Solutions of Homework 4: *Proof Techniques*

Q1. Show that $\sqrt[3]{3}$ is irrational.

Answer

Proof by contradiction: Assume that $\sqrt[3]{3} = \frac{p}{q}$ in its simplest form, i.e., both p and q do not have a common divisor and therefore **the fraction $\frac{p}{q}$ cannot be simplified further**. Thus,

$$
\begin{aligned}
3 &= \frac{p^3}{q^3} && (1)\\
\rightarrow\ p^3 &= 3q^3 && (2)\\
\rightarrow\ 3&|p^3 && (3)\\
\rightarrow\ 3&|p. && (4)
\end{aligned}
$$

Where $3|p$ means that p is divisible by 3........................(I)

From (I), $p = 3k$ for some integer k. Substituting in (2):

$$
\begin{aligned}
&(3k)^3 = 3q^3\\
\rightarrow\ &27k^3 = 3q^3\\
\rightarrow\ &q^3 = 9k^3\\
\rightarrow\ &3|q^3\\
\rightarrow\ &3|q.
\end{aligned}
$$

Thus, q is also divisible by 3........................(II)

From (I) and (II), the fraction $\frac{p}{q}$ is *not* in its simplest form for it can be simplified further by dividing both the numerator and the denominator by 3 which contradicts the original assumption.

Q.2 Show that 3 divides $n^3 + 2n$ whenever n is a nonnegative integer.

Answer

Proof by induction on n

Basis Case ($n = 1$):
$1^3 + 2 \times 1 = 3$ is divisible by 3.

Induction Step:
Assume $P(n)$ is *true*, i.e. 3 divides $n^3 + 2n$. We need to prove that $P(n+1)$ is also *true*:

$$\begin{aligned}(n+1)^3 + 2(n+1) &= (\underline{\underline{n^3}} + \underline{3n^2 + 3n + 1}) + (\underline{\underline{2n}} + \underline{2}) \\ &= (n^3 + 2n) + (3n^2 + 3n + 1 + 2) \\ &= (n^3 + 2n) + 3(n^2 + n + 1).\end{aligned}$$

From the induction hypothesis, n^3+2n is divisible by 3. Moreover, $3(n^2+n+1)$ is also divisible by 3 (because it is a multiple of 3.) Therefore, $(n+1)^3+2(n+1)$ is divisible by 3, i.e. $P(n+1)$ is *true*.

Q.3 Using mathematical induction prove that

$$\sum_{i=1}^{n} i2^i = 2^{n+1}(n-1)+2.$$

Answer

Proof by induction on n

Basis Case ($n = 1$):
$L.H.S. = \sum_{i=1}^{1} i2^i = 2 = R.H.S.$ and, therefore, $P(1)$ is *true*.

Induction Step:
Assume $P(k)$ is *true*, i.e. $\sum_{i=1}^{k} i2^i = 2^{k+1}(k-1)+2$. We need to prove that $P(k+1)$ is also *true*, i.e. $\sum_{i=1}^{k+1} i2^i = 2^{k+2}k + 2$ as follows:

$$\begin{aligned}
\sum_{i=1}^{k+1} i2^i &= \sum_{i=1}^{k} i2^i + 2^{k+1}(k+1) \\
&= 2^{k+1}(k-1) + 2 + 2^{k+1}(k+1) \\
&= 2^{k+1}(k-1+k+1) + 2 \\
&= 2^{k+1}(2k) + 2 \\
&= 2^{k+2}k + 2.
\end{aligned}$$

Q.4 The *harmonic number* H_n is defined as for $n \geq 1$

$$H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

Prove by induction that

$$H_{2^n} \geq 1 + \frac{n}{2}$$

whenever n is nonnegative natural number.

Answer

Proof by induction on n

BASIS CASE ($n = 0$): $H_1 = \sum_{k=1}^{1} \frac{1}{k} = 1 \geq 1 + \frac{0}{2}$.

INDUCTION STEP: Assume $P(n)$ is *true*, i.e., $H_{2^n} \geq 1 + \frac{n}{2}$. We need to prove that $P(n+1)$, which is $H_{2^{n+1}} \geq 1 + \frac{n+1}{2}$, is also *true*:

$$\begin{aligned}
H_{2^{n+1}} &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n} + \frac{1}{2^n + 1} \cdots + \frac{1}{2^{n+1}} \\
&= H_{2^n} + \frac{1}{2^n + 1} \cdots + \frac{1}{2^{n+1}} \\
&\geq (1 + \frac{n}{2}) + \frac{1}{2^n + 1} \cdots + \frac{1}{2^{n+1}} \\
&\geq (1 + \frac{n}{2}) + \frac{1}{2^{n+1}} \cdots + \frac{1}{2^{n+1}} \\
&= (1 + \frac{n}{2}) + 2^n \cdot \frac{1}{2^{n+1}} \\
&= (1 + \frac{n}{2}) + \frac{1}{2} \\
&= 1 + \frac{n+1}{2}.
\end{aligned}$$

Q.5 Derive an explicit formula for the following recurrence for $n \geq 1$

$$a_n = \frac{n}{2}\, a_{n-1}$$

with $a_0 = 1$.

Answer

$$\begin{aligned}
a_n &= \frac{n}{2}\, a_{n-1} \\
&= \frac{n}{2} \times \frac{n-1}{2}\, a_{n-2} \\
&= \frac{n}{2} \times \frac{n-1}{2} \times \frac{n-2}{2}\, a_{n-3} \\
&\vdots \\
&= \underbrace{\frac{n}{2} \times \frac{n-1}{2} \times \frac{n-2}{2} \times \ldots \times \frac{3}{2} \times \frac{2}{2} \times \frac{1}{2}}_{n\ terms}\, a_0 \\
&= \underbrace{\frac{n!}{2^n}} \times 1 \\
&= \frac{n!}{2^n}.
\end{aligned}$$