Sketch Solution of Homework 5 – Problem 3

1.Assume the height of the tree begins at 1.

(i) Upper bound:

The upper bound case happens only when all the N nodes form a linked list. Hence, the internal path length is of $1 + 2 + ... + N = \frac{N(N+1)}{2}$, which is $O(N^2)$.

(ii) Lower bound:

The lower bound case happens when all the N nodes form a complete M-ary tree. For a complete M-ary tree, the number of nodes on level l is M^{l-1}. Hence the internal path length is $\sum_{l=1}^{h} lM^{l-1}$. h is the height of the tree.

For a complete M-ary tree of height h, the total number of nodes in the tree is of

$T(M,h) = 1 + M + M^2 + ... + M^{h-1} = \frac{M^h-1}{M-1}$

Since $\frac{M^h-1}{M-1} = N$, we can get

$M^h = N(M-1)+1 = N(M-1+\frac{1}{N})$

$\Rightarrow h = \log_M N + \log_M (M-1+\frac{1}{N})$

$\Rightarrow \log_M^N < h \leq \log_M^N + 1$, for $0 < log_M^{(M-1+\frac{1}{N})} \leq 1$.

So, the complete M-ary tree is roughly of height $h = \log_M^N$. Therefore, The internal path length of such a tree is roughly

$\sum_{l=1}^{h} lM^{l-1} = 1 + 2M + 3M^2 + ... + hM^{h-1}$

$= \frac{\partial T(M,h+1)}{\partial M}$

$= \frac{hM^{h+1}-(h+1)M^h+1}{(M-1)^2}$

$\approx \frac{log_M^{(N)} \cdot M^{log_M^N+1}-(log_M^{(N)}+1)M^{log_M^{(N)}}+1}{(M-1)^2}$

$= \frac{N log_M^N}{(M-1)} - \frac{N-1}{(M-1)^2}$,

which is $O(NlogN)$ when M is fixed.