Sketch Solution of Homework 1

1.(a) $10n + 1 = O(n)$. When $c = 11$ and $n_0 = 1$, $10n + 1 \le 11n$.

1.(b) $(10n + 1)^2 = O(n^2)$. When $c = 120$ and $n_0 = 2$, $(10n + 1)^2 \le 120n^2$.

1.(c) $6n^3 - 12n^2 - 1 = O(n^3)$. Since $6n^3 - 12n^2 - 1 \le 6n^3$, we can choose $c = 6, n_0 = 1$.

1.(d) $2 + 4 + 6 + ... + 2n = 2(1 + 2 + 3 + ... + n) = n(n + 1) = n^2 + n = O(n^2)$. when $c = 2$ and $n_0 = 2$, $n^2 + n \le 2n^2$.

1.(e) $n^2 + \sqrt{n} + logn \le n^2 + n^2 = 2n^2 = O(n^2)$. We can choose $c = 2, n_0 = 1$.

2.(a) Show it's $O(n^2)$:
If $5n^2 - 6n = O(n)$, then $5n^2 - 6n \le cn^2$, which implies $5n - 6 \le cn$. It holds when $c = 5, n_0 = 1$.
Show it's $\Omega(n^2)$:
if $5n^2 - 6n = \Omega(n^2)$, then $5n^2 - 6n \ge cn^2$, which implies $5n - 6 \ge cn$. It holds when $c = 2, n_0 = 2$.
Hence, $5n^2 - 6n = \Theta(n^2)$.

2.(b)
$$\begin{aligned} n! &= n \times (n-1) \times (n-2) \times ... \times 3 \times 2 \times 1 \\ &< n \times n \times n \times ... \times n \times n \times n \\ &= n^n \end{aligned}$$
Hence, $n! = O(n^n)$.

2.(c) $\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{2n^3+3n^2+n}{6}$.
To show it is $O(n^3)$, $2n^3 + 3n^2 + n \le 6cn^3$. It holds when $c = 1, n_0 = 1$.
To show it is $\Omega(n^3)$, $2n^3 + 3n^2 + n \ge 6cn^3$. It holds when $c = \frac{1}{3}$, $n_0 = 1$.
Hence, $\sum_{i=0}^{n} i^2 = \Theta(n^3)$.

2.(d) $\frac{6n^2}{\log n+1} \le cn^3 \Longrightarrow 6n^2 \le cn^3 \log n + cn^3$, when $n \ge 1$. $\Longrightarrow 6 \le cn \log n + cn$. It holds when $c = 6, n_0 = 1$.

2.(e) $33n^3 + 4n^2 \ge cn^2 \Longrightarrow 33n + 4 \ge c$. Since c is a constant, we can always find a n_0 to make it satisfied. Hence, $33n^3 + 4n^2 = \Omega(n^2)$.

3. Assume it is true, then there would exist a constant c such that $n^2(1+\sqrt{n}) \leq cn^2$. This implies $1+\sqrt{n} \leq c$. For any constant c, we can find a n such that $1+\sqrt{n} > c$. Hence $n^2(1+\sqrt{n}) \neq O(n^2)$.

4. Assume it is true, then $(\log n)^2 \leq c \log n^2 \Longrightarrow \log n \log n \leq 2c \log n \Longrightarrow \log n \leq 2c$, which doesn't hold. Hence, $(\log n)^2 \neq O(\log n^2)$.