Module 6: Basic Counting

Theme 1: Basic Counting Principle

We start with two basic counting principles, namely, the **sum rule** and the **multiplication rule**.

> **The Sum Rule**: If there are n_1 different objects in the first set A_1, n_2 objects in the second set A_2, $\ldots$, n_m objects in the mth set A_m, and if the sets $A_1, A_2, \ldots, A_m$ are *disjoint* (i.e., $A_i \cap A_j = \emptyset$ for any $1 \leq i < j \leq m$), then the total number of ways to select an object from one of the set is
>
> $$n_1 + n_2 + \cdots + n_m;$$
>
> in other words,
>
> $$|A_1 \cup A_2 \cup \ldots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|.$$
>
> **The Multiplication Rule**: Suppose a procedure can be broken into m successive (ordered) stages, with n_1 outcomes in the first stage, n_2 outcomes in the second stage, $\ldots$, n_m outcomes in the mth stage. If the number of outcomes at each stage is independent of the choices in previous stages, and if the composite outcomes are all distinct, then the total procedure has
>
> $$n_1 \cdot n_2 \cdots n_m$$
>
> *different* composite outcomes. Sometimes this rule can be phrased in terms of sets $A_1, \ldots, A_m$ as follows
>
> $$|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \cdot |A_2| \cdots |A_m|.$$

Example 1: There are 40 students in an algebra class and 40 students in a geometry class. How many different students are in both classes combined?

This problem is not well formulated and cannot be answered unless we are told how many students are taking both algebra and geometry. If there is not student taking both algebra and geometry, then by the sum rule the answer is $40 + 40$. But let us assume that there are 10 students taking both algebra and geometry. Then there are 30 students *only* in algebra, 30 students *only* in geometry, and 10 students in *both* algebra and geometry. Therefore, by the sum rule the total number of students is $30 + 30 + 10 = 70$.

Example 2: There are boxes in a postal office labeled with an English letter (out of 26 English characters) and a positive integer not exceeding 80. How many boxes with different labels are possible?

The procedure of labeling boxes consists of two successive stages. In the first stage we assign 26 different English letters, and in the the second stage we assign 80 natural numbers (the second stage does *not* depend on the outcome of the first stage). Thus by the multiplication rule we have $26 \cdot 80 = 2080$ different labels.

Example 3: How many different bit strings are there of length five?

We have here a procedure that assigns two values (i.e., zero or one) in five stages. Therefore, by the multiplication rule we have $2^5 = 32$ different strings.

Exercise 6A: How many binary strings of length 5 are there that start with a 1 and end with a 0?

Example 4: *Counting Functions*. Let us consider functions from a set with m elements to a set with n elements. How many such functions are there?

We can view this as a procedure of successive m stages with n outcomes in each stage, where the outcome of the next stage does not depend on the outcomes of the previous stages. By the multiplication rule there are $n \cdot n \cdots n = n^m$ functions.

But, let us now count the number of *one-to-one* functions from a set of m elements to the set of n elements. Again, we deal here with a procedure of m successive stages. In the first stage we can assign n values. But in the second stage we can only assign $n - 1$ values since for a one-to-one function we are not allowed to select the value used before. In general, in the kth stage we have only $n - k + 1$ elements at our disposal. Thus by (a generalized) multiplication rule we have $n(n - 1) \cdot (n - m + 1)$ one-to-one functions.

Let us now consider some more sophisticated counting problems in which one must use a mixture of the sum and multiplication rules.

Example 5: A valid file name must be six to eight characters long and each name must have at least one digit. How many file names can there be?

If N is the total number of valid file names and N_6, N_7 and N_8 are, respectively, file names of length six, seven, and eight, then by the sum rule

$$N = N_6 + N_7 + N_8.$$

Let us first estimate N_6. We compute it in an indirect way using the multiplication rule together with the sum rule. We first estimate the number of file names of length six without the constraint that there must be at least one digit. By the multiplication rule there are $(26 + 10)^6 = 36^6$ file names. Now the number of file names that consists of *only* letters (no digits) is 26^6. We must subtract these since they are not allowed. Therefore (by the sum rule)

$$N_6 = 36^6 - 26^6 = 1867866560.$$

In a similar way, we compute

$$N_7 = 36^7 - 26^7,$$

$$N_8 = 36^8 - 26^8,$$

so that finally

$$N = N_6 + N_7 + N_8 = 2684483063360.$$

The next example illustrates the inclusion-exclusion principle that we already mentioned in Module 2: For two sets (not necessary disjoint) A and B the following holds

$$|A \cup B| = |A| + |B| - |A \cap B| \tag{1}$$

since in $|A| + |B|$ the part $|A \cap B|$ is counted twice, therefore we must subtract it.

Example 6: How many bit strings of length eight start with 1 **or** end with two bits 00?

We consider two tasks. The first one, constructing a string of length eight with the first bit equal to 1, can be done on $2^7 = 128$ ways (by the multiplication rule after noticing that the first bit is set to be 1 and there are only seven "free" stages). In the second task we count the number of strings that end with 00. Again, by the multiplication rule there are $2^6 = 32$ strings (since the last two stages are set to be 00). By adding these two numbers we would over count since both cases occur twice in this sum. To get it right, let us estimate the number of strings that starts with 1 **and** end with 00. By the multiplication rule we have 2^5 such strings (since three stages are set to be fixed). Therefore, by the inclusion-exclusion rule we find

$$2^7 + 2^6 - 2^5 = 160,$$

that is, it is the sum of strings with the first bit set to 1 and the the last two bits set to 00, minus the number of strings with the first first bit 1 and the last bits 00.

Theme 2: The Pigeonhole Principle

Surprisingly many complex problems in combinatorics can be solved by an easy to state and prove principle called the **pigeonhole principle**.

> **The Pigeonhole Principle**. If $k+1$ objects are placed into k boxes, then there is at least one box containing two or more of the objects.

This principle is easy to prove by contradiction. Assume to the contrary that all boxes have at most one object. Since there are k boxes, we will end up with at most k objects, which contradicts the assumption stating that we have $k+1$ objects.

Example 7: Consider a set of 27 English words. There must be at least two words that begin with the same letter, since there are only 26 letters in the English alphabet.

In some applications the following generalization of the pigeonhole principle is useful.

Theorem 1 [**Generalized Pigeonhole Principle**] *If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects, where $\lceil x \rceil$ is the smallest integer larger or equal to x.*

Proof. Let us assume contrary that all boxes contain at most $\lceil N/k \rceil - 1$ objects. Then the total number of objects is at most

$$k\left(\lceil N/k \rceil - 1\right) < k\left(N/k + 1 - 1\right) = N$$

which is impossible.

Example 8: Consider a group of 100 students. Among them there are at least 9 who were born in the same month. Indeed, by the generalized pigeonhole principle with $N = 100$ and $k = 12$ we have at least $\lceil 100/12 \rceil$ people born in the same month.

Finally, we discuss two more sophisticated examples of the pigeonhole principle.

Lemma 1. *Among any $n+1$ positive integers not exceeding $2n$ there must be an integer that divides one of the other integers from the set of $n+1$ positive integers.*

Proof. Let the $n+1$ integers be $a_1, \ldots, a_{n+1}$. We represent every such an integer as

$$a_j = 2^{k_j} q_j, \quad j = 1, 2, \ldots, n+1$$

where k_j is a nonnegative integer and q_j is an odd integer. For example, if $a_j = 20$, then we can write as $20 = 2^2 \cdot 5$, while $a_j = 15 = 2^0 \cdot 15$. Certainly, the integers $q_1, \ldots, q_{n+1}$ are odd integers smaller than $2n$. Since there are only n odd integers smaller than $2n$, it follows from the pigeonhole principle that two of the odd integers among $n+1$ must be the same. Assume that $q_i = q_j := q$ for i not equal to j. Then

$$a_i = 2^{k_i} q, \qquad a_j = 2^{k_j} q.$$

Clearly, either a_j divides a_i or vice versa since two $2^{k_i}/2^{k_j} = 2^{k_i - k_j}$. The proof is completed.

Exercise 6B: Justify that in any set of $n+1$ positive integers not exceeding $2n$ there must be two that are relative prime (i.e., the greatest common divisor of both numbers is one).

Example 9: Assume that in a group of six people, each pair of individuals consists of two friends or two enemies. We will show that there are either three mutual friends or three mutual enemies in the group.

Indeed, let the group be labeled as A, B, C, D, E and F. Consider now the person labeled as A. The remaining five people can be grouped into friends or enemies of A. Of the five other people (other than A), there are either three or more who are friends of A, or three or more than are enemies of A. Indeed, when a set of 5 objects (persons) is divided into two groups (friends or enemies) there are at least $\lceil 5/2 \rceil = 3$ elements in one of these groups. Consider first the group of friends of A. Call them B, C or D. If any of these three individuals are friends, then these two and A form the group of three mutual friends. Otherwise, B, C and D form a set of three mutual enemies. The proof in the case of three enemies of A proceeds in a similar manner.

This last example is an instance of an important part of combinatorics called **Ramsey theory**. In general, Ramsey theory deals with the distribution of subsets of elements of sets.

Theme 3: Permutations and Combinations

In computer science one often needs to know in how many ways one can arrange certain objects (e.g., how many inputs are there consisting of ten digits?). To answer these questions, we study here permutations and combinations – the simplest arrangements of objects.

A **permutation** of a set of distinct objects is *an ordered arrangements* of these objects. An ordered arrangements of r elements of a set is called an r-**permutation**.

Example 10: Let $S = \{a, b, c\}$. Then abc, acb, bac, bca, cab, cba are permutations of S, while ab, ba, ac, ca, bc and cb are 2-permutations of S.

It is not difficult to compute the number of r-permutations. Let $P(n, r)$ be the number of r-permutations of a set with n distinct elements. Observe that we can choose the first element in the r-permutation in n ways, the second element in $(n-1)$ (since after selecting the first element we can not use it again in the second choice), and so on, finally choosing the r-th element in $n - r + 1$ ways. Therefore, by the multiplication rule the total number of r-permutations is

$$P(n, r) = n(n-1) \cdot (n-2) \cdots (n-r+1) = \prod_{i=0}^{r-1} (n-i). \tag{2}$$

Above we use the product notation $a_1 \cdot a_2 \cdots a_n = \prod_{i=1}^{n} a_i$ introduced in Module 2.

Example 11: On how many ways one can construct a three digits number with all different digits (e.g., 142 is a legitimate digit but 223 is not)? We recognize this problem as a 3-permutation, therefore the answer is $10 \cdot 9 \cdot 8 = 720$.

In an r-permutation the order of elements is important (e.g., ab is different than ba), while in the r-**combination** is not. An r-combination of elements of a set is an *unordered* selection of r elements from the set (i.e., ab and ba are the same 2-combinations). The number of r-combinations of a set with n distinct elements is denoted by $C(n, r)$ or C_n^r. Thus the number of r-permutations is equal to the number of r-combinations times numbers of permutations (within each combinations), that is,

$$P(n, r) = C(n, r) \cdot r!$$

since every r-combination leads to $r! = P(r, r)$ r-permutations.

Example 12: Let $S = \{a, b, c\}$. Consider first 2-combinations. We have the following 2-combinations:

$$\{a, b\}, \{a, c\}, \{b, c\},$$

that generate the following six 2-permutations

$$(a, b), (b, a), (a, c), (c, a), (b, c), (c, b).$$

From the previous formula we immediately obtain

$$\begin{aligned} C(n,r) &= \frac{P(n,r)}{P(r,r)} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} \\ &= \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)!}{r!(n-r)!} \\ &= \frac{n!}{r!(n-r)!}. \end{aligned}$$

The first line above follows from the definition of r-permutations, while in the second line we multiply and divide by $(n-r)!$, and finally in the third line we observed that

$$n! = n(n-1)(n-2)\cdots(n-r+1)(n-r)! = n(n-1)(n-2)\cdots(n-r+1)\cdot(n-r)\cdot(n-r-1)\cdots 2\cdot 1.$$

In summary, we prove

$$C(n,r) = \frac{n!}{r!(n-r)!}. \tag{3}$$

Exercise 6C: In how many ways one can create a four-letter word with all distinct letters (we assume there are 26 letters)?

An astute reader should notice that $C(n,r)$ was already introduced in Module 4 where we wrote it as:

$$C_n^r := C(n,r) := \left(\begin{array}{c} n \\ k \end{array} \right).$$

Hereafter, we shall write $C(n,r)$ for these numbers that are also called **binomial coefficients** or **Newton's coefficients**. In Module 4 we proved several properties of these coefficients algebraically. We now re-prove them using counting or combinatorial arguments.

In particular, in Lemma 2 of Module 2 we proved algebraically that

$$C(n,r) = C(n-1,r) + C(n-1,r-1). \tag{4}$$

We now re-establish it using counting (combinatorial) arguments. In order to obtain all r-combinations ($= C(n,r)$) we pick up one element from the set and put it aside. Call it z. Now we build all r-combinations from the set $S - \{z\}$ of size $n-1$. Clearly, we have $C(n-1,r)$ such r-combinations. Let us now construct $(r-1)$-combinations from the set $S - \{z\}$. We have $C(n-1,r-1)$ such combinations. After adding the element z to such combinations we still have $C(n-1,r-1)$ r-combinations, each different than in the first experiment (i.e., without using z). But combining these two r-combinations we obtain *all* possible r-combinations which is equal to $C(n,r)$. We proved (4).

In a similar fashion we can prove another identity established in Module 4, namely,

$$C(n,r) = C(n,n-r).$$

Indeed, there is one-to-one correspondence between r-combinations and $(n-r)$-combinations: if $\{a, b, \ldots\}$ is an r combination, then the corresponding $(n-r)$-combination is $S - \{a, b, \ldots\}$.

In Module 4 we also proved that

$$\sum_{k=0}^{n} C(n,k) = 2^n.$$

We can re-establish it using counting arguments. Consider a set S of cardinality n. From Module 2 we know that there are 2^n subsets of S. The set of all subsets can be partitioned into subsets of size r, which are in fact r-combinations. There are $C(n,r)$ combinations and they must sum to all subsets, which is 2^n.

Finally, we prove one new identity known as **Vandermonde's Identity**:

$$C(m+n,r) = \sum_{k=0}^{r} C(m,r-k)C(n,k).$$

(In words, the number of r-combinations among $m+n$ elements is the sum of products of k combinations out of n and $r-k$-combinations out of m.) We use a counting argument. Suppose that there are m items in one set and n items in another set. The total number of ways to select r items from the union of these sets is $C(m+n,r)$. Another way of doing the same, is to select k items from the second set (we can do it in $C(n,k)$ ways) and $r-k$ items from the first set (which can be done on $C(m,r-k)$ ways), where $0 \leq k \leq r$. By the multiplication rule these two actions can be done in $C(m,r-k)C(n,k)$ ways, hence the total number of ways to pick r elements is the sum over all k, and the Vandermonde identity is proved.

Theme 4: Generalized Permutations and Combinations

In many counting problems, elements may be used repeatedly. For example, digits $\{0, 1, \ldots, 9\}$ may be used more than once to form a valid number; letters can be repeatedly used in words (e.g., SUCCESS). In the previous section we assumed that the objects were distinguishable, while in this section we consider the case when some elements are indistinguishable. Finally, we also explain how to count the ways to place distinguishable elements in boxes (e.g., in how many ways poker hands can be dealt to four players).

Permutations with Repetition

The are $n!$ permutations of n **distinct** (distinguishable) elements. But in how many ways we can obtain r-permutations when objects (elements) can be repeated?

Example 13: How many words of 7 characters can be created from 26 English letters? Observe that we do allow repetitions, so that SUCCESS is a valid word. By the multiplication rule we have 7^{26} words but there are only $26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20$ words with all different letters.

We can formulate the following general result. Consider r-permutations of a set with n elements when repetition is allowed. *The number of r-permutations of such a set (with repetitions allowed) is*

$$n^r.$$

Indeed, we have r stages with n outcomes in each stage, hence by the multiplication rule the number of outcomes is n^r.

Combinations with Repetitions

How many ways one can pick up (unordered) r elements from a set of n elements when repetitions are allowed? This is a harder problem, and we start with an example.

Example 14: In a bag there are money bills of the following denominations:

$$\$1, \$2, \$5, \$10, \$20, \$50, \text{ and } \$100.$$

We are asked to select *five* bills. In how many ways we can do it assuming that the order in which the bills are chosen does not matter and there are at least five bills of each types?

It is **not** $C(7, 5)$ since we can pick up five bills of the same denomination. To solve this problem we apply an old combinatorial trick: We build an auxiliary device, that of a cash box with seven compartments, each one holding one type of bill. The bins containing the bills are separated by six dividers. Observe that selecting five bills corresponds to placing five markers (denoted usually as a star $\star$) on the compartments holding the bills. For example, the following symbolic figure:

$$\star \mid \; \mid \star\star\star \mid \; \mid \star \mid \; \mid$$

corresponds to the case when one $1 bill, three $5 bills, and one $20 bill are selected.

Therefore, the number of ways to select five bills corresponds to the number of ways to arrange six bars (dividers) and five stars (markers). In other words, this amounts to selecting the position of the five stars from $11(= 6 + 5)$ positions. But this can be done in

$$C(11, 5) = \frac{11!}{5!6!} = 462$$

ways. This is the number of selecting five bills from a bag with seven types of bills.

In general, let us select r-combinations from a set of n elements when repetition of elements is allowed. We represent this problem as a list of $n - 1$ bars and r stars. These $n - 1$ bars are used to mark n cells (bins). We assume that the ith cell contains a star whenever the ith element occurs in the combination. For instance, a 6-combination of a set of four elements has three bars and six stars. In particular,

$$\star\star \mid \; \mid \star \mid \star\star\star$$

corresponds to the combination containing exactly two of the first elements, none of the second element, one of the third element, and three of the fourth element of the set. In general, each different list containing $n - 1$ bars and r stars corresponds to an r-combination of the set with n elements, when repetition is allowed. But the number of such lists is

$$C(n - 1 + r, r) \tag{5}$$

which is also the number of r-combinations from the set of n elements when repetitions is allowed.

Example 15: How many solutions does the following equation

$$x_1 + x_2 + x_3 + x_4 = 15$$

have, where x_1, x_2, x_3 and x_4 are nonnegative integers?

Here is a solution to this problem. We assume we have four types labeled x_1, x_2, x_3 and x_4. There are 15 items or units (since we are looking for an integer solution). Every time an item (unit) is selected it adds one to the type it picked it up. Observe that a solution corresponds to a way of selecting 15 items (units) from a set of four elements. Therefore, it is equal to 15-combinations with repetition allowed from a set with four elements. Thus by (5) we have

$$C(4 + 15 - 1, 15) = C(18, 15) = C(18, 3) = \frac{18 \cdot 17 \cdot 16}{2 \cdot 3} = 816$$

solutions. (We recall that $C(n, k) = C(n, n - k)$.)

Permutations of Sets with Indistinguishable Objects

When counting some care must be exercised to avoid counting indistinguishable objects more than once.

Example 16: How many different strings can be made by reordering the letters of the word TOTTOS?

If all letters in the word TOTTOS would be different, then the answer would be $5!$ but then we would over count. To avoid it, we observe that there are 6 positions. The letter T can be placed among these six positions in $C(6, 3)$ times, while the letter O can be placed in the remaining positions in $C(3, 2)$ ways; finally S can be put in $C(1, 1)$ ways. By the multiplication rule we have

$$\begin{aligned} C(6,3)C(3,2)C(1,1) &= \frac{6!}{3!3!}\frac{3!}{2!1!}\frac{1!}{1!0!} \\ &= \frac{6!}{3!2!1!} = 60 \end{aligned}$$

orderings, where we used the formula

$$C(n, k) = \frac{n!}{k!(n-k)!}$$

learned before.

We can obtain the same result in a different way. Observe that there are $6!$ permutations of six letters, however, there are $3!$ permutations in which permuting the letter T result in the same word; there $2!$ permutations of letter O that results in the same word. In summary, the number of different words is

$$\frac{6!}{3!2!1!}$$

as before.

Let us now generalized the above example. Assume there are n objects with n_1 indistinguishable objects of type 1, n_2 objects of type 2, ..., n_k indistinguishable objects of type k. The number of different permutations are

$$\frac{n!}{n_1!n_2!\cdots n_k!}. \tag{6}$$

There are many ways to prove this result. For example, we know that there $n!$ permutations, but many of these permutations are the same since we have k classes of indistinguishable objects. How many permutations are the same due to n_1 indistinguishable objects. Obviously, there are $n_1!$ such permutations of type 1, $n_2!$ of type 2, ..., $n_k!$ of type k. Thus the result follows.

Balls-and-Urns Model

Finally, we consider throwing n distinguishable balls (objects) into k distinguishable urns (boxes). The combinatorial model will answer such questions as in how many ways five cards from a deck of 52 cards can be distributed to four players.

Consider the following example. There are n balls, and three boxes. We want to know in how many ways we can throw these n balls such that there are n_1 balls in the first box, n_2 in the second box, and n_3 balls in the third box. Of course, there are $C(n, n_1)$ of ways putting n_1 balls from a set of n balls into the first box. For every such an arrangement, the remaining $n - n_1$ balls can be thrown in $C(n - n_1, n_2)$ ways into the second box so that it contains n_2 balls. Finally, the last box will have n_3 balls on $C(n - n_1 - n_2, n_3)$ ways. Therefore, by the multiplication rule we have

$$\begin{aligned} C(n, n_1)C(n - n_1, n_2)C(n - n_1 - n_2, n_3) &= \frac{n!}{n_1!(n - n_1)!}\frac{(n - n_1)!}{n_2!(n - n_1 - n_2)!}\frac{(n - n_1 - n_2)!}{n_3(n - n_1 - n_2 - n_3)!} \\ &= \frac{n!}{n_1!n_2!n_3!}. \end{aligned}$$

In general, let n distinguishable objects be thrown into k distinguishable boxes with n_i objects in the ith box, $i = 1, 2, \ldots, k$. Then, generalizing our example, we obtain

$$\frac{n!}{n_1!n_2!\cdots n_k!(n - n_1 - n_2 - \ldots - n_k)!} \tag{7}$$

ways to distribute these n objects among k boxes.

Example 17: In how many ways we can distribute hands of 5 cards to each of four players from the deck of 52 cards?

We may represent this problem as throwing 52 objects into four boxes each containing 5 cards. Thus the solution is

$$\frac{52!}{5!5!5!5!32!}.$$

since every hand has 5 cards, and after the distribution of $4 \cdot 5$ cards there remain 32 cards.

Theme 5: Linear Recurrences[1]

Consider the following problem, which was originally posed by Leonardo di Pisa, also known as Fibonacci, in the thirteenth century in his book *Lieber abaci*. A young pair of rabbits (one of each sex) is placed on an island. A pair of rabbits does not reproduce until they are 2 months old. After they are 2 month old, each pair of rabbits produces another pair each month. Denote F_n the number of pairs of rabbits after n months, e.g. $F_1 = F_2 = 1$, $F_3 = 2$, $F_4 = 3$, etc. (At the end of the first and second month there is only one pair, but at the end of the third, there is another one and at the end of the fourth once again, an additional one.) To find the number of pairs after n months we just have to add the number of rabbits in the previous month, F_{n-1}, and the number of newborn pairs, which equals F_{n-2}, since each newborn pair comes from a pair at least 2 month old. Consequently, the sequence F_n satisfies the recurrence relation

$$F_n = F_{n-1} + F_{n-2}$$

for $n \geq 3$ together with the initial condition $F_1 = 1$ and $F_2 = 2$. Of course, this recurrence relation and the initial condition uniquely determines the sequence F_n. It should be mentioned that these numbers

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

are called *Fibonacci numbers*. Sometimes the initial conditions for the Fibonacci numbers are $F_0 = 0$ and $F_1 = 1$ (this has some theoretical advantages) but this causes just a shift of 1 in the index.

Internet Exercise: Find on the internet two new applications of Fibonacci numbers and post a paragraph to the forum what you found out.

The Fibonacci numbers occur in various counting problems.

Example 18: Let a_n denote the number of binary strings of length n with the property that there are no two subsequent ones:

$$\begin{aligned} a_1 = 2: &\quad 0, 1 \\ a_2 = 3: &\quad 00, 01, 10 \\ a_3 = 5: &\quad 000, 010, 001, 100, 101 \\ \cdots &\quad \cdots. \end{aligned}$$

Again you can find a recurrence relation for a_n. If the last bit of a sequence of length n (of that kind) is 0, then there are exactly a_{n-1} possible ways for the first $n-1$ bits. However, if the last bit is 1 then the $(n-1)$st bit has to be 0 and, hence, there are exactly a_{n-2} possible ways for the first $n-2$

[1] This material is more advanced and the student should take time to study it carefully.

letters. (For example, there are $a_3 = 5$ strings among which $a_2 = 3$ ends with 0 and $a_1 = 2$ with 1.) Consequently one obtains

$$a_n = a_{n-1} + a_{n-2}$$

for $n \geq 3$. This is the same recurrence relation as for the Fibonacci numbers. Only the initial condition is different: $a_1 = F_3$ and $a_2 = F_4$. This implies a shift by two, that is,

$$a_n = F_{n+2}$$

for all $n \geq 1$.

More generally we define:

> A *linear homogeneous recurrence relation of degree k with constant coefficients* is a recurrence relation of the form
>
> $$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$
>
> where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

The recurrence relation is *linear* since the right-hand side is a sum of multiples of terms $a_n - i$. The recurrence relation is *homogeneous* since no terms occur that are not multiples of the a_js. The coefficients c_j are all *constant*, rather than functions that depend on n. The *degree* is k because a_n is expressed in terms of the previous k terms of the sequence.

It is clear (by induction) that a sequence satisfying the recurrence relation in the definition is uniquely determined by its recurrence relation and the k initial conditions

$$a_0 = C_0, a_1 = C_1, \ldots, a_{k-1} = C_{k-1}.$$

That is, once we set the first $k - 1$ values, then the next values a_n for $n > k$ can be computed from the recurrence.

Example 19: The recurrence relation $a_n = \sqrt{2}\, a_{n-1}$ is a linear homogeneous recurrence relation of degree one. The recurrence relation $a_n = a_{n-1} + a_{n-2}$ is a linear homogeneous recurrence relation of degree two. The recurrence relation $a_n = a_{n-5}$ is a linear homogeneous recurrence relation of degree five.

Example 20: The recurrence relation $a_n = a_{n-1} + a_{n-2}^2$ is not linear because there is term a_{n-2}^2. The recurrence relation $a_n = 2a_{n-1} + 1$ is linear but not homogeneous because of the term 1 (which is not a multiply of a_n). The recurrence relation $a_n = n a_{n-1}$ is linear but does not have constant coefficients.

Exercise 6D: Is the following recurrence

$$a_n = 6a_{n-1} + (n-2)a_{n-2} + 5$$

linear, homogeneous, with constant coefficients?

The basic approach for solving linear homogeneous recurrence relations is to look for solutions of the form

$$a_n = r^n,$$

where r is a constant. Note that $a_n = r^n$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots c_k a_{n-k}$ if and only if

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}.$$

When both sides of this equation are divided by r^{n-k} and the right-hand side is subtracted from the left, we obtain the equivalent equation

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots c_{k-1} r - c_k = 0.$$

Consequently, the sequence $a_n = r^n$ is a solution if and only if r is a solution of this last equation, which is called the **characteristic equation** of the recurrence relation. The solutions of this equation are called the *characteristic roots* of the recurrence relation. As we will see, these characteristic roots can be used to give an explicit formula for all solutions of the recurrence relation.

We will consider in details the case of degree $k = 2$. The general case is similar but more involved and thus we will not stated. For $k = 2$ the characteristic equation is just a quadratic equation of the form $r^2 - c_1 r - c_2 = 0$. We recall that the quadratic equation $r^2 - c_1 r - c_2 = 0$ has two solutions

$$r_{1,2} = \frac{c_1 \pm \sqrt{c_1^2 + 4c_2}}{2}$$

if $c_1^2 + 4c_2 > 0$; if $c_1^2 + 4c_2 = 0$, then the equation has one solution $r_1 = c_1/2$; otherwise there is no real solution to this equation.

First, we consider the case when there are two distinct characteristic roots.

Theorem 2. *Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence a_n is a solution of the recurrence relation*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n, \quad n \geq 0, \tag{8}$$

where α_1 and α_2 are constants.

Proof. We must do two things to prove the theorem. First, it must be shown that if r_1 and r_2 are the roots of the characteristic equation, and α_1 and α_2 are constants, then the sequence $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ is a solution of the recurrence relation. Second, it must be shown that if the sequence a_n is a solution, then $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for some constants α_1 and α_2.

Now we will show that if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$, then the sequence a_n is a solution of the recurrence equation. We have

$$\begin{aligned}
c_1 a_{n-1} + c_2 a_{n-2} &= c_1(\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1}) + c_2(\alpha_1 r_1^{n-2} + \alpha_2 r_2^{n-2}) \\
&= \alpha_1 r_1^{n-2}(cr_1 + c_2) + \alpha_2 r_2^{n-2}(cr_2 + c_2) \\
&= \alpha_1 r_1^n + \alpha_2 r_2^n \\
&= a_n.
\end{aligned}$$

where the third lines is a consequence of the fact that since r_1 and r_2 are roots of $r^2 - c_1 r - c_2 = 0$, then $r_1^2 = c_1 r_1 + c_2$ and $r_2^2 = c_1 r_2 + c_2$. This shows that the sequence a_n with $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ is a solution of the recurrence relation.

To show that every solution a_n of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ is of the form $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for some constants α_1 and α_2, suppose that a_n is a solution of the recurrence relation, and the initial conditions are $a_0 = C_0$ and $a_1 = C_1$. It will be shown that there are constants α_1 and α_2 so that the sequence a_n with $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ satisfies the same initial conditions. This requires that

$$\begin{aligned}
a_0 &= C_0 = \alpha_1 + \alpha_2, \\
a_1 &= C_1 = \alpha_1 r_1 + \alpha_2 r_2.
\end{aligned}$$

We can solve these two equations for α_1 and α_2 and get

$$\alpha_1 = \frac{C_1 - C_0 r_2}{r_1 - r_2}$$

and

$$\alpha_2 = \frac{C_0 r_1 - C_1}{r_1 - r_2},$$

where these expressions depend on the fact that $r_1 \neq r_2$. (When $r_1 = r_2$ this theorem is not true.) Hence, with these values for α_1 and α_2, the sequence a_n and the sequence $\alpha_1 r_1^n + \alpha_2 r_2^n$ satisfy the two initial conditions $a_0 = C_0$ and $a_1 = C_1$. Since the recurrence relation and these initial conditions uniquely determine the sequence, it follows that $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$.

Example 21: We derive an explicit formula for the sequence of Fibonacci numbers F_n defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ with initial conditions $F_0 = 0$ and $F_1 = 1$.

The characteristic equation of the Fibonacci recurrence is $r^2 - r - 1 = 0$. Its solution is

$$r_1 = \frac{1 + \sqrt{1+4}}{2} = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad r_2 = \frac{1-\sqrt{5}}{2}.$$

Therefore, from (8) it follows that the Fibonacci numbers are given by

$$F_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

for some constants α_1 and α_2. The initial conditions $F_0 = 0$ and $F_1 = 1$ can be used to find these constants. We have

$$\begin{aligned} F_0 &= \alpha_1 + \alpha_2 = 0, \\ F_1 &= \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1. \end{aligned}$$

The solution of these simultaneous equations is given by

$$\alpha_1 = \frac{1}{\sqrt{5}} \qquad \alpha_2 = -\frac{1}{\sqrt{5}}.$$

Consequently, the Fibonacci numbers can be explicitly expressed as

$$F_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Now we discuss the case when the characteristic equation has only one root (e.g., $r^2 - 2r + 1 = (r-1)^2 = 0$). We shall omit the proof.

Theorem 3. *Let c_1 and c_2 real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. Then the sequence a_n is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if*

$$a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$$

(for $n \geq 0$), where α_1 and α_2 are constants.

Example 22: What is the solution of the following recurrence relation

$$a_n = 6a_{n-1} - 9a_{n-2}$$

with initial conditions $a_0 = 1$ and $a_1 = 6$?

The only root of $r^2 - 6r + 9 = (r-3)^2 = 0$ is $r = 3$. Hence, the solution to this recurrence relation is

$$a_n = \alpha_1 3^n + \alpha_2 n 3^n$$

for some constants α_1 and α_2. Using the initial conditions, it follows that

$$\begin{aligned} a_0 &= 1 = \alpha_1, \\ a_1 &= 6 = 3\alpha_1 + 3\alpha_2. \end{aligned}$$

Solving these two equations shows that $\alpha_1 = 1$ and $\alpha_2 = 1$. Consequently, the solution to this recurrence relation is

$$a_n = 3^n + n3^n = 3^n(n+1).$$