Solution to Homework 1

Stirlings Approximation

Part 1:

$$\sum_{k=1}^{n} logk$$

The graph at the back represents the above function. The graph does not look much like a logarithmic curve. Looks more like *nlogn*. Plotting the curve for *nlogn* gives a good approximation. The curve for *nlogn* has also been shown in the same graph. It rises above the stirlings approximation curve. The difference between the two curves seems to be proportional to n, so we try (*nlogn - n*)which fits perfectly on the curve.

$$logn! \approx \sum_{k=1}^{n} logk \approx nlogn - n$$

Part 2:

The Euler-Maclaurin formula is

$$\sum_{k=m+1}^{n-1} f(x) + \frac{f(m)+f(n)}{2} = \int_m^n f(x)dx + \int_m^n (x - \lfloor x \rfloor - ½) f'(x)dx$$

We are given that

$$\int_1^{\infty} (x - \lfloor x \rfloor - 1/2)\frac{dx}{x} \approx - 0.080894281$$

now

$$\sum_{k=1}^{n} logk = (\sum_{k=1+1}^{n-1} logk) + log\ 1 + log\ n$$

$$= (\sum_{k=1+1}^{n-1} f(k)) + f(1) + f(n)$$

where $f(x) = logk$, $m = 1$, $n = n$

$$= \left(\sum_{k=1+1}^{n-1} f(k) \right) + 1/2 \left(f(1)+f(n) \right) + 1/2 \left(f(1)+f(n) \right)$$

$$= \int_1^n \log(x)dx + \int_1^n \left(x - \lfloor x \rfloor - 1/2 \right) (\log(x))' dx + 1/2(f(1)+f(n))$$

$$= [\, x\log(x) - x \,]_1^n - \int_m^n \left(x - \lfloor x \rfloor - 1/2 \right) (1/x))dx + 1/2(\log (1) + \log(n))$$

In the above term, we consider n to be a large constant, so the value of the integral can be approximated by the given value.

$$= n\log n - n - 0 - (-0.08089424281) + 1/2(0 + \log n)$$

$$\log(n!) = (n+1/2)\log n - n + 0.91910579$$

which is the anser.

A Very Simple Sum:

Part 1:

$$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)}$$

The graph on the top represents the above function. The graph looks like $\log(n/(n+1))$. We draw a plot for $\log(n/(n+1))$ in the same grid(one on the bottom) and it is missing a constant term. It turns out that $\log n + 1$ very nearly coincides with the graph of the above function. So this could be a good approximation. Hence

$$S_n \approx \log \left(\frac{n}{n+1} \right) + 1$$

Part 2:

The Euler-Maclaurin formula is

$$\sum_{k=m+1}^{n-1} f(x) + \frac{f(m)+f(n)}{2} = \int_m^n f(x)dx + \int_m^n \left(x - \lfloor x \rfloor - ½ \right) f'(x)dx$$

We are given that

$$\int_1^{\infty} (x - \lfloor x \rfloor - ½) \frac{2x+1}{x^2(x+1)^2} \, dx \approx -0.056852656$$

now

$$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1+1}^{n-1} \frac{1}{k(k+1)} + \frac{1}{1(1+1)} + \frac{1}{n(n+1)}$$

$$= (\sum_{k=1+1}^{n-1} f(k)) + f(1) + f(n)$$

where $f(x) = \frac{1}{k(k+1)}$, $m = 1, n = n$

$$= (\sum_{k=1+1}^{n-1} f(k)) + \frac{1}{2} (f(1) + f(n)) + \frac{1}{2} (f(1) + f(n))$$

$$= \int_m^n f(x)dx + \int_m^n (x - \lfloor x \rfloor - 1/2) f'(x)dx + \frac{1}{2} (f(1) + f(n))$$

$$= \int_1^n \frac{1}{x(x+1)} \, dx + \int_1^n (x - \lfloor x \rfloor - 1/2) \left(\frac{1}{x(x+1)} \right)' dx + \frac{1}{2} \left(\frac{1}{1(1+1)} + \frac{1}{n(n+1)} \right)$$

$$= \int_1^n \frac{1}{x} dx - \int_1^n \frac{1}{x+1} \, dx + \int_1^n (x - \lfloor x \rfloor - 1/2) \left(-\frac{2x+1}{x^2(x+1)^2} \right)' dx + \frac{1}{4} + \frac{1}{2n(n+1)}$$

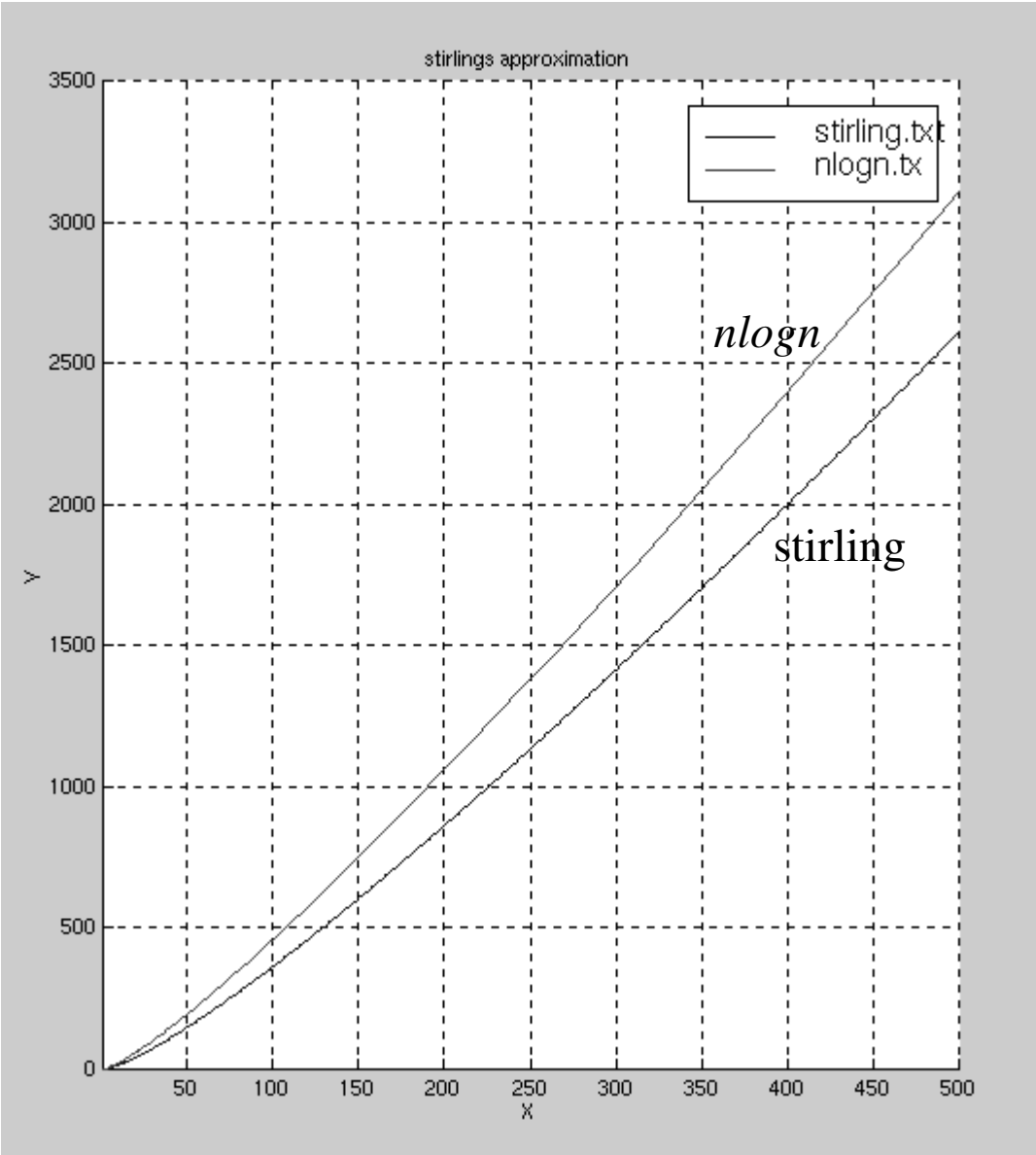
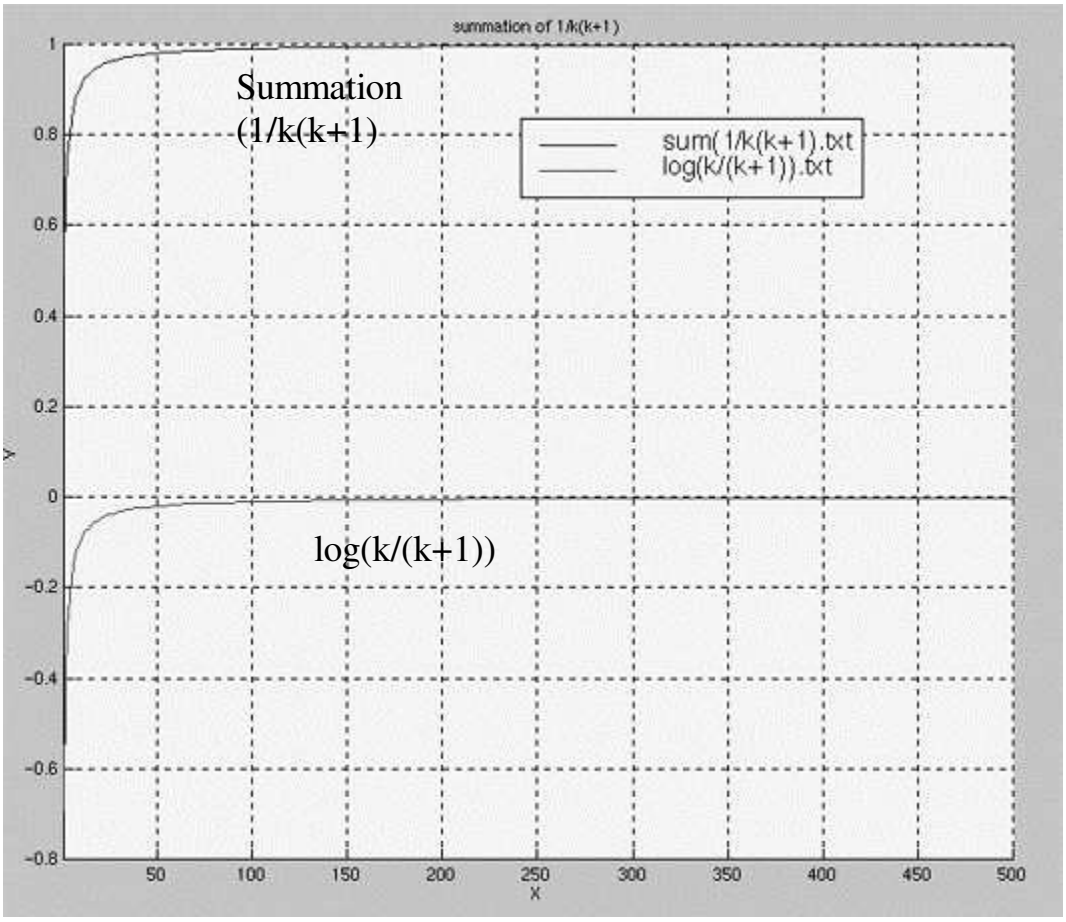
In the above term, we consider n to be a large constant, so the value of the integral can be approximated by the given value.

$$= [\log x]_1^n - [\log(x+1)]_1^n + 0.056852656 + \frac{1}{4} + \frac{1}{2n(n+1)}$$

$$= \log \left(\frac{n}{n+1} \right) + \frac{1}{2n(n+1)} + 0.999999836$$

which is the answer.

Stirling's Approximation:



Simple Sum of 1/k(k+1):