Text

Lipschitz Robustness of Finite-state Transducers

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Joint work with Tom Henzinger and Jan Otop

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Problem Overview

Computational systems in physical environments $\Rightarrow$ uncertain data

Problem Overview

- Corrupt data from sensors in avionics software or medical devices
- Incomplete DNA strings in computational biology
- Wrongly spelled inputs to text processors

Problem Overview

- Functional correctness not enough
- System behaviour must degrade smoothly given input perturbation
- We need continuity or robustness

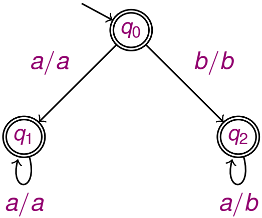
Informal definition

- Focus on finite-state transducers
- Robustness of transducers $\sim$ Lipschitz continuity
- Transducer $\mathcal{T}$ is K-Lipschitz robust if for all inputs s, t:
  $d(s,t) < \infty \Rightarrow d(\mathcal{T}(s), \mathcal{T}(t)) \leq Kd(s,t)$

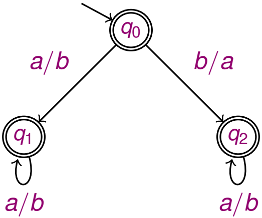
Example

$d(s,t) =$ Hamming distance(s,t) if $|s| = |t|$, else ∞

$\mathcal{T}_{NR}$:



$\mathcal{T}_R$:



Inputs: $d(a^{k+1}, ba^k) = 1$
Outputs: $d(a^{k+1}, b^{k+1}) = k + 1$

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Main contributions

- Undecidability of K-robustness of *deterministic* transducers
- Characterization of *functional* transducers with decidable K-robustness
  - Consider distances computable by weighted automata
  - Reduce K-robustness to emptiness problem for weighted automata
  - Polynomial-time decision procedure
- Formalization/study of K-robustness of *nondeterministic* transducers
- All results hold for transducers over finite or infinite words

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Transducers ($\mathcal{T}$)

- Finite-state device with two tapes
- In each step, a transducer $\mathcal{T}$
  - reads an input letter (alphabet Σ)
  - writes a finite word (alphabet Γ)
  - nondeterministically changes state
- Output of $\mathcal{T}$: defined only if run is *accepting*
  - $dom(\mathcal{T}) \subseteq \Sigma^*$
  - Transduction $[\![\mathcal{T}]\!] \subseteq dom(\mathcal{T}) \times \Gamma^*$
- Functional: at most one output word for every input word
  - $s' = [\![\mathcal{T}]\!](s)$
- Mealy machines: deterministic, letter-to-letter, all states accepting

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Weighted automata

- Finite automaton $\mathcal{A}$ with weighted transitions (weights $\in \mathbb{Q}$)
- A (weighted) run π is a sequence $c(\pi)$ of weights
- Given value-function f, value of a run π: $f(\pi) = f(c(\pi))$
  - Examples for value-functions: sum, discounted sum, limit-average
  - Notation: SUM-WA, DISC$_\delta$-WA, LIMAVG-WA
- Value of a word s: $\mathcal{L}_\mathcal{A}(s) = \inf_{\pi \in \mathrm{Acc}(s)} f(\pi)$

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Weighted automata

Given an f-WA $\mathcal{A}$ and a threshold λ:

- Emptiness: $\exists\, s : \mathcal{L}_{\mathcal{A}}(s) < \lambda$?
- Universality: $\forall\, s : \mathcal{L}_{\mathcal{A}}(s) < \lambda$?

- Emptiness is decidable in polynomial time for SUM-, DISC$_\delta$-, LIMAVG-WA
- Universality is undecidable for SUM-WA.

Similarity functions

$d : S \times S \to \mathbb{Q} \cup \infty$ such that $\forall x, y \in S$:

- $d(x, y) \geq 0$
- $d(x, y) = d(y, x)$

Example: Manhattan distance
$d(s, t) = |\{s[i] \neq t[i]\}|$

Similarity functions

Pairing: $ab \otimes bcaa = \binom{a}{b}\binom{b}{c}\binom{\#}{a}\binom{\#}{a}$

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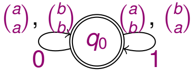

$\mathcal{L}_{\mathcal{A}}(\binom{s[1]}{t[1]}\binom{s[2]}{t[2]}\ldots) = |\{s[i] \neq t[i]\}|$

Robust transducers

Given:

- a functional transducer $\mathcal{T}$, with $[\![\mathcal{T}]\!] \subseteq \Sigma^* \times \Gamma^*$
- similarity function $d_\Sigma : \Sigma^* \times \Sigma^* \rightarrow \mathbb{Q}^\infty$
- similarity function $d_\Gamma : \Gamma^* \times \Gamma^* \rightarrow \mathbb{Q}^\infty$
- constant $K \in \mathbb{Q}$ with $K > 0$

$\mathcal{T}$ is defined to be K-Lipschitz robust w.r.t d_Σ, d_Γ if:

$$\forall s, t \in \mathrm{dom}(\mathcal{T}) : \; d_\Sigma(s,t) < \infty \Rightarrow d_\Gamma([\![\mathcal{T}]\!](s), [\![\mathcal{T}]\!](t)) \leq K d_\Sigma(s,t).$$

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Problem definition

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Check if $\mathcal{T}$ is K-Lipschitz robust w.r.t d_Σ, d_Γ.

Undecidability

K-robustness of functional transducers is undecidable.

1-robustness of deterministic transducers w.r.t. generalized Manhattan distances is undecidable.

Proof hint:
Reduction from PCP.

From K-robustness to robustness

$\mathcal{T}$ is *robust* w.r.t. d_Σ, d_Γ if $\exists K$: $\mathcal{T}$ is K-robust w.r.t. d_Σ, d_Γ.

Let d_Σ, d_Γ be generalized Manhattan distances.

1. Robustness of $\mathcal{T}$ is decidable in co-NP.
2. One can compute $K_\mathcal{T}$ such that $\mathcal{T}$ is robust iff $\mathcal{T}$ is $K_\mathcal{T}$-robust.

Synchronized transducers

$\mathcal{T}$ with $[\![\mathcal{T}]\!] \subseteq \Sigma^\omega \times \Gamma^\omega$ is synchronized iff:
$\exists$ automaton $\mathcal{A}_\mathcal{T}$ over $\Sigma \otimes \Gamma$ recognizing $\{s \otimes [\![\mathcal{T}]\!](s) : s \in \mathrm{dom}(\mathcal{T})\}$.

Synchronicity of a functional transducer can be decided in polynomial time

Example: Mealy machines are synchronized transducers.
$q \xrightarrow{a \otimes a'} q' \in \mathcal{A}_\mathcal{T}$ iff $(q, a, a', q') \in \mathcal{T}$

Robustness of synchronized transducers

If d_Σ, d_Γ are similarity functions computed by functional f-WA $\mathcal{A}_{d_\Sigma}$, $\mathcal{A}_{d_\Gamma}$, K-robustness of synchronized $\mathcal{T}$ w.r.t. d_Σ, d_Γ is decidable in PTIME.

Proof hint:
Construct $\mathcal{A}$ over $s \otimes t \otimes s' \otimes t'$:

$$\bar{\mathcal{A}}^K_{d_\Sigma} \times \bar{\mathcal{A}}^L_{\mathcal{T}} \times \bar{\mathcal{A}}^R_{\mathcal{T}} \times \bar{\mathcal{A}}^{-1}_{d_\Gamma}$$

$\bar{\mathcal{A}}^K_{d_\Sigma} \rightarrow \mathcal{L}_{\mathcal{A}_{d_\Sigma}}(s \otimes t)$

$\bar{\mathcal{A}}^L_{\mathcal{T}} \rightarrow \mathcal{A}_{\mathcal{T}}$ accepts $s \otimes s'$

$\bar{\mathcal{A}}^R_{\mathcal{T}} \rightarrow \mathcal{A}_{\mathcal{T}}$ accepts $t \otimes t'$

$\bar{\mathcal{A}}^{-1}_{d_\Gamma} \rightarrow \mathcal{L}_{\mathcal{A}_{d_\Gamma}}(s' \otimes t')$

$\exists w : \mathcal{L}_{\mathcal{A}}(w) < 0$ iff $\mathcal{T}$ is not K-robust w.r.t. d_Σ, d_Γ

Example

Recall:

- Mealy machines are synchronized transducers
- Manhattan distances are computable by functional SUM-WA

K-robustness of Mealy machines w.r.t. Manhattan distances is decidable.

Nondeterministic transducers

For a nondeterministic transducer $\mathcal{T}$, $|[\![\mathcal{T}]\!](s)| \geq 1$

Nondeterministic transducers

Given:

- a transducer $\mathcal{T}$, with $[\![\mathcal{T}]\!] \subseteq \Sigma^* \times \Gamma^*$,
- similarity function $d_\Sigma : \Sigma^* \times \Sigma^* \rightarrow \mathbb{Q}^\infty$
- set-similarity function $D_\Gamma : 2^{\Gamma^* \times \Gamma^*} \rightarrow \mathbb{Q}^\infty$
- constant $K \in \mathbb{Q}$ with $K > 0$

$\mathcal{T}$ is defined to be K-robust w.r.t d_Σ, D_Γ if:

$$\forall s, t \in \mathrm{dom}(\mathcal{T}) : \; d_\Sigma(s,t) < \infty \Rightarrow D_\Gamma([\![\mathcal{T}]\!](s), [\![\mathcal{T}]\!](t)) \leq K d_\Sigma(s,t).$$

Nondeterministic transducers

Set-similarity functions $D(A, B)$ induced by d
Hausdorff: $max\{\sup_{s \in A} \inf_{t \in B} d(s,t), \sup_{s \in B} \inf_{t \in A} d(s,t)\}$
Inf-inf: $\inf_{s \in A} \inf_{t \in B} d(s,t)$
Sup-sup: $\sup_{s \in A} \sup_{t \in B} d(s,t)$

Nondeterministic transducers

Set-similarity functions $D(A, B)$ induced by d	K-robustness
Hausdorff: $max\{\sup_{s \in A} \inf_{t \in B} d(s,t), \sup_{s \in B} \inf_{t \in A} d(s,t)\}$	Undecidable
Inf-inf: $\inf_{s \in A} \inf_{t \in B} d(s,t)$	Undecidable
Sup-sup: $\sup_{s \in A} \sup_{t \in B} d(s,t)$	Decidable

Thank you.