Announcements

- To make up for the cancelled lecture of March 7th, Lecture 23 will be given Tuesday, April 2 from 5:00–6:00 pm in LWSN B155 and again Wednesday, April 3 from 5:00–6:00 pm in LWSN B134.

- Exam 6 is Thursday, April 18 from noon to 12:25 pm. It covers material from this homework, Sections 5.1–5.5 of the textbook, and Sections 6.1–6.3 of the class notes. You can use only pens, pencils, and erasers; in particular, no written material nor electronic devices is permitted.

Due Thursday, April 11 at noon.

0. Read Sections 5.1–5.5 of the textbook.

1. (10 points) Do Exercises 5.12 from the textbook. For convenience here are the data:

\[
x = [1.02 \ 0.95 \ 0.87 \ 0.77 \ 0.67 \ 0.56 \ 0.44 \ 0.30 \ 0.16 \ 0.01]';
\]
\[
y = [0.39 \ 0.32 \ 0.27 \ 0.22 \ 0.18 \ 0.15 \ 0.13 \ 0.12 \ 0.13 \ 0.15]' ;
\]

Getting just the right x and y ranges for the given contour data requires more than one try. Part (b) has a typographical error; it should say \([-0.005, 0.005]\). There is no need to change the x and y ranges for the perturbed orbit. The function unifrnd is convenient. Hand in your code, your printout of the two sets of coefficients, and a single plot containing both orbits. Use axis equal for your plot.

2. (4 points) Do Exercise 5.5 from the textbook.

3. (5 points) Suppose P and Q are two locations on land near the sea with elevations measured to be 200 meters and 300 meters above sea level, respectively. Also suppose that the difference in elevation between P and Q is measured to be 120 meters. Assume that each of the three measurements contains a random error that is normally distributed and that these errors are independent and of equal standard deviation.

(a) Express the elevations of P and Q as the solution of an overdetermined system of linear equations.

(b) Determine the normal equations for your answer to part (a).

(c) How should your answer to part (a) be modified if the error in measuring the difference in elevation between P and Q has a standard deviation which is 40% of the other two standard deviations?

4. (6 points) Let \( x_* \) minimize \( \| b - Ax \|_2 \) where \( A \) is a matrix having more rows than columns, and let \( A = QR \) be a full QR factorization.
(a) Prove that $x_*$ also minimizes $\|Q^Tb - Rx\|_2$. (Note that $\|v\|_2 = (v^Tv)^{1/2}$.)

(b) What is the least squares solution of the overdetermined system $Rx \approx Q^Tb$? To answer this, you may find it convenient to introduce additional symbols for parts of matrices/vectors.

Problems not to hand in

However, solutions will be provided.

5. Assume the following data

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>2</td>
<td>2</td>
<td>8</td>
</tr>
</tbody>
</table>

are the result of $y$ being a linear polynomial in $x$ except for independent normally distributed random errors of equal standard deviation. Set up an overdetermined system of linear equations whose least squares solution gives the maximum-likelihood estimate of the linear polynomial.

6. True or false? A linear least-squares problem $Ax \approx b$ always has a unique solution $x$ that minimizes the RMS of the residual vector $r = b - Ax$ when normal equations is used. Justify your answer.

7. Consider the tabulated data

<table>
<thead>
<tr>
<th>$k$</th>
<th>$x_k$</th>
<th>$y_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>−2</td>
</tr>
</tbody>
</table>

Determine coefficients $a$ and $b$ such that

$$\sum_{k=1}^{3} (y_k - (ax_k + b))^2$$

is minimum.

Learning objectives

- Explain the assumptions behind least-squares approximation.
- Determine a Householder reflection that reflects a given vector to a given direction.
- Form the normal equations.
- Give the condition for uniqueness of the solution of a linear least-squares problem.
- Define a full QR factorization.
- Given a full QR factorization, obtain a reduced QR factorization.
- Explain how to solve a least squares problem from a QR factorization.
- Explain in general terms how Householder reflections can be used to compute a QR factorization.