Homework 5. Big O, Ω

Q1.

1. $100n + 1 \leq 100n + n = 101n = O(n) \Rightarrow c = 101, n_0 = 1$.
2. $(10n+1)^4 \leq (10n+n)^4 = (11n)^4 = 14641n^4 = O(n) \Rightarrow c = 14641, n_0 = 1$.
3. $3n^3 - 5n^2 - 500 \leq 3n^3 = O(n^3) \Rightarrow c = 3, n_0 = 1$.
4. $n^2 + n + \sqrt{n} + \log^2 n \leq n^2 + n^2 + n^2 + n^2 = 4n^2 = O(n^2) \Rightarrow c = 4, n_0 = 1$.

Q2.

1. $6n^2 - 2n \leq 6n^2 = O(n^2)$.......(1)
$6n^2 - 2n \geq 6n^2 - 2n^2 = 4n^2 = \Omega(n^2)$.......(2)
Form (1) and (2): $6n^2 - 2n = \Theta(n^2)$.
2. $\frac{6n^2}{\log^3 n+1} \leq 6n^2 \leq n^3 = O(n^3)$.
3. $3n^3 + 44n^2 \geq n^2 = \Omega(n^2)$.

Q3. Proof by contradiction: Assume that

$$\begin{aligned} &(\log n)^3 = O(\log n^3) \\ \rightarrow &(\log n)^3 \leq c \log n^3 = k \log n (\text{note that} \log n^3 = 3 \log n) \\ \rightarrow &(\log n)^2 \leq k. \end{aligned}$$

However, it is impossible to find such a constant k which is always greater than $(\log n)^2$ for all possible values of n (taking into account that it is a monotonically increasing function). Therefore, $(\log n)^3 \neq O(\log n^3)$.

Q4. Note that:

- $2^{\log_2 n} = n$.
- $2^{3 \log_2 n} = 2^{\log_2 n^3} = n^3$.
- $(\frac{3}{2})^n \leq 2^n$.
- For $n \geq 4$, we have $2^n = \prod_{i=1}^n 2 \leq \prod_{i=1}^n i = n! \leq \prod_{i=1}^n n = n^n$.
- By sketching the graphs for $\log^4 n$ and n, you can conclude that $\log^4 n \leq n$.
- A constant function (such as 100) does not grow with n as opposed to any other function and, therefore, it is upper bounded by any of these functions regardless the value of this constant.
- The exponential function a^n always dominates polynomial and logarithmic functions.

Hence, the required ascending order is:

$$100 \prec \log^4 n \prec 2^{\log_2 n} \prec 2^{3 \log_2 n} \prec n^3 \log^2 n \prec (\frac{3}{2})^n \prec 2^n \prec n! \prec n^n.$$