

Lipschitz Robustness of Finite-state Transducers

Thomas A. Henzinger, Jan Otop, Roopsha Samanta

IST Austria

Abstract

We investigate the problem of checking if a finite-state transducer is *robust* to uncertainty in its input. Our notion of robustness is based on the analytic notion of Lipschitz continuity — a transducer is K -(Lipschitz) robust if the perturbation in its output is at most K times the perturbation in its input. We quantify input and output perturbation using *similarity functions*. We show that K -robustness is undecidable even for deterministic transducers. We identify a class of functional transducers, which admits a polynomial time automata-theoretic decision procedure for K -robustness. This class includes Mealy machines and functional letter-to-letter transducers. We also study K -robustness of nondeterministic transducers. Since a nondeterministic transducer generates a set of output words for each input word, we quantify output perturbation using *set-similarity functions*. We show that K -robustness of nondeterministic transducers is undecidable, even for letter-to-letter transducers. We identify a class of set-similarity functions which admit decidable K -robustness of letter-to-letter transducers.

1 Introduction

Most computational systems today are embedded in a physical environment. The data processed by such real-world computational systems is often noisy or uncertain. For instance, the data generated by sensors in reactive systems such as avionics software may be corrupted, keywords processed by text processors may be wrongly spelt, the DNA strings processed in computational biology may be incorrectly sequenced, and so on. In the presence of such input uncertainty, it is not enough for a computational system to be functionally correct. An additional desirable property is that of *continuity* or *robustness* — the system behaviour degrades smoothly in the presence of input disturbances [14].

Well-established areas within control theory, such as *robust control* [15], extensively study robustness of systems. However, their results typically involve reasoning about continuous state-spaces and are not directly applicable to inherently discontinuous discrete computational systems. Moreover, uncertainty in robust control refers to differences between a system's model and the actual system; thus robust control focuses on designing controllers that function properly in the presence of perturbation in various internal parameters of a system's model. Given the above, formal reasoning about robustness of computational systems under input uncertainty is a problem of practical as well as conceptual importance.

In our work, we focus on robustness of finite-state transducers, processing finite or infinite words, in the presence of uncertain inputs. Transducers are popular models of input-output computational systems operating in the real world [13, 19, 3, 23]. While many decision problems about transducers have been studied thoroughly over the decades [19, 23], their behaviour under uncertain inputs has only been considered recently [21]. In [21], a transducer was defined to be robust if its output changed proportionally to every change in the input *upto a certain threshold*. In practice, it may not always be possible to determine such a bound on the input perturbation. Moreover, the scope of the work in [21] was limited to the robustness problem for *functional* transducers w.r.t. specific distance functions, and did not consider arbitrary nondeterministic transducers or arbitrary *similarity functions*.

In this paper, we formalize robustness of finite-state transducers as Lipschitz continuity. A function is Lipschitz-continuous if its output changes proportionally to *every* change in the input. Given a constant K and similarity functions d_Σ , d_Γ for computing the input, output

perturbation, respectively, a functional transducer \mathcal{T} is defined to be K -Lipschitz robust (or simply, K -robust) w.r.t. d_Σ, d_Γ if for all words s, t in the domain of \mathcal{T} with finite $d_\Sigma(s, t)$, $d_\Gamma(\mathcal{T}(s), \mathcal{T}(t)) \leq Kd_\Sigma(s, t)$. Let us consider the transducers \mathcal{T}_{NR} and \mathcal{T}_R below. Recall that the Hamming distance between equal length words is the number of positions in which the words differ. Let d_Σ, d_Γ be computed as the Hamming distance for equal-length words, and be ∞ otherwise. Notice that for words a^{k+1}, ba^k in the domain of the Mealy machine \mathcal{T}_{NR} , $d_\Sigma(a^{k+1}, ba^k) = 1$ and the distance between the corresponding output words, $d_\Gamma(a^{k+1}, b^{k+1})$, equals $k + 1$. Thus, \mathcal{T}_{NR} is not K -robust for any K . On the other hand, the transducer \mathcal{T}_R is 1-robust: for words a^{k+1}, ba^k , we have $d_\Sigma(a^{k+1}, ba^k) = d_\Gamma((b)^{k+1}, a(b)^k) = 1$, and for all other words s, t in the domain of \mathcal{T}_R , either $d_\Sigma(s, t) = \infty$ or $d_\Sigma(s, t) = d_\Gamma(\mathcal{T}_R(s), \mathcal{T}_R(t)) = 0$.



While the K -robustness problem is undecidable even for deterministic transducers, we identify interesting classes of finite-state transducers with decidable K -robustness. We first define a class of functional transducers, called synchronized transducers, which admits a polynomial time decision procedure for K -robustness. This class includes Mealy machines and functional letter-to-letter transducers; membership of a functional transducer in this class is decidable in polynomial time. Given similarity functions computable by weighted automata, we reduce the K -robustness problem for synchronized transducers to the emptiness problem for weighted automata.

We extend our decidability results by employing an *isometry approach*. An *isometry* is a transducer, which for all words s, t satisfies $d_\Gamma(\mathcal{T}(s), \mathcal{T}(t)) = d_\Sigma(s, t)$. We observe that if a transducer \mathcal{T}_2 can be obtained from a transducer \mathcal{T}_1 by applying isometries to the input and output of \mathcal{T}_1 , then K -robustness of \mathcal{T}_1 and \mathcal{T}_2 coincide. This observation enables us to reduce K -robustness of various transducers to that of synchronized transducers.

Finally, we study K -robustness of nondeterministic transducers. Since a nondeterministic transducer generates a set of output words for each input word, we quantify output perturbation using *set-similarity functions* and define K -robustness of nondeterministic transducers w.r.t. such set-similarity functions. We show that K -robustness of nondeterministic transducers is undecidable, even for letter-to-letter transducers. We define three classes of set-similarity functions and show decidability of K -robustness of nondeterministic letter-to-letter transducers w.r.t. one class of set-similarity functions.

The paper is organized as follows. We begin by presenting necessary definitions in Sec. 2. We formalize Lipschitz robustness in Sec. 3. In Sec. 4 and Sec. 5, we study the K -robustness problem for functional transducers, showing undecidability of the general problem and presenting two classes with decidable K -robustness. We study K -robustness of arbitrary nondeterministic transducers in Sec. 6, present a discussion of related work in Sec. 7 and conclude in Sec. 8.

2 Preliminaries

In this section, we review definitions of finite-state transducers and weighted automata, and present similarity functions. We use the following notation. We denote input letters by a ,

b etc., input words by s, t etc., output letters by a', b' etc. and output words by s', t' etc. We denote the concatenation of words s and t by $s \cdot t$, the i^{th} letter of word s by $s[i]$, the subword $s[i] \cdot s[i+1] \cdot \dots \cdot s[j]$ by $s[i, j]$, the length of the word s by $|s|$, and the empty word and empty letter by ϵ . Note that for an ω -word s , $|s| = \infty$.

Finite-state Transducers. A finite-state transducer (FST) \mathcal{T} is a tuple $(\Sigma, \Gamma, Q, Q_0, E, F)$ where Σ is the input alphabet, Γ is the output alphabet, Q is a finite nonempty set of states, $Q_0 \subseteq Q$ is a set of initial states, $E \subseteq Q \times \Sigma \times \Gamma^* \times Q$ is a set of transitions¹, and F is a set of accepting states.

A run γ of \mathcal{T} on an input word $s = s[1]s[2]\dots$ is defined in terms of the sequence: $(q_0, w'_1), (q_1, w'_2), \dots$ where $q_0 \in Q_0$ and for each $i \in \{1, 2, \dots\}$, $(q_{i-1}, s[i], w'_i, q_i) \in E$. Let $\text{Inf}(\gamma)$ denote the set of states that appear infinitely often along γ . For an FST \mathcal{T} processing ω -words, a run is accepting if $\text{Inf}(\gamma) \cap F \neq \emptyset$ (Büchi acceptance condition). For an FST \mathcal{T} processing finite words, a run $\gamma: (q_0, w'_1), \dots, (q_{n-1}, w'_n), (q_n, \epsilon)$ on input word $s[1]s[2]\dots s[n]$ is accepting if $q_n \in F$ (final state acceptance condition). The output of \mathcal{T} along a run is the word $w'_1 \cdot w'_2 \cdot \dots$ if the run is accepting, and is undefined otherwise. The transduction computed by an FST \mathcal{T} processing infinite words (resp., finite words) is the relation $\llbracket \mathcal{T} \rrbracket \subseteq \Sigma^\omega \times \Gamma^\omega$ (resp., $\llbracket \mathcal{T} \rrbracket \subseteq \Sigma^* \times \Gamma^*$), where $(s, s') \in \llbracket \mathcal{T} \rrbracket$ iff there is an accepting run of \mathcal{T} on s with s' as the output along that run. With some abuse of notation, we denote by $\llbracket \mathcal{T} \rrbracket(s)$ the set $\{t : (s, t) \in \llbracket \mathcal{T} \rrbracket\}$. The input language, $\text{dom}(\mathcal{T})$, of \mathcal{T} is the set $\{s : \llbracket \mathcal{T} \rrbracket(s) \text{ is non-empty}\}$.

An FST \mathcal{T} is called *functional* if the relation $\llbracket \mathcal{T} \rrbracket$ is a function. In this case, we use $\llbracket \mathcal{T} \rrbracket(s)$ to denote the unique output word generated along any accepting run of \mathcal{T} on input word s . Checking if an arbitrary FST is functional can be done in polynomial time [12]. An FST \mathcal{T} is *deterministic* if for each $q \in Q$ and each $a \in \Sigma$, $|\{q' : (q, a, w', q') \in E\}| \leq 1$. An FST \mathcal{T} is a *letter-to-letter* transducer if for every transition of the form $(q, a, w', q') \in E$, $|w'| = 1$. A *Mealy machine* is a deterministic, letter-to-letter transducer, with every state being an accepting state. In what follows, we use transducers and finite-state transducers interchangeably.

Composition of transducers. Consider transducers $\mathcal{T}_1 = (\Sigma, \Delta, Q_1, Q_{1,0}, E_1, F_1)$ and $\mathcal{T}_2 = (\Delta, \Gamma, Q_2, Q_{2,0}, E_2, F_2)$ such that for every $s \in \text{dom}(\mathcal{T}_1)$, $\llbracket \mathcal{T}_1 \rrbracket(s) \in \text{dom}(\mathcal{T}_2)$. We define $\mathcal{T}_2 \circ \mathcal{T}_1$, the *composition* of \mathcal{T}_1 and \mathcal{T}_2 , as the transducer $(\Sigma, \Gamma, Q_1 \times Q_2, Q_{1,0} \times Q_{2,0}, E, F_1 \times F_2)$, where E is defined as follows: $((q_1, q_2), a, w', (q'_1, q'_2)) \in E$ iff $(q_1, a, t', q'_1) \in E_1$ and upon reading t' , \mathcal{T}_2 generates w' and changes state from q_2 to q'_2 , i.e., iff $(q_1, a, t', q'_1) \in E_1$ and there exist $(q_2, t'[1], w'_1, q_2^1), (q_2^1, t'[2], w'_2, q_2^2), \dots, (q_2^{k-1}, t'[k], w'_k, q_2^k) \in E_2$ such that $k = |t'|$ and $w' = w'_1 \cdot w'_2 \cdot \dots \cdot w'_k$. Observe that if $\mathcal{T}_1, \mathcal{T}_2$ are functional, $\mathcal{T}_2 \circ \mathcal{T}_1$ is functional and $\llbracket \mathcal{T}_2 \circ \mathcal{T}_1 \rrbracket = \llbracket \mathcal{T}_2 \rrbracket \circ \llbracket \mathcal{T}_1 \rrbracket$, where \circ denotes function composition.

Weighted automata. Recall that a finite automaton (with Büchi or final state acceptance) can be expressed as a tuple (Σ, Q, Q_0, E, F) , where Σ is the alphabet, Q is a finite set of states, $Q_0 \subseteq Q$ is a set of initial states, $E \subseteq Q \times \Sigma \times Q$ is a transition relation, and $F \subseteq Q$ is a set of accepting states. A weighted automaton (WA) is a finite automaton whose transitions are labeled by rational numbers. Formally, a WA \mathcal{A} is a tuple $(\Sigma, Q, Q_0, E, F, c)$ such that (Σ, Q, Q_0, E, F) is a finite automaton and $c : E \mapsto \mathbb{Q}$ is a function labeling the transitions of \mathcal{A} . The transition labels are called *weights*.

Recall that a run π of a finite automaton on a word $s = s[1]s[2]\dots$ is defined as a sequence

¹ Note that we disallow ϵ -transitions where the transducer can change state without moving the reading head.

of states: q_0, q_1, \dots where $q_0 \in Q_0$ and for each $i \in \{1, 2, \dots\}$, $(q_{i-1}, s[i], q_i) \in E$. A run π in a finite automaton processing ω -words (resp., finite words) is accepting if it satisfies the Büchi (resp., final state) acceptance condition. The set of accepting runs of an automaton on a word s is denoted $\text{Acc}(s)$. Given a word s , every run π of a WA \mathcal{A} on s defines a sequence $c(\pi) = (c(q_{i-1}, s[i], q_i))_{1 \leq i \leq |s|}$ of weights of successive transitions of \mathcal{A} ; such a sequence is also referred to as a weighted run. To define the semantics of weighted automata we need to define the value of a run (that combines the sequence of weights of the run into a single value) and the value across runs (that combines values of different runs into a single value). To define values of runs, we consider *value functions* f that assign real numbers to sequences of rational numbers, and refer to a WA with a particular value function f as an f -WA. Thus, the value $f(\pi)$ of a run π of an f -WA \mathcal{A} on a word s equals $f(c(\pi))$. The value of a word s assigned by an f -WA \mathcal{A} , denoted $\mathcal{L}_{\mathcal{A}}(s)$, is the *infimum* of the set of values of all accepting runs, i.e., $\mathcal{L}_{\mathcal{A}}(s) = \inf_{\pi \in \text{Acc}(s)} f(\pi)$ (the infimum of an empty set is infinite).

In this paper, we consider the following value functions: (1) the sum function $\text{SUM}(\pi) = \sum_{i=1}^{|\pi|} (c(\pi))[i]$, (2) the discounted sum function $\text{DISC}_{\delta}(\pi) = \sum_{i=1}^{|\pi|} \delta^i (c(\pi))[i]$ with $\delta \in (0, 1)$ and (3) the limit-average function $\text{LIMAVG}(\pi) = \limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{i=1}^k (c(\pi))[i]$. Note that the limit-average value function cannot be used with finite sequences. We define $\text{VALFUNC} = \{\text{SUM}, \text{DISC}_{\delta}, \text{LIMAVG}\}$.

A WA \mathcal{A} is *functional* iff for every word s , all accepting runs of \mathcal{A} on s have the same value.

Decision questions. Given an f -WA \mathcal{A} and a threshold λ , the *emptiness* question asks whether there exists a word s such that $\mathcal{L}_{\mathcal{A}}(s) < \lambda$ and the *universality* question asks whether for all words s we have $\mathcal{L}_{\mathcal{A}}(s) < \lambda$. The following results are known.

► **Lemma 1.** (1) For every $f \in \text{VALFUNC}$, the emptiness problem is decidable in polynomial time for nondeterministic f -automata [11, 10]. (2) The universality problem is undecidable for SUM-automata with weights drawn from $\{-1, 0, 1\}$ [16, 1].

► **Remark.** Weighted automata have been defined over semirings [10] as well as using value functions (along with infimum or supremum) as above [5, 6]. These variants of weighted automata have incomparable expression power. We use the latter definition as it enables us to express long-run average and discounted sum, which are inexpressible using weighted automata over semirings. Long-run average and discounted sum are widely used in quantitative verification and define natural distances (Example 9). Moreover, unlike the semiring-based definition, the value-function-based definition extends easily from finite to infinite words.

Similarity Functions. In our work, we use similarity functions to measure the similarity between words. Let \mathbb{Q}^{∞} denote the set $\mathbb{Q} \cup \{\infty\}$. A similarity function $d : S \times S \rightarrow \mathbb{Q}^{\infty}$ is a function with the properties: $\forall x, y \in S$: (1) $d(x, y) \geq 0$ and (2) $d(x, y) = d(y, x)$. A similarity function d is also a distance (function or metric) if it satisfies the additional properties: $\forall x, y, z \in S$: (3) $d(x, y) = 0$ iff $x = y$ and (4) $d(x, z) \leq d(x, y) + d(y, z)$. We emphasize that in our work we do not need to restrict similarity functions to be distances.

An example of a similarity function is the *generalized Manhattan distance* defined as: $d_M(s, t) = \sum_{i=1}^{\infty} \text{diff}(s[i], t[i])$ for infinite words s, t (resp., $d_M(s, t) = \sum_{i=1}^{\max(|s|, |t|)} \text{diff}(s[i], t[i])$ for finite s, t), where diff is the mismatch penalty for substituting letters. The mismatch penalty is required to be a distance function on the alphabet (extended with a special end-of-string letter $\#$ for finite words). When $\text{diff}(a, b)$ is defined to be 1 for all a, b with $a \neq b$, and 0 otherwise, d_M is called the *Manhattan distance*.

Notation: We use $s_1 \otimes \dots \otimes s_k$ to denote *convolution* of words s_1, \dots, s_k , for $k > 1$. The convolution of k words merges the arguments into a single word over a k -tuple alphabet

(accommodating arguments of different lengths using # letters at the ends of shorter words). Let s_1, \dots, s_k be words over alphabets $\Sigma_1, \dots, \Sigma_k$. Let $\Sigma_1 \otimes \dots \otimes \Sigma_k$ denote the k -tuple alphabet $(\Sigma_1 \cup \{\#\}) \times \dots \times (\Sigma_k \cup \{\#\})$. The convolution $s_1 \otimes \dots \otimes s_k$ is an infinite word (resp., a finite word of length $\max(|s_1|, \dots, |s_k|)$), over $\Sigma_1 \otimes \dots \otimes \Sigma_k$, such that: for each $i \in \{1, \dots, |s_1 \otimes \dots \otimes s_k|\}$, $(s_1 \otimes \dots \otimes s_k)[i] = \langle s_1[i], \dots, s_k[i] \rangle$ (with $s_j[i] = \#$ if $i > |s_j|$). For example, the convolution $aa \otimes b \otimes add$ is the 3 letter word $\langle a, b, a \rangle \langle a, \#, d \rangle \langle \#, \#, d \rangle$.

► **Definition 2** (Automatic Similarity Function). A similarity function $d : \Sigma_1^\omega \times \Sigma_2^\omega \mapsto \mathbb{Q}$ is called automatic if there exists a WA \mathcal{A}_d over $\Sigma_1 \otimes \Sigma_2$ such that $\forall s_1 \in \Sigma_1^\omega, s_2 \in \Sigma_2^\omega$: $d(s_1, s_2) = \mathcal{L}_{\mathcal{A}_d}(s_1 \otimes s_2)$. We say that d is computed by \mathcal{A}_d .

One can similarly define automatic similarity functions over finite words.

3 Problem Definition

Our notion of robustness for transducers is based on the analytic notion of Lipschitz continuity. We first define K -Lipschitz robustness of functional transducers.

► **Definition 3** (K -Lipschitz Robustness of Functional Transducers). Given a constant $K \in \mathbb{Q}$ with $K > 0$ and similarity functions $d_\Sigma : \Sigma^\omega \times \Sigma^\omega \rightarrow \mathbb{Q}^\infty$ (resp., $d_\Sigma : \Sigma^* \times \Sigma^* \rightarrow \mathbb{Q}^\infty$) and $d_\Gamma : \Gamma^\omega \times \Gamma^\omega \rightarrow \mathbb{Q}^\infty$ (resp., $d_\Gamma : \Gamma^* \times \Gamma^* \rightarrow \mathbb{Q}^\infty$), a functional transducer \mathcal{T} , with $\llbracket \mathcal{T} \rrbracket \subseteq \Sigma^\omega \times \Gamma^\omega$ (resp., $\llbracket \mathcal{T} \rrbracket \subseteq \Sigma^* \times \Gamma^*$), is called K -Lipschitz robust w.r.t. d_Σ, d_Γ if:

$$\forall s, t \in \text{dom}(\mathcal{T}) : d_\Sigma(s, t) < \infty \Rightarrow d_\Gamma(\llbracket \mathcal{T} \rrbracket(s), \llbracket \mathcal{T} \rrbracket(t)) \leq K d_\Sigma(s, t).$$

Recall that when \mathcal{T} is an arbitrary nondeterministic transducer, for each $s \in \text{dom}(\mathcal{T})$, $\llbracket \mathcal{T} \rrbracket(s)$ is a set of words in Γ^ω (resp., Γ^*). Hence, we cannot use a similarity function over Γ^ω (resp., Γ^*) to define the similarity between $\llbracket \mathcal{T} \rrbracket(s)$ and $\llbracket \mathcal{T} \rrbracket(t)$, for $s, t \in \text{dom}(\mathcal{T})$. Instead, we must use a *set-similarity function* that can compute the similarity between sets of words in Γ^ω (resp., Γ^*). We define K -Lipschitz robustness of nondeterministic transducers using such set-similarity functions (we use the notation d and D for similarity functions and set-similarity functions, respectively).

► **Definition 4** (K -Lipschitz Robustness of Nondeterministic Transducers). Given a constant $K \in \mathbb{Q}$ with $K > 0$, a similarity function $d_\Sigma : \Sigma^\omega \times \Sigma^\omega \rightarrow \mathbb{Q}^\infty$ (resp., $d_\Sigma : \Sigma^* \times \Sigma^* \rightarrow \mathbb{Q}^\infty$) and a set-similarity function $D_\Gamma : 2^{\Gamma^\omega} \times 2^{\Gamma^\omega} \rightarrow \mathbb{Q}^\infty$ (resp., $D_\Gamma : 2^{\Gamma^*} \times 2^{\Gamma^*} \rightarrow \mathbb{Q}^\infty$), a nondeterministic transducer \mathcal{T} , with $\llbracket \mathcal{T} \rrbracket \subseteq \Sigma^\omega \times \Gamma^\omega$ (resp. $\llbracket \mathcal{T} \rrbracket \subseteq \Sigma^* \times \Gamma^*$), is called K -Lipschitz robust w.r.t. d_Σ, D_Γ if:

$$\forall s, t \in \text{dom}(\mathcal{T}) : d_\Sigma(s, t) < \infty \Rightarrow D_\Gamma(\llbracket \mathcal{T} \rrbracket(s), \llbracket \mathcal{T} \rrbracket(t)) \leq K d_\Sigma(s, t).$$

In what follows, we use K -robustness to denote K -Lipschitz robustness. The results in the remainder of this paper hold both for machines processing ω -words as well as for those processing finite words. To keep the presentation clean, we present all results in the context of machines over ω -words, making a distinction as needed.

4 Synchronized (Functional) Transducers

In this section, we define a class of functional transducers which admits a decision procedure for K -robustness.

► **Definition 5** (Synchronized Transducers). A functional transducer \mathcal{T} with $\llbracket \mathcal{T} \rrbracket \subseteq \Sigma^\omega \times \Gamma^\omega$ is synchronized iff there exists an automaton $\mathcal{A}_\mathcal{T}$ over $\Sigma \otimes \Gamma$ recognizing the language $\{s \otimes \llbracket \mathcal{T} \rrbracket(s) : s \in \text{dom}(\mathcal{T})\}$.

Let \mathcal{T} be an arbitrary functional transducer. In each transition, \mathcal{T} reads a single input letter and may generate an empty output word or an output word longer than a single letter. To process such non-aligned input and output words, the automaton $\mathcal{A}_{\mathcal{T}}$ needs to internally implement a buffer. Thus, \mathcal{T} is synchronized iff there is a bound B on the required size of such a buffer. We can use this observation to check if \mathcal{T} is synchronized.

► **Proposition 6.** *Synchronicity of a functional transducer is decidable in polynomial time.*

Synchronized transducers admit an automata-theoretic decision procedure for checking K -robustness w.r.t. similarity functions satisfying certain properties.

► **Theorem 7.** *For every $f \in \text{VALFUNC}$, if d_{Σ}, d_{Γ} are similarity functions computed by functional f -WA $\mathcal{A}_{d_{\Sigma}}, \mathcal{A}_{d_{\Gamma}}$, respectively, and \mathcal{T} is a synchronized transducer, K -robustness of \mathcal{T} w.r.t. d_{Σ}, d_{Γ} is decidable in polynomial time in the sizes of $\mathcal{T}, \mathcal{A}_{d_{\Sigma}}$ and $\mathcal{A}_{d_{\Gamma}}$.*

We show that for every $f \in \text{VALFUNC}$, if the conditions of Theorem 7 are met, K -robustness of \mathcal{T} can be reduced to the emptiness problem for f -weighted automata, which is decidable in polynomial time.

Similarity functions computed by nondeterministic automata. If we permit the weighted automata computing the similarity functions d_{Σ}, d_{Γ} to be nondeterministic, K -robustness becomes undecidable. We can show that the universality problem for nondeterministic weighted automata reduces to checking 1-robustness. Indeed, given a nondeterministic weighted automaton \mathcal{A} , consider (1) d_{Σ} such that $\forall s, t \in \Sigma^{\omega}: d_{\Sigma}(s, t) = \lambda$ if $s = t$, and undefined otherwise, (2) \mathcal{T} encoding the identity function, and (3) d_{Γ} such that $\forall s', t' \in \Sigma^{\omega}: d_{\Gamma}(s', t') = \mathcal{L}_{\mathcal{A}}(s')$ if $s' = t'$, and undefined otherwise. Note that d_{Γ} is computed by a nondeterministic weighted automaton obtained from \mathcal{A} by changing each transition (q, a, q') in \mathcal{A} to $(q, (a, a), q')$ while preserving the weight. Then, \mathcal{T} is 1-robust w.r.t. d_{Σ}, d_{Γ} iff for all words s , $\mathcal{L}_{\mathcal{A}}(s) \leq \lambda$. Since the universality problem for f -weighted automata is undecidable (e.g., for $f = \text{SUM}$), it follows that checking 1-robustness of transducers with similarity functions computed by nondeterministic weighted automata is undecidable.

We now present examples of synchronized transducers and automatic similarity functions satisfying the conditions of Theorem 7.

► **Example 8. Mealy machines and generalized Manhattan distances.** Mealy machines are perhaps the most widely used transducer model. Prior work [21] has shown decidability of robustness of Mealy machines with respect to generalized Manhattan distances given a fixed bound on the amount of input perturbation. In what follows, we argue the decidability of robustness of Mealy machines (processing infinite words) with respect to generalized Manhattan distances in the presence of unbounded input perturbation.

A Mealy machine $\mathcal{T} : (\Sigma, \Gamma, Q, \{q_0\}, E_{\mathcal{T}}, Q)$ is a synchronized transducer with $\mathcal{A}_{\mathcal{T}}$ given by $(\Sigma \otimes \Gamma, Q, \{q_0\}, E_{\mathcal{A}_{\mathcal{T}}}, Q)$, where $E_{\mathcal{A}_{\mathcal{T}}} = \{(q, a \otimes a', q') : (q, a, a', q') \in E_{\mathcal{T}}\}$. The generalized Manhattan distance $d_M : \Sigma^{\omega} \times \Sigma^{\omega} \rightarrow \mathbb{Q}^{\infty}$ can be computed by a functional SUM-weighted automaton \mathcal{A}_M given by the tuple $(\Sigma \otimes \Sigma, \{q_0\}, \{q_0\}, E_M, \{q_0\}, c)$. Here, q_0 is the initial as well as the accepting state, $E_M = \{(q_0, a \otimes b, q_0) : a \otimes b \in \Sigma \otimes \Sigma\}$, and the weight of each transition $(q_0, a \otimes b, q_0)$ equals $\text{diff}(a, b)$.

Thus, all the conditions of Theorem 7 are satisfied. K -robustness of Mealy machines, when d_{Σ} and d_{Γ} are computed as the generalized Manhattan distance, is decidable in polynomial time.

► **Example 9. Piecewise-linear functions.** Let us use \underline{q} to denote an infinite word over $\{0, \dots, 9, +, -\}$ representing the fractional part of a real number in base 10. E.g.,

$-0.21 = -21$ and $\pi - 3 = 1415\dots$. Then, $\underline{q_1} \otimes \dots \otimes \underline{q_k}$ is a word over $\{0, \dots, 9, +, -\} \otimes \dots \otimes \{0, \dots, 9, +, -\}$ that represents a k -tuple of real numbers q_1, \dots, q_k from the interval $[-1, 1]$. Now, observe that one can define letter-to-letter transducers that compute the following functions: (1) swapping of arguments, $\llbracket \mathcal{T} \rrbracket(\underline{q_1}, \dots, \underline{q_l}, \dots, \underline{q_m}, \dots, \underline{q_k}) = (\underline{q_1}, \dots, \underline{q_m}, \dots, \underline{q_l}, \dots, \underline{q_k})$, (2) addition, $\llbracket \mathcal{T} \rrbracket(\underline{q_1}, \dots, \underline{q_k}) = (\underline{q_1 + q_2}, \underline{q_2}, \dots, \underline{q_k})$, (3) multiplication by a constant c , $\llbracket \mathcal{T} \rrbracket(\underline{q_1}, \dots, \underline{q_k}) = (\underline{cq_1}, \dots, \underline{cq_k})$, (4) projection, $\llbracket \mathcal{T} \rrbracket(\underline{q_1}, \dots, \underline{q_k}) = (\underline{q_1}, \dots, \underline{q_{k-1}})$, and (5) conditional expression, $\llbracket \mathcal{T} \rrbracket(\underline{q_1}, \dots, \underline{q_k})$ equals $\llbracket \mathcal{T}_1 \rrbracket(\underline{q_1}, \dots, \underline{q_k})$, if $\underline{q_1} > 0$, and $\llbracket \mathcal{T}_2 \rrbracket(\underline{q_1}, \dots, \underline{q_k})$ otherwise. We can model a large class of piecewise-linear functions using transducers obtained by composition of transducers (1)-(5). The resulting transducers are functional letter-to-letter transducers.

Now, consider d_Σ, d_Γ defined as the $L1$ -norm over \mathbb{R}^k , i.e., $d_\Sigma(\underline{q_1} \otimes \dots \otimes \underline{q_k}, \underline{q'_1} \otimes \dots \otimes \underline{q'_k}) = d_\Gamma(\underline{q_1} \otimes \dots \otimes \underline{q_k}, \underline{q'_1} \otimes \dots \otimes \underline{q'_k}) = \sum_{i=1}^k \text{abs}(q_i - q'_i)$. Observe that d_Σ, d_Γ can be computed by deterministic DISC_δ -weighted automata, with $\delta = \frac{1}{10}$. Therefore, 1-robustness of \mathcal{T} can be decided in polynomial time (Theorem 7). Finally, note that K -robustness of a transducer computing a piecewise-linear function h w.r.t. the above similarity functions is equivalent to Lipschitz continuity of h with coefficient K .

5 Functional Transducers

It was shown in [21] that checking K -robustness of a functional transducer w.r.t. to a fixed bound on the amount of input perturbation is decidable. In what follows, we show that when the amount of input perturbation is unbounded, the robustness problem becomes undecidable even for deterministic transducers.

► **Theorem 10.** *1-robustness of deterministic transducers is undecidable.*

Proof. The Post Correspondence Problem (PCP) is defined as follows. Given a set of word pairs $\{\langle v_1, w_1 \rangle, \dots, \langle v_k, w_k \rangle\}$, does there exist a sequence of indices i_1, \dots, i_n such that $v_{i_1} \dots v_{i_n} = w_{i_1} \dots w_{i_n}$? PCP is known to be undecidable.

Let $\mathcal{G}_{pre} = \{\langle v_1, w_1 \rangle, \dots, \langle v_k, w_k \rangle\}$ be a PCP instance with $v_i, w_i \in \{a, b\}^*$ for each $i \in [1, k]$. We define a new instance $\mathcal{G} = \mathcal{G}_{pre} \cup \{\langle v_{k+1}, w_{k+1} \rangle\}$, where $\langle v_{k+1}, w_{k+1} \rangle = \langle \$, \$ \rangle$. Observe that for $i_1, \dots, i_n \in [1, k]$, $i_1, \dots, i_n, k+1$ is a solution of \mathcal{G} iff i_1, \dots, i_n is a solution of \mathcal{G}_{pre} . We define a deterministic transducer \mathcal{T} processing finite words and generalized Manhattan distances d_Σ, d_Γ such that \mathcal{T} is *not* 1-robust w.r.t. d_Σ, d_Γ iff \mathcal{G} has a solution of the form $i_1, \dots, i_n, k+1$, with $i_1, \dots, i_n \in [1, k]$.

We first define \mathcal{T} , which translates indices into corresponding words from the PCP instance \mathcal{G} . The input alphabet Σ is the set of indices from \mathcal{G} , marked with a *polarity*, L or R , denoting whether an index i , corresponding to a pair $\langle v_i, w_i \rangle \in \mathcal{G}$, is translated to v_i or w_i . Thus, $\Sigma = \{1, \dots, k+1\} \times \{L, R\}$. The output alphabet Γ is the alphabet of words in \mathcal{G} , marked with a polarity. Thus, $\Gamma = \{a, b, \$\} \times \{L, R\}$. The domain of $\llbracket \mathcal{T} \rrbracket$ is described by the following regular expression: $\text{dom}(\mathcal{T}) = \Sigma_L^* \langle k+1, L \rangle + \Sigma_R^* \langle k+1, R \rangle$, where for $P \in \{L, R\}$, $\Sigma_P = \{1, \dots, k\} \times \{P\}$. Thus, \mathcal{T} only processes input words over letters with the same polarity, rejecting upon reading an input letter with a polarity different from that of the first input letter. Moreover, \mathcal{T} accepts iff the first occurrence of $\langle k+1, L \rangle$ or $\langle k+1, R \rangle$ is in the last position of the input word. Note that the domain of \mathcal{T} is *prefix-free*, i.e., if $s, t \in \text{dom}(\mathcal{T})$ and s is a prefix of t , then $s = t$. Let u^P denote the word $u \otimes P^{|u|}$. Along accepting runs, \mathcal{T} translates each input letter $\langle i, L \rangle$ to v_i^L and each letter $\langle i, R \rangle$ to w_i^R , where $\langle v_i, w_i \rangle$ is the

i^{th} word pair of \mathcal{G} . Thus, the function computed by \mathcal{T} is:

$$\begin{aligned} \llbracket \mathcal{T} \rrbracket(\langle i_1, L \rangle \dots \langle i_n, L \rangle \langle k+1, L \rangle) &= v_{i_1}^L \dots v_{i_n}^L v_{k+1}^L \\ \llbracket \mathcal{T} \rrbracket(\langle i_1, R \rangle \dots \langle i_n, R \rangle \langle k+1, R \rangle) &= w_{i_1}^R \dots w_{i_n}^R w_{k+1}^R \end{aligned}$$

We define the output similarity function d_Γ as a generalized Manhattan distance with the following *symmetric* diff_Γ where $P, Q \in \{L, R\}$ and $\alpha, \beta \in \{a, b, \$\}$ with $\alpha \neq \beta$:

$$\begin{aligned} \text{diff}_\Gamma(\langle \alpha, P \rangle, \langle \alpha, P \rangle) &= 0 & \text{diff}_\Gamma(\langle \alpha, L \rangle, \langle \alpha, R \rangle) &= 2 \\ \text{diff}_\Gamma(\langle \alpha, P \rangle, \langle \beta, Q \rangle) &= 1 & \text{diff}_\Gamma(\langle \alpha, P \rangle, \#) &= 1 \end{aligned}$$

Note that for $s', t' \in \Gamma^*$ with different polarities, $d_\Gamma(s', t')$ equals the sum of $\max(|s'|, |t'|)$ and $\mathcal{N}(s', t')$, where $\mathcal{N}(s', t')$ is the number of positions in which s' and t' agree on the first components of their letters.

Let us define a projection π as $\pi(\langle i_1, P_1 \rangle \langle i_2, P_2 \rangle \dots \langle i_n, P_n \rangle) = i_1 i_2 \dots i_n$, where $i_1, \dots, i_n \in [1, k+1]$ and $P_1, \dots, P_n \in \{L, R\}$. We define the input similarity function d_Σ as a generalized Manhattan distance such that $d_\Sigma(s, t)$ is finite iff $\pi(s)$ is a prefix of $\pi(t)$ or vice versa. We define d_Σ using the following *symmetric* diff_Σ where $P, Q \in \{L, R\}$ and $i, j \in [1, k+1]$ with $i \neq j$:

$$\begin{aligned} \text{diff}_\Sigma(\langle i, P \rangle, \langle i, P \rangle) &= 0 & \text{diff}_\Sigma(\langle i, P \rangle, \langle j, Q \rangle) &= \infty \\ \text{diff}_\Sigma(\langle i, L \rangle, \langle i, R \rangle) &= |v_i| + |w_i|, \text{ if } i \in [1, k] & \text{diff}_\Sigma(\langle i, P \rangle, \#) &= \infty \\ \text{diff}_\Sigma(\langle k+1, L \rangle, \langle k+1, R \rangle) &= 1 \end{aligned}$$

Thus, for all $s, t \in \text{dom}(\mathcal{T})$, $d_\Sigma(s, t) < \infty$ iff one of the following holds:

- (i) for some $P \in \{L, R\}$, $s = t = \langle i_1, P \rangle \dots \langle i_n, P \rangle \langle k+1, P \rangle$, or,
- (ii) $s = \langle i_1, L \rangle \dots \langle i_n, L \rangle \langle k+1, L \rangle$ and $t = \langle i_1, R \rangle \dots \langle i_n, R \rangle \langle k+1, R \rangle$.

In case (i), $d_\Sigma(s, t) = d_\Gamma(\llbracket \mathcal{T} \rrbracket(s), \llbracket \mathcal{T} \rrbracket(t)) = 0$. In case (ii), $d_\Sigma(s, t) = |\llbracket \mathcal{T} \rrbracket(s)| + |\llbracket \mathcal{T} \rrbracket(t)| - 1$ and $d_\Gamma(\llbracket \mathcal{T} \rrbracket(s), \llbracket \mathcal{T} \rrbracket(t)) = \max(|\llbracket \mathcal{T} \rrbracket(s)|, |\llbracket \mathcal{T} \rrbracket(t)|) + \mathcal{N}(\llbracket \mathcal{T} \rrbracket(s), \llbracket \mathcal{T} \rrbracket(t))$. Thus, $d_\Gamma(\llbracket \mathcal{T} \rrbracket(s), \llbracket \mathcal{T} \rrbracket(t)) > d_\Sigma(s, t)$ iff $\mathcal{N}(\llbracket \mathcal{T} \rrbracket(s), \llbracket \mathcal{T} \rrbracket(t)) = \min(|\llbracket \mathcal{T} \rrbracket(s)|, |\llbracket \mathcal{T} \rrbracket(t)|)$. Since the letters $\langle \$, L \rangle, \langle \$, R \rangle$ occur exactly once in $\llbracket \mathcal{T} \rrbracket(s), \llbracket \mathcal{T} \rrbracket(t)$, respectively, at the end of each word, $\mathcal{N}(\llbracket \mathcal{T} \rrbracket(s), \llbracket \mathcal{T} \rrbracket(t)) = \min(|\llbracket \mathcal{T} \rrbracket(s)|, |\llbracket \mathcal{T} \rrbracket(t)|)$ iff $|\llbracket \mathcal{T} \rrbracket(s)| = |\llbracket \mathcal{T} \rrbracket(t)|$ and $\pi(\llbracket \mathcal{T} \rrbracket(s)) = \pi(\llbracket \mathcal{T} \rrbracket(t))$, which holds iff \mathcal{G} has a solution. Therefore, \mathcal{T} is not 1-robust w.r.t. d_Σ, d_Γ iff \mathcal{G} has a solution. \blacktriangleleft

We have shown that checking 1-robustness w.r.t. generalized Manhattan distances is undecidable. Observe that for every $K > 0$, K -robustness can be reduced to 1-robustness by scaling the output distance by K . We conclude that checking K -robustness is undecidable for any fixed K . In contrast, if K is not fixed, checking if there exists K such that \mathcal{T} is K -robust w.r.t. d_Σ, d_Γ is decidable for transducers processing finite words.

Let us define a functional transducer \mathcal{T} to be *robust* w.r.t. d_Σ, d_Γ if there exists K such that \mathcal{T} is K -robust w.r.t. d_Σ, d_Γ .

► **Proposition 11.** *Let \mathcal{T} be a given functional transducer processing finite words and d_Σ, d_Γ be instances of the generalized Manhattan distance.*

1. *Robustness of \mathcal{T} is decidable in CO-NP.*
2. *One can compute $K_\mathcal{T}$ such that \mathcal{T} is robust iff \mathcal{T} is $K_\mathcal{T}$ -robust.*

Proof sketch. Given \mathcal{T} , one can easily construct a *trim*² functional transducer $\mathcal{P}_\mathcal{T}$ such that $\llbracket \mathcal{P}_\mathcal{T} \rrbracket(s, t) = (s', t')$ iff $\llbracket \mathcal{T} \rrbracket(s) = s'$ and $\llbracket \mathcal{T} \rrbracket(t) = t'$. We show that \mathcal{T} is not robust

² $\mathcal{P}_\mathcal{T}$ is trim if every state in $\mathcal{P}_\mathcal{T}$ is reachable from the initial state and some final state is reachable from every state in $\mathcal{P}_\mathcal{T}$.

w.r.t. generalized Manhattan distances iff there exists some cycle in $\mathcal{P}_{\mathcal{T}}$ satisfying certain properties. Checking the existence of such a cycle is in NP. If such a cycle exists, one can construct paths in $\mathcal{P}_{\mathcal{T}}$ through the cycle, labeled with input words (s, t) and output words (s', t') , with $d_{\Gamma}(s', t') > K d_{\Sigma}(s, t)$ for *any* K . Conversely, if there exists no such cycle, one can compute $K_{\mathcal{T}}$ such that \mathcal{T} is $K_{\mathcal{T}}$ -robust. It follows that one can compute $K_{\mathcal{T}}$ such that \mathcal{T} is robust iff \mathcal{T} is $K_{\mathcal{T}}$ -robust.

5.1 Beyond Synchronized transducers

In this section, we present an approach for natural extensions of Theorem 7.

Isometry approach. We say that a transducer \mathcal{T} is a $(d_{\Lambda}, d_{\Delta})$ -isometry if and only if for all $s, t \in \text{dom}(\mathcal{T})$ we have $d_{\Lambda}(s, t) = d_{\Delta}(\llbracket \mathcal{T} \rrbracket(s), \llbracket \mathcal{T} \rrbracket(t))$.

► **Proposition 12.** *Let $\mathcal{T}, \mathcal{T}'$ be transducers with $\llbracket \mathcal{T} \rrbracket \subseteq \Sigma^{\omega} \times \Gamma^{\omega}$ and $\llbracket \mathcal{T}' \rrbracket \subseteq \Lambda^{\omega} \times \Delta^{\omega}$. Assume that there exist transducers \mathcal{T}^I and \mathcal{T}^O such that \mathcal{T}^I is a $(d_{\Sigma}, d_{\Lambda})$ -isometry, \mathcal{T}^O is a (d_{Δ}, d_{Γ}) -isometry and $\llbracket \mathcal{T} \rrbracket = \llbracket \mathcal{T}^O \circ (\mathcal{T}' \circ \mathcal{T}^I) \rrbracket$. Then, for every $K > 0$, \mathcal{T} is K -robust w.r.t. d_{Σ}, d_{Γ} if and only if \mathcal{T}' is K -robust w.r.t. d_{Λ}, d_{Δ} .*

► **Example 13 (Stuttering).** For a given word w we define the *stuttering pruned* word $\text{STUTTER}(w)$ as the result of removing from w letters that are the same as the previous letter. E.g. $\text{STUTTER}(\mathbf{baacccaab}) = \mathit{bacab}$.

Consider a transducer \mathcal{T} and a similarity function d_{Σ} over finite words that are *stuttering invariant*, i.e., for all $s, t \in \text{dom}(\mathcal{T})$, if $\text{STUTTER}(s) = \text{STUTTER}(t)$, then $\llbracket \mathcal{T} \rrbracket(s) = \llbracket \mathcal{T} \rrbracket(t)$ and for every $u \in \Sigma^*$, $d_{\Sigma}(s, u) = d_{\Sigma}(t, u)$. In addition, we assume that for every $s \in \text{dom}(\mathcal{T})$, $|\llbracket \mathcal{T} \rrbracket(s)| = |\text{STUTTER}(s)|$.

Observe that these assumptions imply that: (1) the projection transducer \mathcal{T}^{π} defined such that $\llbracket s \rrbracket = \text{STUTTER}(s)$ is a (d_{Σ}, d_{Σ}) -isometry, (2) the transducer \mathcal{T}^S obtained by restricting the domain of \mathcal{T} to *stuttering-free* words, i.e., the set $\{w \in \text{dom}(\mathcal{T}) : \text{STUTTER}(w) = w\}$, is a synchronized transducer, and (3) $\llbracket \mathcal{T} \rrbracket = \llbracket \mathcal{T}^I \circ (\mathcal{T}^S \circ \mathcal{T}^{\pi}) \rrbracket$, where \mathcal{T}^I defines the identity function over Γ^* . Therefore, by Proposition 12, in order to check K -robustness of \mathcal{T} , it suffices to check K -robustness of \mathcal{T}^S . Since \mathcal{T}^S is a synchronized transducer, K -robustness of \mathcal{T}^S can be effectively checked, provided the similarity functions d_{Σ}, d_{Γ} satisfy the conditions of Theorem 7.

► **Example 14 (Letter-to-multiple-letters transducers).** Consider a transducer \mathcal{T} which on every transition outputs a 2-letter word³. Although, \mathcal{T} is not synchronized, it can be transformed to a letter-to-letter transducer \mathcal{T}^D , whose output alphabet is $\Gamma \times \Gamma$. The transducer \mathcal{T}^D is obtained from \mathcal{T} by substituting each output word ab to a single letter $\langle a, b \rangle$ from $\Gamma \times \Gamma$. We can use \mathcal{T}^D to decide K -robustness of \mathcal{T} in the following way. First, we define transducers $\mathcal{T}^I, \mathcal{T}^{\text{pair}}$ such that \mathcal{T}^I computes the identity function over Σ^{ω} and $\mathcal{T}^{\text{pair}}$ is a transducer representing the function $\llbracket \mathcal{T}^{\text{pair}} \rrbracket(\langle a_1, b_1 \rangle \langle a_2, b_2 \rangle \dots) = a_1 b_1 a_2 b_2 \dots$. Observe that $\llbracket \mathcal{T} \rrbracket = \llbracket \mathcal{T}^{\text{pair}} \circ (\mathcal{T}^D \circ \mathcal{T}^I) \rrbracket$. Second, we define d_{Γ}^D as follows: $\forall s, t \in (\Sigma \times \Sigma)^{\omega}$, $d_{\Gamma}^D(s, t) = d_{\Gamma}(\llbracket \mathcal{T}^{\text{pair}} \rrbracket(s), \llbracket \mathcal{T}^{\text{pair}} \rrbracket(t))$. Observe that \mathcal{T}^I is a (d_{Σ}, d_{Σ}) -isometry and $\mathcal{T}^{\text{pair}}$ is a $(d_{\Gamma}^D, d_{\Gamma})$ -isometry. Thus, K -robustness of \mathcal{T} w.r.t. d_{Σ}, d_{Γ} reduces to K -robustness of the letter-to-letter transducer \mathcal{T}^D w.r.t. d_{Σ}, d_{Γ}^D , which can be effectively checked (Theorem 7).

³ One can easily generalize this example to any fixed number.

6 Nondeterministic Transducers

Let \mathcal{T} be a nondeterministic transducer with $\llbracket \mathcal{T} \rrbracket \subseteq \Sigma^\omega \times \Gamma^\omega$. Let d_Σ be an automatic similarity function for computing the similarity between input words in Σ^* . As explained in Sec. 3, the definition of K -robust nondeterministic transducers involves set-similarity functions that can compute the similarity between sets of output words in Γ^ω . In this section, we examine the K -robustness problem of \mathcal{T} w.r.t. d_Σ and three classes of such set-similarity functions.

Let d_Γ be an automatic similarity function for computing the similarity between output words in Γ^ω . We first define three set-similarity functions induced by d_Γ .

► **Definition 15.** Given sets A, B of words in Γ^ω , we consider the following set-similarity functions induced by d_Γ :

- (i) Hausdorff set-similarity function $D_\Gamma^H(A, B)$ induced by d_Γ :

$$D_\Gamma^H(A, B) = \max\{ \sup_{s \in A} \inf_{t \in B} d_\Gamma(s, t), \sup_{s \in B} \inf_{t \in A} d_\Gamma(s, t) \}$$

- (ii) Inf-inf set-similarity function $D_\Gamma^{\text{inf}}(A, B)$ induced by d_Γ :

$$D_\Gamma^{\text{inf}}(A, B) = \inf_{s \in A} \inf_{t \in B} d_\Gamma(s, t)$$

- (iii) Sup-sup set-similarity function $D_\Gamma^{\text{sup}}(A, B)$ induced by d_Γ :

$$D_\Gamma^{\text{sup}}(A, B) = \sup_{s \in A} \sup_{t \in B} d_\Gamma(s, t)$$

Of the above set-similarity functions, only the Hausdorff set-similarity function is a distance function (if d_Γ is a distance function).

Note that when \mathcal{T} is a functional transducer, each set-similarity function above reduces to d_Γ . Hence, K -robustness of a functional transducer \mathcal{T} w.r.t. d_Σ , D_Γ and K -robustness of \mathcal{T} w.r.t. d_Σ , d_Γ coincide. As K -robustness of functional transducers is undecidable (Theorem 10), K -robustness of nondeterministic transducers w.r.t. the above set-similarity functions is undecidable as well.

Recall from Theorem 7 that K -robustness of a synchronized (functional) transducer is decidable w.r.t. certain automatic similarity functions. In particular, K -robustness of Mealy machines is decidable when d_Σ , d_Γ are generalized Manhattan distances. In contrast, K -robustness of nondeterministic letter-to-letter transducers is undecidable w.r.t. the Hausdorff and Inf-inf set-similarity functions even when d_Σ , d_Γ are generalized Manhattan distances. Among the above defined set-similarity functions, K -robustness of nondeterministic transducers is decidable only w.r.t. the Sup-sup set-similarity function.

► **Theorem 16.** Let d_Σ , d_Γ be computed by functional weighted-automata. Checking K -robustness of nondeterministic letter-to-letter transducers w.r.t. d_Σ , D_Γ induced by d_Γ is

- (i) undecidable if D_Γ is the Hausdorff set-similarity function,
- (ii) undecidable if D_Γ is the Inf-inf set-similarity function, and
- (iii) decidable if D_Γ is the Sup-sup set-similarity function and d_Σ, d_Γ satisfy the conditions of Theorem 7.

Proof. [of (iii)] We can encode nondeterministic choices of \mathcal{T} , with $\llbracket \mathcal{T} \rrbracket \subseteq \Sigma^\omega \times \Gamma^\omega$, in an extended input alphabet $\Sigma \times \Lambda$. We construct a deterministic transducer \mathcal{T}^e such that for every $s \in \Sigma^\omega$, $\{\llbracket \mathcal{T}^e \rrbracket(\langle s, \lambda \rangle) : \langle s, \lambda \rangle \in \text{dom}(\mathcal{T}^e)\} = \llbracket \mathcal{T} \rrbracket(s)$. We also define d_Σ^e such that for all $\langle s, \lambda_1 \rangle, \langle t, \lambda_2 \rangle \in (\Sigma \times \Lambda)^\omega$, $d_\Sigma^e(\langle s, \lambda_1 \rangle, \langle t, \lambda_2 \rangle) = d_\Sigma(s, t)$. Then, \mathcal{T} is K -robust w.r.t. $d_\Sigma, D_\Gamma^{\text{sup}}$ induced by d_Γ iff \mathcal{T}^e is K -robust w.r.t. d_Σ^e, d_Γ . Indeed, a nondeterministic transducer \mathcal{T} is K -robust w.r.t. $d_\Sigma, D_\Gamma^{\text{sup}}$ induced by d_Γ iff for all input words $s, t \in \text{dom}(\mathcal{T})$ and for all outputs $s' \in \llbracket \mathcal{T} \rrbracket(s), t' \in \llbracket \mathcal{T} \rrbracket(t)$, $d_\Sigma(s, t) < \infty$ implies $d_\Gamma(s', t') \leq K d_\Sigma(s, t)$. ◀

7 Related Work

In early work [18], [7, 8] on continuity and robustness analysis, the focus is on software programs manipulating numbers. In [18], the authors compute the maximum deviation of a program's output given the maximum possible perturbation in a program input. In [7], the authors formalize $\epsilon - \delta$ continuity of programs and present sound proof rules to prove continuity of programs. In [8], the authors formalize robustness of programs as Lipschitz continuity and present a sound program analysis for robustness verification. While arrays of numbers are considered in [8], the size of an array is immutable.

More recent papers have aimed to develop a notion of robustness for reactive systems. In [22], the authors present polynomial-time algorithms for the analysis and synthesis of robust transducers. Their notion of robustness is one of input-output stability, that bounds the output deviation from disturbance-free behaviour under bounded disturbance, as well as the persistence of the effect of a sporadic disturbance. Their distances are measured using cost functions that map *each* string to a nonnegative integer. In [17, 4, 2], the authors develop different notions of robustness for reactive systems, with ω -regular specifications, interacting with uncertain environments. In [9], the authors present a polynomial-time algorithm to decide robustness of sequential circuits modeled as Mealy machines, w.r.t. a *common suffix distance* metric. Their notion of robustness also bounds the persistence of the effect of a sporadic disturbance.

Recent work in [20] and [21] formalized and studied robustness of systems modeled using transducers, in the presence of bounded perturbation. The work in [20] focussed on the outputs of synchronous networks of Mealy machines in the presence of channel perturbation. The work in [21] focussed on the outputs of functional transducers in the presence of input perturbation. Both papers presented decision procedures for robustness verification w.r.t. specific distance functions such as Manhattan and Levenshtein distances.

8 Conclusion

In this paper, we studied the K -Lipschitz robustness problem for finite-state transducers. While the general problem is undecidable, we identified decidability criteria that enable reduction of K -robustness to the emptiness problem for weighted automata.

In the future, we wish to extend our work in two directions. We plan to study robustness of other computational models. We also wish to investigate synthesis of robust transducers.

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A Proofs

► **Proposition 6.** *Synchronicity of a functional transducer is decidable in polynomial time.*

Proof. We prove the result for transducers processing finite words. The proof for transducers processing infinite words is similar, but a bit more technical.

The proof consists of two claims:

(1) a functional transducer \mathcal{T} is synchronized iff there are $B > 0$ and a finite set of words \mathcal{L}_{fin} such that for every word $s = s[1]s[2]\dots \in \text{dom}(\mathcal{T})$ there is an accepting run $(q_0, u_1)(q_1, u_2)\dots(q_{n-1}, u_n)(q_n, \epsilon)$ on s such that for every $i \in \{1, \dots, |s|\}$

(a) $|u_1u_2\dots u_i| - i \leq B$, and

(b) if $i - |u_1u_2\dots u_i| > B$, then $u_{i+1}u_{i+2}\dots u_{|s|} \in \mathcal{L}_{\text{fin}}$.

(2) it is decidable in polynomial time whether such B exists.

(1) \Leftarrow : Given B and a finite language \mathcal{L}_{fin} we can construct an automaton that simulates runs of \mathcal{T} . The automaton implements a buffer of size B used to align input and output words. Due to assumptions on \mathcal{T} , the buffer will not overflow with output letters (cond. (a)), and if it overflows with input letters, the remaining words belongs to a finite language (cond. (b), language \mathcal{L}_{fin}). In the latter case, once the overflow is detected, the automaton can nondeterministically guess a word from \mathcal{L}_{fin} , and check correctness of that guess.

\Rightarrow : Assume towards contradiction that such $B, \mathcal{L}_{\text{fin}}$ do not exist but \mathcal{T} is synchronized, i.e., there exists an automaton \mathcal{A} satisfying: for all s, t we have $\llbracket \mathcal{T} \rrbracket(s) = t$ iff $s \otimes t \in \mathcal{L}_{\mathcal{A}}$. First suppose (a) is violated. Then, there is a finite word s such that $|\llbracket \mathcal{T} \rrbracket(s)| - |s| > |\mathcal{A}|$. Consider an accepting run of \mathcal{A} on $s \otimes \llbracket \mathcal{T} \rrbracket(s)$. That run can be pumped between positions $|s|$ and $|s| + |\mathcal{A}|$ to get an accepting run on $s \otimes u'$, where $|u'| > |\llbracket \mathcal{T} \rrbracket(s)|$, which contradicts functionality of \mathcal{T} .

Suppose (b) is violated. Consider a state \hat{q} of \mathcal{T} such that there exist $k = |\mathcal{A}| + 1$ inputs t_1, \dots, t_k corresponding to k pairwise different outputs v_1, \dots, v_k , i.e., the transducer starting in the state \hat{q} upon reading t_i , produces the word v_i . Let $M = \max(|v_1|, \dots, |v_k|)$. Suppose that there are $s = s[1]\dots s[i]$ and a run $(q_0, u_1)\dots(q_{i-2}, u_{i-1})(\hat{q}, u_i)$ such that $|s| - |u_1u_2\dots u_i| > M$. We have $\llbracket \mathcal{T} \rrbracket(st_1) = uv_1, \dots, \llbracket \mathcal{T} \rrbracket(st_k) = uv_k$, where $u = u_1u_2\dots u_i$. Thus, \mathcal{A} accepts $st_1 \otimes uv_1, \dots, st_k \otimes uv_k$. As $k > |\mathcal{A}|$ there are two different indices i, j such that \mathcal{A} is in the same state after reading $|s|$ -letter prefixes of $st_i \otimes uv_i$ and $st_j \otimes uv_j$. Since $|v_i|, |v_j| < M$, $|uv_i|, |uv_j| < |s|$. Therefore, \mathcal{A} accepts $st_i \otimes uv_j$, which contradicts functionality of \mathcal{T} .

(2) Consider a functional transducer \mathcal{T} . Let Q' be a subset of states of Q consisting of states that are reachable from the initial state and from which some final states are reachable. The condition (a) holds if and only if there are no cycles in which the output word is longer than the input word. The condition (b) holds if and only if all states reachable from cycles with empty output word have output languages finite, i.e., once transducer enters such a state, there is a bounded number of possible words it can output. ◀

► **Theorem 7.** *For every $f \in \text{VALFUNC}$, if d_Σ, d_Γ are similarity functions computed by functional f -WA $\mathcal{A}_{d_\Sigma}, \mathcal{A}_{d_\Gamma}$, respectively, and \mathcal{T} is a synchronized transducer, K -robustness of \mathcal{T} w.r.t. d_Σ, d_Γ is decidable in polynomial time in the sizes of $\mathcal{T}, \mathcal{A}_{d_\Sigma}$ and \mathcal{A}_{d_Γ} .*

Proof. Let \mathcal{A}_{d_Σ} be an f -weighted automaton computing the similarity function d_Σ . Let \mathcal{A}_{d_Γ} be a functional f -weighted automaton computing the similarity function d_Γ . Let $\mathcal{A}_\mathcal{T}$ be the automaton corresponding to \mathcal{T} as defined in Definition 5. We define variants of $\mathcal{A}_{d_\Sigma}, \mathcal{A}_{d_\Gamma}$ and $\mathcal{A}_\mathcal{T}$ to enable these automata to operate over a common alphabet $\Lambda = \Sigma \otimes \Sigma \otimes \Gamma \otimes \Gamma$. We obtain $\bar{\mathcal{A}}_{d_\Sigma}$ by replacing each transition $(q, \langle a, b \rangle, q')$ in \mathcal{A}_{d_Σ} with a set $\{(q, \langle a, b, a', b' \rangle, q') :$

$a', b' \in \Gamma$] of transitions and setting the weight of each transition $(q, \langle a, b, a', b' \rangle, q')$ in this set to the weight of $(q, \langle a, b \rangle, q')$. Thus, the value of a run of $\bar{\mathcal{A}}_{d_\Sigma}$ on a word $s \otimes t \otimes s' \otimes t'$ over Λ equals the value of the corresponding run (over the same sequence of states) of \mathcal{A}_{d_Σ} on word $s \otimes t$. This implies that the value of $\bar{\mathcal{A}}_{d_\Sigma}$ on $s \otimes t \otimes s' \otimes t'$ is equal to the value of \mathcal{A}_{d_Σ} on $s \otimes t$. In a similar way, we define automata $\bar{\mathcal{A}}_{d_\Gamma}, \bar{\mathcal{A}}_{d_\Sigma}^L, \bar{\mathcal{A}}_{d_\Sigma}^R$ on Λ such that for all s, t, s', t' :

1. the value of $\bar{\mathcal{A}}_{d_\Gamma}$ on $s \otimes t \otimes s' \otimes t'$ is equal to the value of \mathcal{A}_{d_Γ} on $s' \otimes t'$,
2. $\bar{\mathcal{A}}_{d_\Sigma}^L$ accepts $s \otimes t \otimes s' \otimes t'$ iff \mathcal{A}_{d_Σ} accepts $s \otimes s'$, and
3. $\bar{\mathcal{A}}_{d_\Sigma}^R$ accepts $s \otimes t \otimes s' \otimes t'$ iff \mathcal{A}_{d_Σ} accepts $t \otimes t'$.

Let $\bar{\mathcal{A}}_{d_\Sigma}^K, \bar{\mathcal{A}}_{d_\Gamma}^{-1}$ are f -weighted automata obtained by multiplying each transition weight of $\bar{\mathcal{A}}_{d_\Sigma}, \bar{\mathcal{A}}_{d_\Gamma}$ by $K, -1$, respectively. Consider the f -weighted automaton \mathcal{A} defined as $\bar{\mathcal{A}}_{d_\Sigma}^K \times \bar{\mathcal{A}}_{d_\Sigma}^L \times \bar{\mathcal{A}}_{d_\Sigma}^R \times \bar{\mathcal{A}}_{d_\Gamma}^{-1}$, the synchronized product of automata $\bar{\mathcal{A}}_{d_\Sigma}^K, \bar{\mathcal{A}}_{d_\Sigma}^L, \bar{\mathcal{A}}_{d_\Sigma}^R, \bar{\mathcal{A}}_{d_\Gamma}^{-1}$ where the weight of each transition is equal the sum of the weights of the corresponding transitions in $\bar{\mathcal{A}}_{d_\Sigma}^K$ and $\bar{\mathcal{A}}_{d_\Gamma}^{-1}$.

Now, we show that there exists a word with the value below 0 assigned by \mathcal{A} iff \mathcal{T} is not K -robust w.r.t. d_Σ, d_Γ . We perform case distinction: (1) $f \in \{\text{SUM}, \text{DISC}_\lambda^{\text{fin}}, \text{DISC}_\lambda^{\text{inf}}\}$ and (2) $f = \text{LIMAVG}$.

In proofs of cases (1) and (2) we use the following notation. Given sequences of real numbers σ_1, σ_2 and a real number c we denote by $\sigma_1 + \sigma_2$ the component-wise sum of sequences σ_1 and σ_2 , and by $c \cdot \sigma_1$ the component-wise multiplication of σ_1 by c .

(1): Let $f \in \{\text{SUM}, \text{DISC}_\lambda^{\text{fin}}, \text{DISC}_\lambda^{\text{inf}}\}$. Observe that for all sequences of real numbers σ_1, σ_2 and every real number c we have (a) $f(\sigma_1) + f(\sigma_2)$ equals $f(\sigma_1 + \sigma_2)$, and (b) $f(c \cdot \sigma_1) = c \cdot f(\sigma_1)$.

Consider words s, t, s', t' such that \mathcal{A} accepts $s \otimes t \otimes s' \otimes t'$. The conditions (a), (b) imply that $\mathcal{L}_{\mathcal{A}}(s \otimes t \otimes s' \otimes t')$ equals $Kd_\Sigma(s, t) + \inf_{\pi \in \text{Acc}} -f(\pi)$, where Acc is the set of accepting runs of \mathcal{A}_{d_Γ} on $s' \otimes t'$. As \mathcal{A}_{d_Γ} is functional, each run in Acc has the same value $d_\Gamma(s', t')$. Thus, $\mathcal{L}_{\mathcal{A}}(s \otimes t \otimes s' \otimes t')$ equals $Kd_\Sigma(s, t) - d_\Gamma(s', t')$, and, $\mathcal{L}_{\mathcal{A}}(s \otimes t \otimes s' \otimes t') < 0$ implies \mathcal{T} is not K -robust. Conversely, if \mathcal{T} is not K -robust there are words s, t such that $d_\Gamma(\llbracket \mathcal{T} \rrbracket(s), \llbracket \mathcal{T} \rrbracket(t)) > Kd_\Sigma(s, t)$. This implies \mathcal{A} accepts $s \otimes t \otimes \llbracket \mathcal{T} \rrbracket(s) \otimes \llbracket \mathcal{T} \rrbracket(t)$ and $\mathcal{L}_{\mathcal{A}}(s \otimes t \otimes \llbracket \mathcal{T} \rrbracket(s) \otimes \llbracket \mathcal{T} \rrbracket(t)) < 0$. Thus, nonemptiness of \mathcal{A} and K -robustness of \mathcal{T} w.r.t. d_Σ, d_Γ coincide

(2): Let $f = \text{LIMAVG}$. Observe that for sequences of real numbers σ_1, σ_2 , $\limsup_{k \rightarrow \infty} K \cdot \sigma_1[k] - \sigma_2[k] \geq K \limsup_{k \rightarrow \infty} \sigma_1[k] - \limsup_{k \rightarrow \infty} \sigma_2[k]$. Therefore, for sequences of real numbers σ_1, σ_2 , $\text{LIMAVG}(K\sigma_1 - \sigma_2) \leq K\text{LIMAVG}(\sigma_1) - \text{LIMAVG}(\sigma_2)$. It follows that if \mathcal{A} assigns to every accepted word the value greater or equal to 0, \mathcal{T} is K -robust w.r.t. d_Σ, d_Γ . Conversely, assume that \mathcal{A} has a run of the value below 0. Then, it has also a run π on some word $s \otimes t \otimes s' \otimes t'$ of the value below 0 which is a lasso. The partial averages of weights is lasso converge, i.e., $\limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{i=1}^k (c(\pi))[i] = \liminf_{k \rightarrow \infty} \frac{1}{k} \sum_{i=1}^k (c(\pi))[i]$. Hence, $0 > \text{LIMAVG}(\pi) = \text{LIMAVG}(\pi_1) - \text{LIMAVG}(\pi_2)$, where π_1 (resp. π_2) is the projection of the run π on states of $\bar{\mathcal{A}}_{d_\Sigma}^K$ (resp. $\bar{\mathcal{A}}_{d_\Gamma}^{-1}$). As $\text{LIMAVG}(\pi_1) \geq Kd_\Sigma(s, t)$ and $\text{LIMAVG}(\pi_2) \geq d_\Gamma(s', t')$, we have $Kd_\Sigma(s, t) < d_\Gamma(s', t')$. It follows that if \mathcal{A} assigns to some word the value below 0, \mathcal{T} is not K -robust w.r.t. d_Σ, d_Γ . ◀ ◀

Let $\mathcal{P}_{\mathcal{T}}$ denote the trim functional transducer obtained from \mathcal{T} such that $\llbracket \mathcal{P}_{\mathcal{T}} \rrbracket(s, t) = (s', t')$ iff $\llbracket \mathcal{T} \rrbracket(s) = s'$ and $\llbracket \mathcal{T} \rrbracket(t) = t'$. Let q_{0_p} denote an initial state of $\mathcal{P}_{\mathcal{T}}$ and q_{F_p} denote a final state of $\mathcal{P}_{\mathcal{T}}$. We say two words s, t are *rotations* of each other if $|s| = |t|$ and there exist words u and v such that $s = u.v$ and $t = v.u$; ($|u|, |v|$) are referred to as the *gap parameters*. Note that words s, t with $s = t$ are also rotations of each other.

► **Lemma 17.** *Let d be a Manhattan distance on $(\Sigma \cup \Gamma)^*$ and let \mathcal{T} be a transducer. The following conditions are equivalent:*

1. *there exists a cycle $c : (q_c, (w_c, v_c), (w'_c, v'_c), q_c)$ in $\mathcal{P}_{\mathcal{T}}$ with $d(w_c, v_c) = 0$, satisfying one of the following conditions:*
 - a. *w'_c, v'_c are not rotations of each other, or,*
 - b. *w'_c, v'_c are rotations of each other with gap parameters (r, s) , and there exists a path of \mathcal{P} through $c : (q_{0_p}, (w_{pre}, v_{pre}), (w'_{pre}, v'_{pre}), q_c) \cdot c$ with $\text{abs}(|v'_{pre}| - |w'_{pre}|) \neq r$ and $\text{abs}(|v'_{pre}| - |w'_{pre}|) \neq s$,*
2. *\mathcal{T} is not robust w.r.t. Manhattan distances, and*
3. *\mathcal{T} is not $K^{\mathcal{T}}$ -robust w.r.t. Manhattan distances, where $K^{\mathcal{T}} = |\mathcal{P}_{\mathcal{T}}|^2$.*

Proof. (i) \implies (ii): Suppose there exists a cycle $c : (q_c, (w_c, v_c), (w'_c, v'_c), q_c)$ in \mathcal{P} with $d(w_c, v_c) = 0$ satisfying condition 1 or 2 in Lemma 17. If c satisfies condition 1, let π denote any path $\pi_{pre} \cdot c$ in \mathcal{P} through c , with π_{pre} given by $(q_{0_p}, (w_{pre}, v_{pre}), (w'_{pre}, v'_{pre}), q_c)$. If c satisfies condition 2, let $\pi : \pi_{pre} \cdot c$, with $\pi_{pre} : (q_{0_p}, (w_{pre}, v_{pre}), (w'_{pre}, v'_{pre}), q_c)$, be a path such that $\text{abs}(|v'_{pre}| - |w'_{pre}|) \neq r$ and $\text{abs}(|v'_{pre}| - |w'_{pre}|) \neq s$.

Then, given an arbitrary $K \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that for the path $\pi_{pre} \cdot (c)^n$, $d(w'_{pre} \cdot (w'_c)^n, v'_{pre} \cdot (v'_c)^n) > Kd(w_{pre} \cdot (w_c)^n, v_{pre} \cdot (v_c)^n)$. Hence, \mathcal{T} is not robust.

(ii) \implies (iii): Follows directly from the definition of robustness.

(iii) \implies (i): Assume towards contradiction that no cycle in $\mathcal{P}_{\mathcal{T}}$ satisfies (a) or (b) in condition 1.

Let $K_{cyc} = \max_c(\max(|w'_c|, |v'_c|))$ be the maximum length of an output word over all cycles $c : (q_c, (w_c, v_c), (w'_c, v'_c), q_c)$ in \mathcal{P} with $d(w_c, v_c) \neq 0$ i.e., $d(w_c, v_c) \geq 1$. Then, for any such cycle, $d(w'_c, v'_c) \leq K_{cyc} \leq K_{cyc}d(w_c, v_c)$.

Let $K_{acyc} = (|Q_{\mathcal{P}}| - 1)\ell_{max}$, where $Q_{\mathcal{P}}$ is the set of states of \mathcal{P} and ℓ_{max} is the length of the longest output word along any transition of \mathcal{T} . Then, for any acyclic fullpath $\pi : (q_{0_p}, (w_{\pi}, v_{\pi}), (w'_{\pi}, v'_{\pi}), q_{F_p})$ in \mathcal{P} , either $d(w_{\pi}, v_{\pi}) = 0$ and consequently, since \mathcal{T} is functional, $d(w'_{\pi}, v'_{\pi}) = 0$, or, $d(w_{\pi}, v_{\pi}) \geq 1$ and consequently, $d(w'_{\pi}, v'_{\pi}) \leq K_{acyc} \leq K_{acyc}d(w_{\pi}, v_{\pi})$. Note that the maximum possible distance, K_{acyc} , between the output words along an acyclic fullpath of \mathcal{P} , is exhibited along an acyclic path of maximum length $|Q_{\mathcal{P}}| - 1$, with the output word pair along each transition being ϵ and a word of length ℓ_{max} .

Let $K = \max(K_{cyc}, K_{acyc})$. We prove below that \mathcal{T} is K -robust. Observe that $K_{cyc} \leq |\mathcal{P}|$, $K_{acyc} \leq |\mathcal{P}| \cdot |\mathcal{P}|$. Thus, $K \leq K^{\mathcal{T}}$ and if \mathcal{T} is K -robust, it is $K^{\mathcal{T}}$ robust.

For an arbitrary fullpath $\pi : (q_{0_p}, (w_{\pi}, v_{\pi}), (w'_{\pi}, v'_{\pi}), q_{F_p})$ of \mathcal{P} , we will prove by induction on the number of cycles in π that $d(w'_{\pi}, v'_{\pi}) \leq Kd(w_{\pi}, v_{\pi})$.

When the number of cycles is 0, i.e., when π is acyclic, it is obvious from the definitions of K_{acyc} and K that the induction hypothesis holds. Suppose the induction hypothesis holds for any fullpath with $N \geq 0$ cycles. Consider a fullpath $(q_{0_p}, (w_{pre}, v_{pre}), (w'_{pre}, v'_{pre}), q_c) \cdot c \cdot (q_c, (w_{post}, v_{post}), (w'_{post}, v'_{post}), q_{F_p})$ with $N+1$ cycles, where cycle $c : (q_c, (w_c, v_c), (w'_c, v'_c), q_c)$ is the $N+1^{\text{th}}$ cycle. From the induction hypothesis, we have: $d(w'_{pre} \cdot w'_{post}, v'_{pre} \cdot v'_{post}) \leq Kd(w_{pre} \cdot w_{post}, v_{pre} \cdot v_{post})$.

If $d(w_c, v_c) = 0$, then $d(w_{pre} \cdot w_c \cdot w_{post}, v_{pre} \cdot v_c \cdot v_{post}) = d(w_{pre} \cdot w_{post}, v_{pre} \cdot v_{post})$, and since c does not satisfy any of the conditions in Lemma 17, $d(w'_{pre} \cdot w'_c \cdot w'_{post}, v'_{pre} \cdot v'_c \cdot v'_{post}) = d(w'_{pre} \cdot w'_{post}, v'_{pre} \cdot v'_{post})$. Hence, from the induction hypothesis, it follows that $d(w'_{\pi}, v'_{\pi}) \leq Kd(w_{\pi}, v_{\pi})$.

If $d(w_c, v_c) \neq 0$, then we have:

$$\begin{aligned} d(w'_\pi, v'_\pi) &\leq d(w'_{pre} \cdot w'_{post}, v'_{pre} \cdot v'_{post}) + d(w'_c, v'_c) \\ &\leq K(d(w_{pre} \cdot w_{post}, v_{pre} \cdot v_{post})) + Kd(w_c, v_c) \\ &= Kd(w_\pi, v_\pi). \end{aligned}$$

The first inequality follows from the definition of the Manhattan distance. The second inequality follows from the induction hypothesis, and the definitions of K_{cyc} and K . The last equality follows from the definition of the Manhattan distance and the facts — $|w_{pre}| = |v_{pre}|$, $|w_c| = |v_c|$ and $|w_{post}| = |v_{post}|$. ◀

► **Proposition 11.** *Let \mathcal{T} be a given functional transducer processing finite words and d_Σ, d_Γ be instances of the generalized Manhattan distance.*

1. *Robustness of \mathcal{T} is decidable in CO-NP.*
2. *One can compute $K_\mathcal{T}$ such that \mathcal{T} is robust iff \mathcal{T} is $K_\mathcal{T}$ -robust.*

Proof. First, observe that for the generalized Manhattan distance d_M , there are constants c, C such that for all s, t , $c \cdot d_1(s, t) \leq d_M(s, t) \leq C \cdot d_1(s, t)$, where d_1 is the Manhattan distance. Thus, \mathcal{T} is not robust w.r.t. generalized Manhattan distances d_Σ, d_Γ iff \mathcal{T} is not robust w.r.t. Manhattan distances on the input and the output words. Thus, it suffices to focus on robustness of \mathcal{T} w.r.t. Manhattan distances.

(1): It follows from Lemma 17 that \mathcal{T} is not robust w.r.t. Manhattan distances iff there exists a cycle satisfying Condition 1 of Lemma 17. Existence of such a cycle can be checked in NP. Therefore, robustness of \mathcal{T} can be checked in CO-NP.

(2): This follows directly from Lemma 17. ◀

► **Theorem 16.** *Let d_Σ, d_Γ be computed by functional weighted-automata. Checking K -robustness of nondeterministic letter-to-letter transducers w.r.t. d_Σ, D_Γ induced by d_Γ is*

- (i) *undecidable if D_Γ is the Hausdorff set-similarity function,*
- (ii) *undecidable if D_Γ is the Inf-inf set-similarity function, and*
- (iii) *decidable if D_Γ is the Sup-sup set-similarity function and d_Σ, d_Γ satisfy the conditions of Theorem 7.*

Proof. [of (i)] To show undecidability, it suffices to consider transducers processing finite words. We show a reduction of the universality problem for SUM-automata, with weights $\{-1, 0, 1\}$ and threshold 1, to robustness of non-deterministic letter-to-letter transducers w.r.t. the Hausdorff set-similarity function.

Let us fix a SUM-automaton \mathcal{A} with alphabet $\{a, b\}$ and weights $\{-1, 0, 1\}$. Without loss of generality, we assume that for every word $s \in \{a, b\}^*$, \mathcal{A} has a run of the value $|s|$, i.e., a run in which each transition taken by \mathcal{A} has weight 1. Let $\Sigma = \{a, b\} \times \{\perp, \top\}$, let $\Gamma = \{a, b\} \times \{\perp, \top\} \times \{-1, 0, 1\}$. To refer conveniently to words over Σ or Γ observe that for $s \in \{a, b\}^*$, $\theta \in \{\perp, \top\}^*$, $\tau \in \{-1, 0, 1\}^*$, if $|s| = |\theta| = |\tau|$, then $s \otimes \theta \otimes \tau$ is a word over Γ and $s \otimes \theta$ is a word over Σ .

We define generalized Manhattan distances d_Σ, d_Γ the Hausdorff set-similarity function D_Γ , and a nondeterministic letter-to-letter transducer \mathcal{T} such that \mathcal{T} is 1-robust iff for every word $s \in \{a, b\}^*$, $\mathcal{L}_\mathcal{A}(s) < 1$. The transducer \mathcal{T} is defined as follows: for all $s \in \{a, b\}^*$ and $\theta \in \{\perp, \top\}^*$, $\llbracket \mathcal{T} \rrbracket(s \otimes \theta) = \{s \otimes \theta \otimes \tau : \tau \text{ is a sequence of weights of some run of } \mathcal{A} \text{ on } s\}$. We define the generalized Manhattan distance d_Σ as follows: for all $i, j \in \{a, b\}$ and $P, Q \in \{\perp, \top\}$ we have

$$1. \text{diff}_\Sigma(\langle i, P \rangle, \#) = \text{diff}_\Sigma(\#, \langle i, P \rangle) = \infty$$

$$2. \text{diff}_\Sigma(\langle i, P \rangle, \langle j, Q \rangle) = \begin{cases} \infty & \text{if } i \neq j \\ 1 & \text{if } i = j, P \neq Q \\ 0 & \text{otherwise} \end{cases}$$

We define the generalized Manhattan distance d_Γ as follows: for all $i, j \in \{a, b\}$, $P, Q \in \{\perp, \top\}$ and $c, e \in \{-1, 0, 1\}$ we have

$$1. \text{diff}_\Gamma(\langle i, P, \underline{c} \rangle, \#) = \text{diff}_\Gamma(\#, \langle i, P, \underline{c} \rangle) = \infty$$

$$2. \text{diff}_\Gamma(\langle i, P, \underline{c} \rangle, \langle j, Q, \underline{e} \rangle) = \begin{cases} \infty & \text{if } i \neq j \\ c + e & \text{if } i = j, P \neq Q \\ 0 & \text{if } i = j, P = Q, c = e \\ 1 & \text{if } i = j, P = Q, c \neq e \end{cases}$$

Observe that for each $s, t \in \{a, b\}^*$, $s \otimes \theta \in \text{dom}(\mathcal{T})$ iff $\theta \in \{\perp^{|s|}, \top^{|s|}\}$ and $d_\Sigma(s \otimes \theta, t \otimes \vartheta) < \infty$ iff $s = t$ (and $\theta, \vartheta \in \{\perp^{|s|}, \top^{|s|}\}$). Further note that $d_\Sigma(s \otimes \theta, s \otimes \theta) = 0 = D_\Gamma^H(\llbracket \mathcal{T} \rrbracket(s \otimes \theta), \llbracket \mathcal{T} \rrbracket(s \otimes \theta))$. Thus, we only consider pairs of inputs of the form $s \otimes \theta$ and $s \otimes \vartheta$, where $\theta \neq \vartheta$. Then, $d_\Sigma(s \otimes \theta, s \otimes \vartheta) = |s|$. Let us denote the set $\{\tau : \tau \text{ is a sequence of weights of some run of } \mathcal{A} \text{ on } s\}$ as W . Thus, $D_\Gamma^H(\llbracket \mathcal{T} \rrbracket(s \otimes \theta), \llbracket \mathcal{T} \rrbracket(s \otimes \vartheta))$ equals $\sup_{\tau \in W} \inf_{v \in W} d_\Gamma(s \otimes \theta \otimes \tau, s \otimes \vartheta \otimes v)$, which in turn equals $\sup_{\tau \in W} \inf_{v \in W} \text{SUM}(\tau) + \text{SUM}(v)$. Since for every $s \in \{a, b\}^*$, \mathcal{A} has a run of the value $|s|$, $\sup_{\tau \in W} \inf_{v \in W} \text{SUM}(\tau) + \text{SUM}(v)$ equals the sum of $|s|$ and the value of an accepting run of \mathcal{A} on s . Therefore, \mathcal{T} is 1-robust iff for every word s in $\{a, b\}^*$, \mathcal{A} has an accepting run on s of value < 1 . ◀

Proof. [of (ii)] As before, it suffices to consider transducers processing finite words. We show a reduction of the universality problem for SUM-automata, with weights $\{-1, 0, 1\}$ and threshold 1, to robustness of non-deterministic letter-to-letter transducers w.r.t. the Inf-inf set similarity function.

Let us fix a SUM-automaton \mathcal{A} with alphabet $\{a, b\}$ and weights $\{-1, 0, 1\}$. Without loss of generality, we assume that for every word $s \in \{a, b\}^*$, \mathcal{A} has a run of the value $|s|$, i.e., a run in which each transition taken by \mathcal{A} has weight 1. Let $\Sigma = \{\perp, a, b\}$ and $\Gamma = \{\perp, -1, 0, 1\}$. Let d_Σ, d_Γ be the generalized Manhattan distances and D_Γ be the Inf-inf set-similarity function. We define a transducer \mathcal{T} such that \mathcal{T} is 1-robust iff for every word $s \in \{a, b\}^*$, $\mathcal{L}_\mathcal{A}(s) < 1$. \mathcal{T} is defined as follows: for every $k > 0$, $\llbracket \mathcal{T} \rrbracket(\perp^k) = \{\perp^k\}$ and for every $s \in \{a, b\}^*$, $\llbracket \mathcal{T} \rrbracket(s) = \{\tau : \tau \text{ is a sequence of weights of some run of } \mathcal{A} \text{ on } s\}$.

Let d_Σ be a generalized Manhattan distance defined as follows: for all $i, j \in \{a, b\}$ we have

$$1. \text{diff}_\Sigma(i, \#) = \text{diff}_\Sigma(\#, i) = \infty$$

$$2. \text{diff}_\Sigma(i, j) = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{otherwise} \end{cases}$$

Let d_Γ be a generalized Manhattan distance defined as follows: for all $i, j \in \{-1, 0, 1\}$ we have

$$1. \text{diff}_\Sigma(i, \#) = \text{diff}_\Sigma(\#, i) = \infty$$

$$2. \text{diff}_\Sigma(\perp, \#) = \text{diff}_\Sigma(\#, \perp) = \infty$$

$$3. \text{diff}_\Gamma(i, j) = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{if } i \neq j \end{cases}$$

$$4. \text{diff}_\Gamma(\perp, i) = i + 1$$

Now, observe that for every $k > 0$, $D_\Gamma^{\text{inf}}(\llbracket \mathcal{T} \rrbracket(\perp^k), \llbracket \mathcal{T} \rrbracket(\perp^k)) = 0 = d_\Sigma(\perp^k, \perp^k)$, and for all $s, t \in \{a, b\}^*$ with $|s| = |t|$, $D_\Gamma^{\text{inf}}(\llbracket \mathcal{T} \rrbracket(s), \llbracket \mathcal{T} \rrbracket(t)) = 0 \leq d_\Sigma(s, t)$. Indeed, $1^{|s|} \in \llbracket \mathcal{T} \rrbracket(s)$, $1^{|t|} \in \llbracket \mathcal{T} \rrbracket(t)$ and $|s| = |t|$. We now consider the case when $s \in \{a, b\}^*$ and $t = \perp^{|s|}$. Then, $d_\Sigma(s, t) = |s|$ and $D_\Gamma^{\text{inf}}(\llbracket \mathcal{T} \rrbracket(s), \llbracket \mathcal{T} \rrbracket(t))$ equals the sum of $|s|$ and the minimal value of an

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accepting run of \mathcal{A} on s (notice that for $i \in \{-1, 0, 1\}$, $\text{diff}_\Gamma(\perp, i) = 1 + i$). Therefore, \mathcal{T} is 1-robust iff for every word s in $\{a, b\}^*$, \mathcal{A} has an accepting run on s of value < 1 . ◀