1.

$$
\begin{aligned}
T(n) &= 3T(n-1)+2 \\
&= 3(3T(n-2)+2)+2 \\
&= 3^2 T(n-2)+3\cdot 2+2 \\
&= 3^2(3T(n-3)+2)+3\cdot 2+2 \\
&= 3^3 T(n-3)+3^2\cdot 2+3\cdot 2+2
\end{aligned}
$$

*This can be generalized to:*

$$T(n) = 3^i T(n-i) + 2\sum_{j=0}^{i-1} 3^j$$

*This is trivially true for* $i=1$. *Now* sup*pose that* $i>1$ *and that*

$$T(n) = 3^{i-1} T(n-i+1) + 2\sum_{j=0}^{i-2} 3^j$$

*Then*,

$$
\begin{aligned}
T(n) &= 3^{i-1} T(n-i+1) + 2\sum_{j=0}^{i-2} 3^j \\
&= 3^{i-1}(3T(n-i)+2) + 2\sum_{j=0}^{i-2} 3^j \\
&= 3^i T(n-i) + 2\sum_{j=0}^{i-1} 3^j
\end{aligned}
$$

*as required. Suppose we take* $i = n-1$. *Then*,

$$
\begin{aligned}
T(n) &= 3^{n-1} T(1) + 2\sum_{j=0}^{n-2} 3^j \\
&= 3^{n-1} + 3^{n-1} - 1 \\
&= 2\cdot 3^{n-1} - 1.
\end{aligned}
$$

2.

$$
\begin{aligned}
T(n) &= 9T(n/3)+n^3; T(1)=1 \\
&= 9(9T(n/3^2)+(n/3)^3)+n^3 \\
&= 9^2 T(n/3^2)+9(n/3)^3+n^3 \\
&= 9^2 T(n/3^2)+n^3/3+n^3
\end{aligned}
$$

*Similarly the next step comes out to be*

$$= 9^3 T(n/3^3) + n^3/3^2 + n^3/3 + n^3$$

This can be generalized to:

$$T(n) = 9^k T(n/3^k) + \sum_{i=0}^{k-1} (n^3/3^i)$$

*The latter part of the expression is a geometric progression(after taking out the common term* $n^3$*) with* $r = 1/3$ *which is less than* 1*. So assuming that* $k$ *can be large, the value of the summation will be* $a/(1-r)$ *(where* $a = 1$ *and* $r = 1/3$*) which is* $3/2$*.*

$T(n/3^k)$ *disappears when* $n = 3^k$ *i.e* $\log_3 n = k$
*Therefore, substituting* $k$ *with* $\log_3 n$ *we get*

$$T(n) = n^2 + \frac{3}{2} n^3$$

$$= O(n^3)$$

3.

$$T(n) = 2T(n/2) + 2n\log_2 n; \quad T(2) = 4$$

$$= 2(2T(\frac{n}{4}) + 2\frac{n}{2}\log\frac{n}{2}) + 2n\log n$$

$$= 2^2 T(\frac{n}{2^2}) + 2n\log\frac{n}{2} + 2n\log n$$

$$= 2^2 T(\frac{n}{2^2}) + 2n\log(\frac{n}{2}n)$$

$$= 2^2 T(\frac{n}{2^2}) + 2n\log(\frac{n^2}{2^1})$$

*Similarly the next step would result in*

$$= 2^3 T(\frac{n}{2^3}) + 2n\log(\frac{n^3}{2^2})$$

*This can be generalized as*

$$= 2^k T(\frac{n}{2^k}) + 2n\log(\frac{n^k}{2^{k-1}})$$

$T(\frac{n}{2^k})$ *disappears when* $\frac{n}{2^k} = 2$

*i.e* $n = 2^{k+1}$. *Therefore* $k = \log n - 1$

*Substituting the value of* $k$ *and* $T(2)$ *in the result we get*

$$T(n) = 2^{\log n-1}4 + 2n\log(\frac{n^{\log n-1}}{2^{\log n-2}})$$

$$\begin{aligned}
&= 2^{\log n}2 + 2n\log(n^{\log n-1}) + 2n\log(2^{\log n-2}) \\
&= 2n + 2n(\log n - 1)(\log n) + 2n(\log n - 2) \\
&= \mathrm{O}(n\log^2 n)
\end{aligned}$$

4.

$$T(n) = \sum_{i=1}^{n-1} T(i) + 1; \quad T(1) = 1, n \geq 2$$

$$\begin{aligned}
T(n) - T(n-1) &= (\sum_{i=1}^{n-1} T(i) + 1) - (\sum_{i=1}^{n-2} T(i) + 1) \\
&= T(n-1)
\end{aligned}$$

*Therefore*, $T(n) = 2T(n-1)$. *Thus, we have reduced a recurrence with full history to a simple recurrence that can be solved by repeated substitution to give the solution* $T(n) = 2^{n-1}$.

BY: TU