

$$\lim_{n \rightarrow \infty} \frac{F_n}{\log n} = \frac{1}{\log p_{\min}^{-1}} \quad (\text{pr.}),$$

where $p_{\min} = \min\{p, 1 - p\}$.