1) $n^{\ln n}$ or $\ln n^n$

both take ln

$\ln n^{\ln n} = \ln n * \ln n = (\ln n)^2$

$\ln \ln^n = n (\ln \ln n)$

$\lim_{n\to\infty} \frac{(\ln n)^2}{n(\ln \ln n)}$, Let $\ln n = t$, then $n = e^t$

$\lim_{n\to\infty} \frac{(\ln n)^2}{n(\ln \ln n)} = \lim_{t\to\infty} \frac{t^2}{e^t \ln t}$

$\lim_{t\to\infty} \frac{(t^2)'}{(e^t \ln t)'} = \lim_{t\to\infty} \frac{2t}{e^t/t + e^t \ln t}$

$\lim_{t\to\infty} \frac{(2t)'}{(e^t \ln t + e^t/t)'} = \lim_{t\to\infty} \frac{2}{e^t/t + e^t \ln t + f(t)} = 0$, where $f(t) = (e^t/t)'$

$\therefore \lim_{t\to\infty} \frac{(\ln n)^2}{n(\ln \ln n)} = 0$

$\therefore$ $(\ln n)^2 < n \ln \ln n$

$\therefore$ $n^{\ln} < \ln n^n$

2) $n^{\ln \ln \ln n}$, $(\log n)!$

$(\log n)! \approx \log n^{\log n + ½} (e^{-\log n} \sqrt{2\pi})$

take log on both function

$\log n^{\ln \ln \ln n} = (\ln \ln \ln n) * \log n = \log \log \log n * \log n * 1/c$

$\log(\log n^{\log n + ½} . e^{-\log n} \sqrt{2\pi}) = \log(\log n^{\log n + ½}) + \log(e^{-\log n} \sqrt{2\pi})$

$= (\log n + ½) \log \log n) + \log(e^{-\log n} \sqrt{2\pi}) \geq \log n * \log \log n + \log(e^{-\log n} \sqrt{2\pi})$

$\therefore$ $\log \log \log n \leq \log \log n$

$\therefore$ $\log n * \log \log \log n \leq \log n * \log \log n$

$\therefore$ $\log n^{\ln \ln \ln n} \leq \log n * \log \log n + \log(e^{-\log n} \sqrt{2\pi}) \leq \log(\log n^{\log n + 1/2}) e^{\log n} \sqrt{2\pi}$

$\therefore \quad n^{\ln \ln \ln n} \leq (\log n)!$

Q2) we know that

$$\int_1^n i^5 \, di \leq \sum_1^n i^5 \leq \int_1^{n+1} i^5 \, di$$

$$\int_1^n i^5 \, di = (1/6)\, i^6 \Big|_1^n = (1/6)\, n^6 - (1/6)$$

$$\therefore \sum_{k=1}^{n} i^5 \geq (1/6)\, n^6 - (1/6)$$

We assume $\sum_{k=1}^{n} i^5 = \theta(n^5)$,

$$\text{so } \sum_{k=1}^{n} i^5 \leq c\, n^5$$

$\therefore$ $(1/6)\, n^6 - (1/6) \leq c\, n^5$ must be true,
then $(1/6)\, n^6 - (1/6) \leq c\, n^5 \rightarrow (1/6)n - (1/6n^5) \leq c$

since c is constant, this inequality cannot hold for all n, so $\sum_{n=1}^{n} \neq \theta(n^2)$

3) $(n + 2 + O(n-1))^n$

$= (n(1 + 2/n + O(n^{-2}))^n$

$= (n^n (1 + 2/n + O(n^{-2}))^n$

for large n. $(1 + \alpha/n)^n = e^{\alpha}$

$\therefore n^n (1 + 2/n + O(n^{-2}))^n$

$= n^n (e^2 + O(n^{-1}))$

$= e^2 n^n (1 + O(n^{-1}))$

Q 4: The table below is showing the relationship between input value n and return value k.

Input n	k	Relationship between n and k
2	1	$2=2^0$ — exception
4	2	$4 = 2^2 = 2^{2^1}$
16	3	$16 = 2^{2x2} = 2^{2^2}$
256	4	$256 \leftarrow 2^{2x2x2} = 2^{2^3}$
256^2	5	$256 \leftarrow 2^{2x2x2x2} = 2^{2^4}$
…..	….	
n	k	$n = 2^{2^{k-1}}$

$\therefore$ $n = 2^{2^{k-1}}$ for $n \geq 4$

$\log n = 2^{k-1}$

$\log \log n = k - 1$

$k = \log \log n + 1 \leq \log \log n + \log \log n = 2 \log \log n$ for $n \geq 4$

$\therefore$ $k = O(\log \log n)$ $n \geq 4$