3. For the following pseudo-code determine the value returned in terms of n. Use big-Oh notation. Show your work.

int function sqr(int n)
  if (n <= 2)
  then return (1)
  else return (1 + sqr($n^{1/2}$))

[Solution]

Let's call our function F(n) for convenience:

$F(n) = 1$, if $n <= 2$ .... [1]

$F(n) = 1 + F(n^{1/2})$, if $n > 2$ ....[2]

We can keep expanding F(n) with equation [2]:

$F(n) = 1 + F(n^{1/2}) = 1 + (1 + F(n^{1/4})) = 2 + F(n^{1/4}) = 3 + F(n^{1/8}) = ... = k + F(n^{1/2\wedge k})$.

Note: 2^k means 2^k.

When $n^{1/2\wedge k} <= 2$, we have $F(n^{1/2\wedge k}) = 1$ from equation [1], and thus $F(n) = k + 1$.
So (k + 1) will be the returned value. Now let's find how to express k as a big-Oh notation in terms of n:

Since $n^{1/2\wedge k} <= 2$, we obtain $n <= 2^{2\wedge k}$ immediately by squaring both sides k times. Then we compute logarithms of both sides twice:

$\log_2 n <= 2\wedge k = 2^k$.

$\log_2 \log_2 n <= k$.

Therefore, k = O(log log n), and F(n) = k + 1 = O(log log n).

CS 251 Homework #2 Solution: *Mathematical Induction*

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1. Prove by induction that

$$\mathbf{1} + \alpha^1 + \alpha^2 + \alpha^3 + ... + \alpha^{\mathbf{n}} = (\alpha^{\mathbf{n+1}} - 1) / (\alpha - 1)$$

Show the basis step, the inductive hypothesis and the inductive step.

[Proof]

Basis step:

For $n = 1$, we have $1 + \alpha^1 = (\alpha^2 - 1) / (\alpha - 1)$.

Inductive hypothesis:

Assume that the following equation holds for some $n > 1$:

$1 + \alpha^1 + \alpha^2 + \alpha^3 + ... + \alpha^n = (\alpha^{n+1} - 1) / (\alpha - 1)$,

Let's prove this equation also holds for n+1:

Inductive step:

$1 + \alpha^1 + \alpha^2 + \alpha^3 + ... + \alpha^n + \alpha^{n+1} = (\alpha^{n+1} - 1) / (\alpha - 1) + \alpha^{n+1}$

$= (\alpha^{n+1} - 1 + \alpha^{n+2} - \alpha^{n+1}) / (\alpha - 1) = (\alpha^{n+2} - 1) / (\alpha - 1)$.

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2. Using induction, verify that each equation is true for every positive integer n.

$$\mathbf{1^3 + 2^3 + 3^3 + ... + n^3 = [\ n\ (n + 1) / 2\ ]^2}$$

[Proof]

Basis step:

For $n = 1$, we have $1^3 = [1 * (1 + 1) / 2]^2 = 1$.

Inductive hypothesis:

Assume that the following equation holds for some $n > 1$:

$1^3 + 2^3 + 3^3 + ... + n^3 = [n (n + 1) / 2]^2$,

Let's prove this equation also holds for n+1:

Inductive step:

$1^3 + 2^3 + 3^3 + ... + n^3 + (n + 1)^3 = [n (n + 1) / 2]^2 + (n + 1)^3$

$= (n + 1)^2 [(n / 2)^2 + (n + 1)] = (n + 1)^2 (n^2 + 4 n + 4) / 4 = [(n + 1) (n + 2) / 2]^2$.
