1.

$$T(n) = 3T(n-1) + 2$$
$$= 3(3T(n-2) + 2) + 2$$
$$= 3^2 T(n-2) + 3 \cdot 2 + 2$$
$$= 3^2 (3T(n-3) + 2) + 3 \cdot 2 + 2$$
$$= 3^3 T(n-3) + 3^2 \cdot 2 + 3 \cdot 2 + 2$$

This can be generalized to:

$$T(n) = 3^i T(n-i) + 2\sum_{j=0}^{i-1} 3^j$$

This is trivially true for $i = 1$. *Now* sup *pose that* $i > 1$ *and that*

$$T(n) = 3^{i-1} T(n-i+1) + 2\sum_{j=0}^{i-2} 3^j$$

Then,

$$T(n) = 3^{i-1} T(n-i+1) + 2\sum_{j=0}^{i-2} 3^j$$
$$= 3^{i-1} (3T(n-i) + 2) + 2\sum_{j=0}^{i-2} 3^j$$
$$= 3^i T(n-i) + 2\sum_{j=0}^{i-1} 3^j$$

as required. *Suppose we take* $i = n-1$. *Then*,

$$T(n) = 3^{n-1} T(1) + 2\sum_{j=0}^{n-2} 3^j$$
$$= 3^{n-1} + 3^{n-1} - 1$$
$$= 2 \cdot 3^{n-1} - 1.$$

2.

$$T(n) = 9T(n/3) + n^3; T(1) = 1$$
$$= 9(9T(n/3^2) + (n/3)^3) + n^3$$
$$= 9^2 T(n/3^2) + 9(n/3)^3 + n^3$$
$$= 9^2 T(n/3^2) + n^3/3 + n^3$$

Similarly the next step comes out to be

$$= 9^3 T(n/3^3) + n^3/3^2 + n^3/3 + n^3$$

This can be generalized to:

$$T(n) = 9^k T(n/3^k) + \sum_{i=0}^{k-1} (n^3/3^i)$$

$T(n/3^k)$ *disappears when* $n = 3^k$ *i.e* $\log_3 n = k$
Therefore, substituting k *with* $\log_3 n$ *we get*

$$T(n) = n^2 + n^3 \frac{1 - \frac{1}{3^k}}{1 - \frac{1}{3}}$$

$$= n^2 + \frac{3}{2} n^3 (1 - \frac{1}{3^k})$$

$$= n^2 + \frac{3}{2} n^3 - \frac{3}{2} n^2$$

$$= \frac{3}{2} n^3 - \frac{1}{2} n^2$$

$$= O(n^3)$$

3.

$$T(n) = 2T(n/2) + 2n \log_2 n; \quad T(2) = 4$$

$$= 2(2T(\frac{n}{4}) + 2\frac{n}{2} \log \frac{n}{2}) + 2n \log n$$

$$= 2^2 T(\frac{n}{2^2}) + 2n \log \frac{n}{2} + 2n \log n$$

$$= 2^2 (2T(\frac{n}{2^3}) + \frac{n}{2} \log_2 \frac{n}{4}) + 2n \log_2 \frac{n}{2} + 2n \log_2 n$$

$$= 2^3 T(\frac{n}{2^3}) + 2n(\log_2 \frac{n}{2^2} + \log_2 \frac{n}{2^1} + \log \frac{n}{2^0})$$

$$= 2^3 T(\frac{n}{2^3}) + 2n(\log_2 n^3 - (2+1))$$

This can be generalized as

$$= 2^k T(\frac{n}{2^k}) + 2n(\log_2 n^k - \sum_{i=1}^{k-1} i)$$

$$= 2^k T(\frac{n}{2^k}) + 2n \log_2 n^k - n(k-1)k$$

$T(\frac{n}{2^k})$ *disappears when* $\frac{n}{2^k} = 2$

i.e $n = 2^{k+1}$. *Therefore* $k = \log n - 1$

Substituting the value of k *and* $T(2)$ *in the result we get*

$$\begin{aligned}
T(n) &= 2^{\log n-1}4 + 2n\log_2 n^{\log n-1} - n(\log_2 n - 2)(\log_2 n - 1) \\
&= 2n + 2n(\log_2 n - 1)\log_2 n - n(\log_2^2 n - 3\log_2 n + 2) \\
&= 2n + 2n\log_2^2 n - 2n\log_2 n - n\log_2^2 n + 3n\log_2 n - 2n \\
&= n\log_2^2 n + n\log_2 n \\
&= O(n\log_2^2 n)
\end{aligned}$$

4.

$$T(n) = \sum_{i=1}^{n-1} T(i) + 1; \quad T(1) = 1, n \geq 2$$

$$\begin{aligned}
T(n) - T(n-1) &= (\sum_{i=1}^{n-1} T(i) + 1) - (\sum_{i=1}^{n-2} T(i) + 1) \\
&= T(n-1)
\end{aligned}$$

Therefore, $T(n) = 2T(n-1)$. *Thus, we have reduced a recurrence with full history to a simple recurrence that can be solved by repeated substitution to give the solution* $T(n) = 2^{n-1}$.

Note: This file may not print correctly in *ix operating systems but should print properly on windows machines. There is a html version of this file on: http://www.cs.purdue.edu/homes/awan/381/hw4.html