

# Modeling Reachability Types with Logical Relations

Semantic Type Soundness, Termination, and Equational Theory

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Reachability types are a recent proposal to bring Rust-style reasoning about memory properties to higher-level languages. While key type soundness results for reachability types have been established using syntactic techniques in prior work, stronger metatheoretic properties have so far been unexplored. This paper presents an alternative semantic model of reachability types using logical relations, providing a framework in which to study key properties of interest such as (1) semantic type soundness, including of not syntactically well-typed code fragments, (2) termination, especially in the presence of higher-order state, and (3) program equivalence, especially reordering of non-interfering expressions for parallelization or compiler optimization.

## 1 INTRODUCTION

Reachability types [Bao et al. 2021] are a recent proposal to bring Rust-style reasoning about memory properties to higher-level languages, specifically about sharing and the absence of sharing: separation. While key type soundness results for reachability types have been established using the usual syntactic techniques in prior work, including a syntactic preservation of separation property [Bao et al. 2021; Wei et al. 2023], stronger metatheoretic properties have so far been left unexplored.

We address this gap by presenting an alternative semantic model of reachability types using the technique of logical relations [Ahmed et al. 2009; Benton et al. 2007; Timany et al. 2022], providing a framework in which to study key properties of interest such as semantic type soundness, termination, and program equivalence.

We present the corresponding developments in detail as follows:

- We introduce the syntax and the typing rules of our reachability type and effect system, as well as its operational semantics (Section 2).
- We present the definition of a unary logical relation that establishes semantic type soundness as well as termination (Section 3).
- We define a binary logical relation over reachability types to support relational reasoning with respect to the observational equivalence of two programs (Section 4).

The semantic type soundness result is useful in addition to the existing syntactic results [Bao et al. 2021; Wei et al. 2023]. First, it does not require terms to be syntactically well-typed. Thus, it provides a foundation for studying potentially unsafe features and interactions with the outside world. Second, unlike prior results (e.g., preservation of separation) that are established with respect to a small-step operational semantics based on substitution, we adopt a big-step operational semantics based on closures and environments. Thus, our results map more closely to real language implementations.

The termination result is interesting as our logical relations do not rely on step indexing. This is in contrast to other recent work, which introduced a form of transfinite step indexing to prove

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termination for a feature-rich language with higher-order state [Spies et al. 2021]. In comparison, our model is entirely elementary and does not rely on advanced set-theoretic concepts or classical reasoning.

The program equivalence result is significant as it provides a foundation for parallelization and for a variety of effect-based compiler optimizations in the style of Benton et al. [2007] or Birkedal et al. [2016]. Specifically, Bračevac et al. [2023a] have proposed a novel Graph IR for impure higher-order languages with a dependency analysis based on reachability types, and use the logical relations model presented here to justify the correctness of their optimizations rules.

The results in this paper have been mechanized in Coq, and are available online.<sup>1</sup>

## 2 THE $\lambda_{\varepsilon}^*$ -CALCULUS

The base language in this paper is the  $\lambda_{\varepsilon}^*$ -calculus, a variant of Bao et al.’s  $\lambda^*$ -calculus. Part of the description here is reproduced from Bračevac et al. [2023b], the supplemental technical report accompanying Bračevac et al. [2023a]. The original system features an effect system based on Gordon [2021]’s effect quantale framework. For simplicity, we only consider a stripped-down effect system corresponding to a trivial effect quantale just tracking whether an effect is induced on reachable variables, effectively making effects just another qualifier (i.e., a set of variables) in the typing judgment. This version also lacks a  $\perp$  qualifier for untracked values, and recursive  $\lambda$ -abstractions. To keep the discussion focused and on point, we omitted those features which do not add much to the discussion of the core ideas apart from additional proof cases.

### 2.1 Syntax

Figure 1 shows the syntax of  $\lambda_{\varepsilon}^*$  which is based on the simply-typed  $\lambda$ -calculus with mutable references and subtyping. We denote general term variables by the meta variables  $x, y, z$ , and reserve  $\ell, w$  for store locations.

Terms consist of constants of base types, variables, functions  $\lambda x.t$ , function applications, reference allocations, dereferences, assignments and sequence statement.

Reachability qualifiers  $p, q, r$  are finite sets of variables. For readability, we often drop the set notation for qualifiers and write them down as comma-separated lists of atoms.

We distinguish ordinary types  $T$  from qualified types  $T^q$ , where the latter annotates a qualifier  $q$  to an ordinary type  $T$ . The types consist of Boolean type  $B$  (to streamline the presentation, we omit other base types), dependent function types  $(x : T^q) \rightarrow^{\varepsilon} S^p$ , where both argument and return type are qualified. The codomain  $S^p$  may depend on the argument  $x$  in its qualifier and type. Function types carry an annotation  $\varepsilon$  for its latent effect, which is a set of variables and locations, akin to qualifiers.

An *observation*  $\varphi$  is a finite set of variables which is part of the term typing judgment (Section 2.2). It specifies which variables and locations in the typing context  $\Gamma$  are observable, where the typing context assigns qualified typing assumptions to variables.

### 2.2 Type Rules

The term typing judgment  $\Gamma^{\varphi} \vdash t : T^q \ \varepsilon$  in Figure 1 states that term  $t$  has qualified type  $T^q$  and may induce effect  $\varepsilon$ , and may only access the typing assumptions of  $\Gamma$  observable by  $\varphi$ . One may think of  $t$  as a computation that incurs effect  $\varepsilon$  and yields a result value of type  $T$  aliasing no more than  $q$ , if it terminates.

Different from Bao et al. [2021], we internalize the filter  $\varphi$  as part the typing relation. Alternatively, we could formulate the typing judgment without internalizing  $\varphi$ , and instead have an explicit

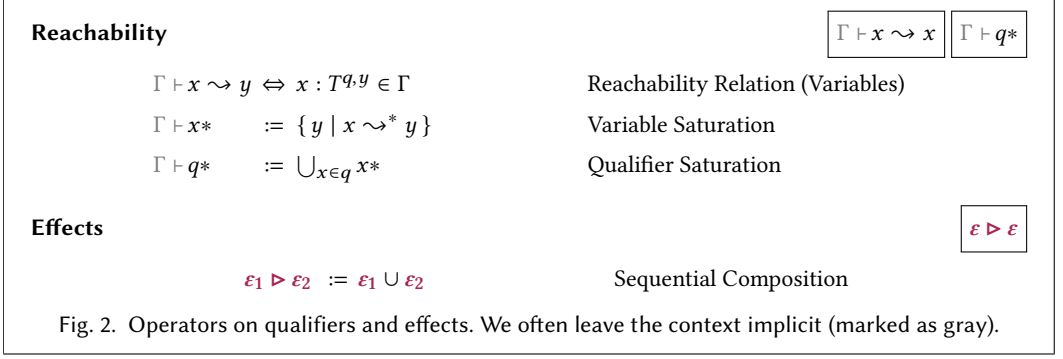
<sup>1</sup><https://github.com/tiarkrumpf/reachability>

<b>Syntax</b>		$\lambda_{\varepsilon}^*$
$x, y, z$	$\in \text{Var}$	Variables
$t$	$::= x \mid (\lambda x.t)^q \mid t t \mid \mathbf{ref} t \mid ! t \mid t := t \mid t; t$	Terms
$p, q, r, \varepsilon, \varphi$	$\in \mathcal{P}_{\text{fin}}(\text{Var})$	Qualifiers/Effects/Observations
$S, T, U, V$	$::= B \mid (x : T^q) \rightarrow^{\varepsilon} T^q \mid \text{Ref } B$	Types
$\Gamma$	$::= \emptyset \mid \Gamma, x : T^q$	Typing Environments
<b>Term Typing</b>		$\Gamma^{\varphi} \vdash t : T^q \varepsilon$
		$\frac{\Gamma^{\varphi} \vdash t : B^q \varepsilon}{\Gamma^{\varphi} \vdash \mathbf{ref} t : (\text{Ref } B)^q \varepsilon}$ (T-REF)
	$\frac{c \in B}{\Gamma^{\varphi} \vdash c : B^{\emptyset} \emptyset}$ (T-CST)	$\frac{\Gamma^{\varphi} \vdash t : (\text{Ref } B)^q \varepsilon}{\Gamma^{\varphi} \vdash !t : B^{\emptyset} \varepsilon \triangleright q}$ (T-!)
	$\frac{x : T^q \in \Gamma \quad x \subseteq \varphi}{\Gamma^{\varphi} \vdash x : T^x \emptyset}$ (T-VAR)	$\frac{\Gamma^{\varphi} \vdash t_1 : (\text{Ref } B)^q \varepsilon_1 \quad \Gamma^{\varphi} \vdash t_2 : B^p \varepsilon_2}{\Gamma^{\varphi} \vdash t_1 := t_2 : B^{\emptyset} \varepsilon_1 \triangleright \varepsilon_2 \triangleright q}$ (T-:=)
	$\frac{(\Gamma, x : T^p)^{q \cdot x} \vdash t : U^r \varepsilon \quad q \subseteq \varphi}{\Gamma^{\varphi} \vdash (\lambda x.t)^{p \cap q} : (x : T^p \rightarrow^{\varepsilon} U^r)^q \emptyset}$ (T-ABS)	$\frac{\Gamma^{\varphi_1} \vdash t_1 : B^q \varepsilon_1 \quad \Gamma^{\varphi_2} \vdash t_2 : B^p \varepsilon_2 \quad \varphi_1 \subseteq \varphi \quad \varphi_2 \subseteq \varphi}{\Gamma^{\varphi} \vdash t_1; t_2 : B^{\emptyset} \varepsilon_1 \triangleright \varepsilon_2 \triangleright q}$ (T-SEQ)
	$\frac{\Gamma^{\varphi} \vdash t_1 : (x : T^{p^* \cap q^*} \rightarrow^{\varepsilon_3} U^r)^q \varepsilon_1 \quad \Gamma^{\varphi} \vdash t_2 : T^p \varepsilon_2 \quad \theta = [p/x] \quad x \notin \text{fv}(U) \quad \varepsilon_3 \subseteq q, x \quad r \subseteq \varphi, x}{\Gamma^{\varphi} \vdash t_1 t_2 : (U^r \varepsilon_1 \triangleright \varepsilon_2 \triangleright \varepsilon_3) \theta}$ (T-APP)	$\frac{\Gamma^{\varphi} \vdash t : S^p \varepsilon_1 \quad \Gamma \vdash S^p \varepsilon_1 <: T^q \varepsilon_2 \quad q, \varepsilon_2 \subseteq \varphi}{\Gamma^{\varphi} \vdash t : T^q \varepsilon_2}$ (T-SUB)
<b>Subtyping</b>		$\Gamma \vdash q <: q \quad \Gamma \vdash T <: T \quad \Gamma \vdash T^q \varepsilon <: T^q \varepsilon$
	$\frac{p \subseteq q \subseteq \text{dom}(\Gamma)}{\Gamma \vdash p <: q}$ (Q-SUB)	$\frac{\Gamma \vdash U^q \emptyset <: S^{\emptyset} \emptyset \quad \Gamma, x : U^p \mid \Sigma \vdash T^q \varepsilon_1 <: V^r \varepsilon_2}{\Gamma \vdash (x : S^{\emptyset}) \rightarrow^{\varepsilon_1} T^q <: (x : U^p) \rightarrow^{\varepsilon_2} V^r}$ (S-FUN)
	$\frac{}{\Gamma \vdash B <: B}$ (S-BASE)	$\frac{\Gamma \vdash S <: T \quad \Gamma \vdash p <: q \quad \Gamma \vdash \varepsilon_1 <: \varepsilon_2}{\Gamma \vdash S^p \varepsilon_1 <: T^q \varepsilon_2}$ (SQE-SUB)
	$\frac{}{\Gamma \vdash \text{Ref } B <: \text{Ref } B}$ (S-REF)	

Fig. 1. The  $\lambda_{\varepsilon}^*$ -calculus.

context filter operation  $\Gamma^{\varphi} := \{x : T^q \in \Gamma \mid q, x \subseteq \varphi\}$  for restricting the context in subterms, just like Bao et al. [2021] which loosely takes inspiration from substructural type systems. Internalizing  $\varphi$  (1) makes observability an explicit notion, which facilitates reasoning about separation and overlap, and (2) greatly simplifies the Coq mechanization. Context filtering is only needed for term typing, but not for subtyping, so as to keep the formalization simple.

**2.2.1 Functions and Lightweight Polymorphism.** Function typing (T-ABS) implements the observable separation guarantee, i.e., the body  $t$  can only observe what the function type's qualifier  $q$  specifies, plus the argument  $x$ , and is otherwise oblivious to anything else in the environment. We model this by setting the observation to  $q, x, f$  when typing the body. Thus, its observation  $q$  at least includes



the free variables of  $t$ . To ensure well-scopedness,  $q$  must be a subset of the observation  $\varphi$  on the outside. In essence, a function type *implicitly* quantifies over anything that is not observed by  $q$ , achieving a lightweight form of qualifier polymorphism, following [Wei et al. \[2023\]](#).

**2.2.2 Dependent Application, Separation and Overlap.** Function applications ( $\tau$ -APP) are qualifier-dependent in that the result qualifier can depend on the argument.

Function applications also establish an *observable separation* between the argument reachable set  $p$  and the function reachable set  $q$ , as denoted as  $p^* \cap q^*$ . The intersection between  $p^*$  and  $q^*$  specifies the permitted overlap. We are careful to intersect the transitive reachability closure (a.k.a. saturated version, Figure 2) of the two qualifiers. This is necessary in the lazy reachability assignment, because we might miss common, indirect overlap between the sets otherwise. If the intersection declared in the function type is empty, then it means complete separation between the argument and the entities observed by the function from the environment.

**2.2.3 Effects.** Our effect system is a simple flow-insensitive instantiation of [Gordon \[2021\]](#)'s effect quantale system. An effect  $\varepsilon$  denotes the set of variables that might be used during the computation. For a compound term, the final effect is computed by composing the effects of sub-terms with the intrinsic effect of this term. For example, the effect of assignments has two parts: (1)  $\varepsilon_1, \varepsilon_2$  the effects of sub-terms, and (2)  $q$  the variables being modified. The final effect is obtained by composing these effects.

Although the typing rules presented in Figure 1 pretend to use the sequential effect composition operator  $\triangleright$ , its definition  $\cup$  computes an upper bound of two effects and is *not* flow-sensitive (Figure 2), *i.e.* the composed effect is not sensitive to the order of composition.

### 2.3 Semantics

Fig. 3 defines the big-step semantics with a value environment  $H$  and a store  $M$ . The definitions are mostly standard. A value environment,  $H$ , is a partial function that maps from variables to values. A store,  $M$ , is a partial function that maps from locations to values. A program state is a pair of a value environment and a store. We write  $t, H, M \Downarrow v, M'$  to mean term  $t$  is evaluated to value  $v$ , resulting a store transition from  $M$  to  $M'$ . Note that the value environment is immutable.

Following [[Amin and Rompf 2017](#); [Ernst et al. 2006](#); [Siek 2013](#); [Wang and Rompf 2017](#)], in the proof of semantic type soundness, we extend the big-step semantics  $\Downarrow$  to a total evaluation function by adding a numeric fuel value and explicit timeout and Error results. Note, however, that while the operational semantics is step-indexed in this way, the logical relations we define are not, and could be defined just as well with a partial (big-step or small-step) evaluation semantics.

<b>Value, Value Environment, Store</b>		$\lambda_{\varepsilon}^*$
$\ell \in \text{Loc}$	Locations	
$v ::= b \mid \ell \mid \langle H, (\lambda x.t)^q \rangle$	Values	
$\Sigma ::= \emptyset \mid \Sigma, \ell : T$	Store Typing	
$H ::= \emptyset \mid H, x : v$	Value Environment	
$M, \sigma ::= \emptyset \mid M, l : v$	Store	
<b>Well-Formed State</b>		$[\Sigma \mid \Gamma]^\varphi \vdash H, M$
$\frac{([\Sigma \mid \Gamma]^\varphi \vdash \sigma(\ell) : \Sigma(\ell))_{\ell \in \text{dom}(\Sigma)} \quad \text{dom}(H) = \text{dom}(\Gamma) \quad \text{dom}(\Sigma) = \text{dom}(M)}{[\Sigma \mid \Gamma]^\varphi \vdash H, M}$		
<b>Big-Step Semantics</b>		$t, H, M \Downarrow v, M'$
$\frac{}{b, H, M \Downarrow b, M}$	(E-CST)	$\frac{t_1, H, M \Downarrow \ell, M' \quad t_2, H, M' \Downarrow v, M''}{t_1 := t_2, H, M \Downarrow M''(\ell), M''[\ell \mapsto v]}$ (E-:=)
$\frac{H(x) = v}{x, H, M \Downarrow v, M}$	(E-VAR)	$\frac{t_1, H, M' \Downarrow \langle H', (\lambda x.t)^q \rangle, M' \quad t_2, H, M' \Downarrow v, M''}{t, H'; (x, v), M'' \Downarrow v, M''}$ (E-APP)
$\frac{t, H, M \Downarrow v, M' \quad \ell \notin \text{dom}(M')}{\text{ref } t, H, M \Downarrow l, M'; (\ell, v)}$	(E-REF)	$\frac{t_1, H, M \Downarrow b_1, M' \quad t_2, H, M' \Downarrow b_2, M''}{t_1; t_2, H, M \Downarrow b_1 \& \& b_2, M''}$ (E-SEQ)
$\frac{t, H, M \Downarrow \ell, M' \quad M'(\ell) = v}{!t, H, M \Downarrow v, M'}$	(E-!)	
$\frac{(\lambda x.t)^q, H, M \Downarrow \langle H, (\lambda x.t)^q \rangle, M}{(\lambda x.t)^q, H, M \Downarrow \langle H, (\lambda x.t)^q \rangle, M}$	(E-ABS)	

Fig. 3. The big-step semantics of the  $\lambda_{\varepsilon}^*$ -calculus.

### 3 SEMANTIC TYPE SOUNDNESS

In this section, we define a unary logical relation that establishes semantic type soundness as well as termination.

#### 3.1 High-Level Overview of the Proofs

The semantic typing is written as  $\Gamma^\varphi \models t : T^q \varepsilon$ . The high-level structure of the proof is the following:

- Semantic soundness (Theorem 3.1). We show every syntactically well-typed term is semantically well-typed:

$$\Gamma^\varphi \vdash t : T^q \varepsilon \text{ implies } \Gamma^\varphi \models t : T^q \varepsilon$$

- Adequacy of unary logical relation (Theorem 3.2). Every closed semantically well-typed term  $t$  is safe:

$$\emptyset \models t : T^q \varepsilon \text{ implies } \exists v, M. t, \emptyset, \emptyset \Downarrow v, M'$$

#### 3.2 Interpretation of Reachability

In the  $\lambda_{\varepsilon}^*$ -calculus, reachability qualifiers are used to specify desired separation or permissible overlapping of reachable locations from a function's argument and its body. Fig. 4 shows the interpretation of reachability qualifiers. As in the  $\lambda_{\varepsilon}^*$  calculus, values cannot be cyclic, we axiomatize the definition of reachability, without proving termination.

**Interpretation of Value Reachability**

$$\text{locs}(b) = \emptyset \quad \text{locs}(\ell) = \{\ell\} \quad \text{locs}(\langle H, (\lambda x.t)^q \rangle) = \text{locs}_H(q)$$

**Interpretation of Reachability Qualifiers**

$$\text{locs}_H(q) \stackrel{\text{def}}{=} \{\ell \mid \ell \in H(q) \wedge \ell \in \text{Loc}\}$$

**Reachability Predicates**

$$v \rightsquigarrow^\sigma L \stackrel{\text{def}}{=} (\text{dom}(\sigma) \cap \text{locs}(v)) \subseteq L$$

Fig. 4. Interpretation of reachability qualifiers.

**Value Interpretation of Types and Terms**

$$\begin{aligned} V[[B]] &= \{(H, \Sigma, v) \mid v = \text{true} \vee v = \text{false}\} \\ V[[\text{Ref } B]] &= \{(H, \Sigma, l) \mid \forall M : \Sigma, (H, \Sigma, M(l)) \in V[[B]]\} \\ V[[T^p \rightarrow^\varepsilon U^r]] &= \{(H, \Sigma, \langle H', (\lambda x.t)^q \rangle) \mid \text{locs}(\langle H', (\lambda x.t)^q \rangle) \subseteq \text{dom}(\Sigma) \wedge \\ &\quad (\forall v, M', \Sigma'. \Sigma \sqsubseteq_{\text{locs}(\langle H', (\lambda x.t)^q \rangle)} \Sigma' \wedge M' : \Sigma' \Rightarrow (H, \Sigma', v) \in V[[T]]) \wedge \\ &\quad \text{locs}(\langle H_1, (\lambda x.t)^q \rangle) \cap \text{locs}(v) \subseteq \text{locs}_H(p) \Rightarrow \\ &\quad \exists \Sigma'', M'', v'. H'; (x, v), M', t \Downarrow v', M'' \wedge \Sigma' \sqsubseteq \Sigma'' \wedge M'' : \Sigma'' \wedge (H, \Sigma'', v') \in V[[U]] \wedge \\ &\quad (x \in r \Rightarrow v' \rightsquigarrow^M (\text{locs}_H(r) \cap \text{locs}(\langle H', (\lambda x.t)^q \rangle) \cup \text{locs}(v_1))) \wedge \\ &\quad (x \notin r \Rightarrow v' \rightsquigarrow^M (\text{locs}_H(r) \cap \text{locs}(\langle H', (\lambda x.t)^q \rangle))) \wedge \\ &\quad (x \in \varepsilon \Rightarrow M \hookrightarrow_{\text{locs}_H((\varepsilon \triangleright p))} M') \wedge (x \notin \varepsilon \Rightarrow M \hookrightarrow_{\text{locs}_H(\varepsilon)} M')\} \\ M \hookrightarrow^\varepsilon M' &\stackrel{\text{def}}{=} \forall l \in \text{dom}(M). M(l) = M'(l) \vee l \in \varepsilon \\ E[[T^q \varepsilon]]_\varphi &= \{(H, \Sigma, t) \mid \forall \Sigma', M. M : \Sigma \wedge \exists \Sigma', M', v'. t, H, M \Downarrow v', M' \wedge \Sigma \sqsubseteq \Sigma' \wedge M' : \Sigma' \wedge \\ &\quad (H, \Sigma', v') \in V[[T]] \wedge v \rightsquigarrow^M (\text{locs}_H(\varphi \cap q)) \wedge M' \hookrightarrow_{\text{locs}_H(\varepsilon)} M'\} \end{aligned}$$

Fig. 5. Unary logical relations for the  $\lambda_\varepsilon^*$ -calculus.

We use  $\text{locs}(v)$  to define the set of locations that are reachable from a given value  $v$ . Boolean type values, *i.e.*, true and false, do not reach any store locations. Thus, they reach the empty set of locations. A location  $\ell$  can only reach itself. Thus, its reachable set is the singleton set  $\{\ell\}$ . The set of locations that are reachable from a closure record  $\langle H, (\lambda x.t)^q \rangle$  are the set of the locations in appearing in the function body, which are computed by  $\text{locs}_H(q)$ .

The notation  $\text{locs}_H(q)$  means the set of locations reachable from qualifier  $q$ , which are the set of the locations appeared in  $q$ . The notation  $H(q)$  means retrieving the location for each free variable in  $q$  from  $H$ . A bound variable may appear in  $q$ , and serves as a placeholder to specify the set of locations that a function's return value may reach. See Section 4.4 for details. The notation  $v \rightsquigarrow^M L$  is a predicate that asserts the set of locations that are reachable from  $v$  in store  $M$  is a subset of  $L$ , where  $L$  is a set of locations.

**3.3 Unary Logical Relation**

This section presents the definition of unary logical relations for  $\lambda_\varepsilon^*$ . We first define the relation on two store typing. Given two store typing  $\Sigma'$  and  $\Sigma$ , and a set of locations  $L$ , we define the relation of  $\Sigma$  and  $\Sigma'$  (written as  $\Sigma \sqsubseteq_L \Sigma'$ ) as follows:

$$\Sigma \sqsubseteq_L \Sigma' \stackrel{\text{def}}{=} L \subseteq \text{dom}(\Sigma) \wedge L \subseteq \text{dom}(\Sigma') \wedge (\forall l \in L. l \in \Sigma \Rightarrow l \in \Sigma')$$

We write  $\Sigma \sqsubseteq \Sigma'$  to mean  $\Sigma \sqsubseteq_{\text{dom}(\Sigma)} \Sigma'$ .

Now we define the interpretation of typing contexts:

$$\begin{aligned} G[[\emptyset^\varphi]] &= \emptyset \\ G[[\Gamma, x : T^q]^\varphi] &= \{(\Sigma, H; (x \mapsto v)) \mid (\Sigma, H) \in G[[\Gamma^\varphi]] \wedge \varphi \subseteq \text{dom}(\Gamma) \wedge q \subseteq \text{dom}(\Gamma) \wedge (H, \Sigma, v) \in V[[T]] \wedge \\ &\quad (\forall q, q'. q \subseteq \varphi \wedge q' \subseteq \varphi \wedge \Rightarrow (\text{locs}_H(q^*) \cap \text{locs}_H(q'^*) \subseteq \text{locs}_H(q^* \cap q'^*)))\} \end{aligned}$$

*The Value Interpretation.* The definition of value interpretation of types is shown in Fig. 5. The interpretation of type  $T$ , written as  $V[[T]]$ , is a tripe of form  $(H, \Sigma, v)$ , where  $H$  is a value environment,  $v$  is a value, and  $\Sigma$  is a store typing.

*Ground Types.* The value interpretation of ground types is straightforward. The value of the Boolean type are true and false; the value of the reference type Ref B is store locations  $\ell$ , which store a value, whose type is always B.

*Function Types.* The value interpretation of the function types  $T^p \xrightarrow{\varepsilon} U^r$  with respect to a store typing  $\Sigma$  are closure records in a form  $\langle H, (\lambda x.t)^q \rangle$ , meaning that it satisfies the followings:

- The set of locations reachable from a closure record are well-formed with respect to the store typing, i.e.,  $\text{locs}(\langle H, (\lambda x.t)^q \rangle) \subseteq \text{dom}(\Sigma)$ .
- The argument is allowed if
  - the argument  $v$  has type  $T$  with respect  $\Sigma'$ , for all  $\Sigma'$ , such that  $\Sigma \sqsubseteq_{\text{locs}(\langle H, (\lambda x.t)^q \rangle)} \Sigma'$ ; and
  - the overlapping locations reachable from the function and its argument are permissible by the argument's qualifier  $p$ , i.e.,  $\text{locs}(\langle H, (\lambda x.t)^q \rangle) \cap \text{locs}(v) \subseteq \text{locs}_H(p)$ .
- Under the extended value environment, the term  $t$  is reduced to some value  $v'$  with some final stores  $M'$ .
- $M'$  respects the store typing  $\Sigma''$ , where  $\Sigma' \sqsubseteq \Sigma''$ , for some  $\Sigma''$ .
- $v'$  has type  $U$  with respect to store typing  $\Sigma''$ .
- If the return value's qualifier  $r$  depends on the argument (i.e.,  $x \in r$ ), then the locations reachable from  $v'$  is subsets of those reachable both from the function and  $r$ , plus those reachable from the arguments; otherwise (i.e.,  $x \notin r$ ), they are just subset of those reachable both from the function and  $r$ .
- If a bound variable  $x$  appears in the effect  $\varepsilon$ , meaning the function body may modify the argument, then the effect will include the qualifier that may reach the value of function argument  $p$ ; otherwise it is just  $\varepsilon$ .

*The Term Interpretation.* A term,  $t$  is defined based on their computational behaviors, i.e., returned values, reachability qualifiers and effects, which is defined by  $E[[T^q \ \varepsilon]]_\varphi$ . It means given a store with respect to store typing,  $M : \Sigma$ , if

- $t$  is evaluated to some value  $v$  with some final store  $M'$ ;
- $M'$  respects the store typing  $\Sigma'$ , where  $\Sigma \sqsubseteq \Sigma'$ ;
- $v$  has type  $T$  with respect to store typing  $\Sigma'$ ;
- $M'$  is the store with respect to the store typing  $\Sigma'$ .
- The locations reachable from the values in the domain of pre-stores are subset of those reachable from  $\text{locs}_H(\varphi^* \cap q^*)$  for the term.
- The effect captures what may be read/modified in the pre-state store.

*Semantic Typing.* Fig. 6 shows the semantic typing rules. The proofs are quite similar to those of compatibility lemmas in Section 4.6, thus are omitted.

*The Fundamental Theorem and Adequacy.*

Semantic Typing		$\Gamma^\varphi \models t : T^q \varepsilon$
$\frac{c \in B}{\Gamma^\varphi \models c : B^\varnothing \varnothing} \quad (\text{T-CST})$	$\frac{\Gamma^\varphi \models t : B^q \varepsilon}{\Gamma^\varphi \models \mathbf{ref} \ t : (\mathbf{Ref} \ B)^q \varepsilon} \quad (\text{T-REF})$	
$\frac{x : T^q \in \Gamma \quad x \subseteq \varphi}{\Gamma^\varphi \models x : T^x \varnothing} \quad (\text{T-VAR})$	$\frac{\Gamma^\varphi \models t : (\mathbf{Ref} \ B)^q \varepsilon}{\Gamma^\varphi \models !t : B^\varnothing \varepsilon \triangleright q} \quad (\text{T-!})$	
$\frac{(\Gamma, x : T^p)^{q,x} \models t : U^r \varepsilon \quad q \subseteq \varphi}{\Gamma^\varphi \vdash (\lambda x.t)^{p \cap q} : (x : T^p \rightarrow^\varepsilon U^r)^q \varnothing} \quad (\text{T-ABS})$	$\frac{\Gamma^\varphi \models t_1 : (\mathbf{Ref} \ B)^q \varepsilon_1 \quad \Gamma^\varphi \models t_2 : B^p \varepsilon_2}{\Gamma^\varphi \models t_1 := t_2 : B^\varnothing \varepsilon_1 \triangleright \varepsilon_2 \triangleright q} \quad (\text{T-:=})$	
$\frac{\Gamma^\varphi \vdash t_1 : (x : T^{p^* \cap q^*} \rightarrow^{\varepsilon_3} U^r)^q \varepsilon_1 \quad \Gamma^\varphi \models t_2 : T^p \varepsilon_2 \quad \theta = [p/x] \quad x \notin \text{fv}(U) \quad \varepsilon_3 \subseteq q, x \quad r \subseteq \varphi, x}{\Gamma^\varphi \models t_1 \ t_2 : (U^r \varepsilon_1 \triangleright \varepsilon_2 \triangleright \varepsilon_3)\theta} \quad (\text{T-APP})$	$\frac{\Gamma^{\varphi_1} \models t_1 : B^q \varepsilon_1 \quad \Gamma^{\varphi_2} \models t_2 : B^p \varepsilon_2 \quad \varphi_1 \subseteq \varphi \quad \varphi_2 \subseteq \varphi}{\Gamma^\varphi \models t_1; t_2 : B^\varnothing \varepsilon_1 \triangleright \varepsilon_2 \triangleright q} \quad (\text{T-SEQ})$	
	$\frac{\Gamma^\varphi \models t : S^p \varepsilon_1 \quad \Gamma \models S^p \varepsilon_1 < : T^q \varepsilon_2 \quad q, \varepsilon_2 \subseteq \varphi}{\Gamma^\varphi \models t : T^q \varepsilon_2} \quad (\text{T-SUB})$	

Fig. 6. Semantic typing rules of the  $\lambda_\varepsilon^*$ -calculus.

**THEOREM 3.1.** (*Fundamental Theorem of Unary Logical Relations*) Every syntactically well-typed term is semantically well-typed, i.e., if  $\Gamma^\varphi \vdash t : T^q \varepsilon$ , then  $\Gamma^\varphi \models t : T^q \varepsilon$ .

**THEOREM 3.2.** (*Adequacy of Unary Logical Relations*) Every closed semantically well-typed term  $t$  is safe: if  $\varnothing \models t : T^q \varepsilon$ , then  $\exists v, M, t, \varnothing, \varnothing \Downarrow v, M'$ .

From Theorem 3.2 (Adequacy), termination of all semantically well-typed terms is immediate.

## 4 CONTEXTUAL EQUIVALENCE - THE DIRECT-STYLE $\lambda_\varepsilon^*$ -CALCULUS

We apply a logical relations approach following [Ahmed et al. 2009; Benton et al. 2007; Timany et al. 2022] to support relational reasoning with respect to the *observational equivalence* of two programs. We define binary logical relations over reachability types (the  $\lambda_\varepsilon^*$ -calculus in Section 2), and prove the soundness of the equational rules. To avoid technical complications, we choose a model that allows mutable references to contain only first-order values, consistent with the previous section.

### 4.1 High-level Overview of the Proofs

A program  $t_1$  is said to be *contextually equivalent* to another program  $t_2$ , written as  $\Gamma^\varphi \models t_1 \approx_{\text{ctx}} t_2 : T^p \varepsilon$ , if for any program context  $C$  with a hole of type  $T^p \varepsilon$ , if  $C[t_1]$  has some (observable) behavior, then so does  $C[t_2]$ . The definition of context  $C$  can be found in Section 4.2.

Following the approach of Timany et al. [2022] and related prior works [Ahmed et al. 2009], we define a judgement for logical equivalence using binary logical relations, written as  $\Gamma^\varphi \models t_1 \approx_{\text{log}} t_2 : T^q \varepsilon$ .

The high-level structure of the proof is the following:

- Soundness (Theorem 4.39, Section 4.7). We show that the logical relation is sound with respect to contextual equivalence:

$$\Gamma^\varphi \models t_1 \approx_{\text{log}} t_2 : T^q \varepsilon \text{ implies } \Gamma^\varphi \models t_1 \approx_{\text{ctx}} t_2 : T^q \varepsilon.$$



**Context for Contextual Equivalence**

$$C ::= \square \mid C \ t \mid t \ C \mid \lambda x.C \mid \mathbf{ref} \ C \mid ! \ C \mid C := t \mid t := C \mid t; C \mid C;$$
**Context Typing Rules**

$$C : (\Gamma^\varphi; T^q \ \varepsilon) \Rightarrow (\Gamma'^\varphi; T^q \ \varepsilon')$$

$$\frac{\Gamma^\varphi \vdash T^q \ \varepsilon <: T'^q \ \varepsilon'}{\square : (\Gamma^\varphi; T^q \ \varepsilon) \Rightarrow (\Gamma'^\varphi; T'^q \ \varepsilon')} \quad (\text{C-HOLE})$$

$$\frac{C : (\Gamma^\varphi; U^r \ \varepsilon_1) \Rightarrow (\Gamma'^\varphi; ((x : T^{p^* \cap q^*}) \rightarrow^{\varepsilon_3} U'^r r')^q \ \varepsilon_4) \quad \Gamma'^\varphi \vdash t_2 : T^p \ \varepsilon_2 \quad x \notin \text{fv}(U') \quad r' \subseteq \varphi', x \quad \varepsilon_3 \subseteq \varphi', x \quad \theta = [p/x]}{C \ t_2 : (\Gamma^\varphi; U^r \ \varepsilon_1) \Rightarrow (\Gamma'^\varphi; (U'^r \ \varepsilon_4 \triangleright \varepsilon_2 \triangleright \varepsilon_3) \theta)} \quad (\text{C-APP-1})$$

$$\frac{\Gamma'^\varphi \vdash t_1 : ((x : T^{p^* \cap q^*}) \rightarrow^{\varepsilon_4} U'^r r')^q \ \varepsilon_2 \quad C : (\Gamma^\varphi; U^r \ \varepsilon_1) \Rightarrow (\Gamma'^\varphi; T^p \ \varepsilon_3) \quad x \notin \text{fv}(U') \quad r' \subseteq \varphi', x \quad \varepsilon_3 \subseteq \varphi', x \quad \theta = [p/x]}{t_1 \ C : (\Gamma^\varphi; U^r \ \varepsilon_1) \Rightarrow (\Gamma'^\varphi; (U'^r \ \varepsilon_2 \triangleright \varepsilon_3 \triangleright \varepsilon_4) \theta)} \quad (\text{C-APP-2})$$

$$\frac{C : (\Gamma^\varphi; S^r \ \varepsilon) \Rightarrow ((\Gamma', x : T^p)^{q,x}; U^r \ \varepsilon') \quad q \subseteq \varphi}{\lambda x.C : (\Gamma^\varphi; S^r \ \varepsilon) \Rightarrow (\Gamma'^\varphi; ((x : T^p) \rightarrow^{\varepsilon'} U^r)^q \ \emptyset)} \quad (\text{C-}\lambda)$$

$$\frac{C : (\Gamma^\varphi; T^r \ \varepsilon) \Rightarrow (\Gamma'^\varphi; B^q \ \varepsilon')}{\vdash \mathbf{ref} \ C : (\Gamma^\varphi; T^r \ \varepsilon) \Rightarrow (\Gamma'^\varphi; (\mathbf{Ref} \ B)^q \ \varepsilon')} \quad (\text{C-REF})$$

$$\frac{C : (\Gamma^\varphi; T^r \ \varepsilon) \Rightarrow (\Gamma'^\varphi; (\mathbf{Ref} \ B)^q \ \varepsilon')}{\vdash ! \ C : (\Gamma^\varphi; T^r \ \varepsilon) \Rightarrow (\Gamma'^\varphi; B^\emptyset \ \varepsilon')} \quad (\text{C-!})$$

$$\frac{C : (\Gamma^\varphi; T^r \ \varepsilon) \Rightarrow (\Gamma'^\varphi; (\mathbf{Ref} \ B)^q \ \varepsilon') \quad \Gamma'^\varphi \vdash t_2 : B^\emptyset \ \varepsilon'}{C := t_2 : (\Gamma^\varphi; T^r \ \varepsilon) \Rightarrow (\Gamma'^\varphi; \mathbf{Unit}^\emptyset \ \varepsilon')} \quad (\text{C-:=1})$$

$$\frac{\Gamma'^\varphi \vdash t_1 : (\mathbf{Ref} \ B)^q \ \varepsilon' \quad C : (\Gamma^\varphi; T^r \ \varepsilon) \Rightarrow (\Gamma'^\varphi; B^\emptyset \ \varepsilon')}{t_1 := C : (\Gamma^\varphi; T^r \ \varepsilon) \Rightarrow (\Gamma'^\varphi; B^\emptyset \ \varepsilon')} \quad (\text{C-:=2})$$

$$\frac{C : (\Gamma^\varphi; T^r \ \varepsilon) \Rightarrow (\Gamma'^\varphi; B^q \ \varepsilon') \quad \Gamma'^\varphi \vdash t_2 : B^\emptyset \ \varepsilon'}{C; t_2 : (\Gamma^\varphi; T^r \ \varepsilon) \Rightarrow (\Gamma'^\varphi; B^\emptyset \ \varepsilon')} \quad (\text{C-SEQ-1})$$

$$\frac{\Gamma'^\varphi \vdash t_1 : B^q \ \varepsilon' \quad C : (\Gamma^\varphi; T^r \ \varepsilon) \Rightarrow (\Gamma'^\varphi; B^\emptyset \ \varepsilon')}{t_1; C : (\Gamma^\varphi; T^r \ \varepsilon) \Rightarrow (\Gamma'^\varphi; B^\emptyset \ \varepsilon')} \quad (\text{C-SEQ-2})$$

Fig. 7. Context typing rules for the  $\lambda_{\varepsilon}^*$ -Calculus.

- Compatibility lemmas (Section 4.6). We show that the logical relation is compatible with syntactic typing.

These results can be used to prove the soundness of the re-ordering rule (Section 4.8).

## 4.2 Contextual Equivalence

Unlike reduction contexts, contexts  $C$  for reasoning about equivalence allow a “hole” to appear in any place. We write  $C : (\Gamma^\varphi; T^q \ \varepsilon) \Rightarrow (\Gamma'^\varphi; T'^q \ \varepsilon')$  to mean that the context  $C$  is a program of type  $T'^q \ \varepsilon'$  (closed under  $\Gamma'^\varphi$ ) with a hole that can be filled with any program of type  $T^q \ \varepsilon$  (closed under  $\Gamma^\varphi$ ). The typing rules for well-typed contexts imply that if  $\Gamma^\varphi \vdash t : T^q \ \varepsilon$  and

$C : (\Gamma^\varphi; T^q \varepsilon) \Rightarrow (\Gamma'^{\varphi'}; T'^{q'} \varepsilon')$  hold, then  $\Gamma'^{\varphi'} \vdash C[t] : T'^{q'} \varepsilon'$ . Fig. 7 shows the typing rules for well-typed contexts.

Two well-typed terms,  $t_1$  and  $t_2$ , under type context  $\Gamma^\varphi$ , are *contextually equivalent* if any occurrences of the first term in a closed term can be replaced by the second term without affecting the *observable results* of reducing the program, which is formally defined as follows:

*Definition 4.1 (Contextual Equivalence).* We say  $t_1$  is *contextually equivalent* to  $t_2$ , written as  $\Gamma^\varphi \models t_1 \approx_{\text{ctx}} t_2 : T^p \varepsilon$ , if  $\Gamma^\varphi \vdash t_1 : T^q \varepsilon$ , and  $\Gamma^\varphi \vdash t_2 : T^q \varepsilon$ , and:

$$\forall C : (\Gamma^\varphi; T^q \varepsilon) \Rightarrow (\emptyset; \text{Unit}^\emptyset \emptyset). C[t_1] \downarrow \iff C[t_2] \downarrow.$$

We write  $t \downarrow$  to mean term  $t$  terminates, if  $t, \emptyset, \emptyset \Downarrow v, \sigma$ , for some value  $v$  and final store  $\sigma$ .

The above definition is standard [Ahmed et al. 2009] and defines a partial program equivalence. However, since we focus on a total fragment of the  $\lambda_\varepsilon^*$ -calculus here, program termination can not be used as an observer for program equivalence. We will thus rely on the following refined version of contextual equivalence using Boolean contexts:

$$\begin{aligned} \forall C : (\Gamma^\varphi; T^q \varepsilon) \Rightarrow (\emptyset; B^\emptyset \emptyset). \exists \sigma, \sigma', v. \\ C[t_1], \emptyset, \emptyset \Downarrow v, \sigma \wedge C[t_2], \emptyset, \emptyset \Downarrow v, \sigma'. \end{aligned}$$

That is to say, we consider two terms contextually equivalent if they yield the same answer value in all Boolean contexts.

### 4.3 The Model

Following other prior works [Ahmed 2004; Benton et al. 2007; Thamsborg and Birkedal 2011], we apply Kripke logical relations to the  $\lambda_\varepsilon^*$ -calculus. Our logical relations are indexed by types and store layouts via *worlds*. This allows us to interpret Ref B as an allocated location that holds values of type B. The invariant that all allocated locations hold well-typed values with respect to the world must hold in the pre-state and be re-established in the post-state of a computation. The world may grow as more locations may be allocated. It is important that this invariant must hold in future worlds, which is commonly referred as *monotonicity*.

Considering the restriction to first-order references here, our store layouts are always “flat”, *i.e.*, free of cycles. The notion of world for the  $\lambda_\varepsilon^*$ -calculus is defined in the following:

*Definition 4.2 (World).* A world  $W$  is a triple  $(L_1, L_2, f)$ , where

- $L_1$  and  $L_2$  are finite sets of locations,
- $f \subseteq (L_1 \times L_2)$  is a partial bijection.

A world is meant to define relational stores. The partial bijection captures the fact that a relation holds under permutation of locations.

If  $W = (L_1, L_2, f)$  is a world, we refer to its components as follows:

$$\begin{aligned} W(\ell_1, \ell_2) &= \begin{cases} (\ell_1, \ell_2) \in f & \text{when defined} \\ \emptyset & \text{otherwise} \end{cases} \\ \text{dom}_1(W) &= L_1 \\ \text{dom}_2(W) &= L_2 \end{aligned}$$

If  $W$  and  $W'$  are worlds, such that  $\text{dom}_1(W) \cap \text{dom}_1(W') = \text{dom}_2(W) \cap \text{dom}_2(W') = \emptyset$ , then  $W$  and  $W'$  are called disjoint, and we write  $W; W'$  to mean extending  $W$  with a disjoint world  $W'$ .

Let  $\sigma_1$  and  $\sigma_2$  be two stores. We write  $(\sigma_1, \sigma_2) : W$  to mean  $W = (\text{dom}(\sigma_1), \text{dom}(\sigma_2), f)$ .

Our world definition allows us to specify that the domains of two relational stores may grow during a computation, but does not cover store operations, which is important when proving the

soundness of equational rules. Like prior works (e.g., [Benton et al. 2007; Thamsborg and Birkedal 2011]), we use effects as a refinement for the definition of world. The notation  $\epsilon$  denotes read/write effects.<sup>2</sup> Local reasoning is enabled by reachability qualifiers and read/write effects, meaning that what is preserved during an effectful computation are the locations that are *not* mentioned in the read/write effects. This is a common technique used in reasoning about frames in Hoare-style logics, e.g., separation logic [Reynolds 2002]. This treatment is also applicable to our refined effect system (i.e., the  $\lambda^*$ 's effect system in [Bao et al. 2021]), where framing is achieved through write effects – an established technique in Dafny [Leino 2010] and region logics [Banerjee et al. 2013; Bao et al. 2015]. In this case, a frame indirectly describes the locations that a computation may not change [Borgida et al. 1995]. Framing allows the proof to carry properties of effectful terms, such as function applications, since properties that are true for unchanged locations will remain valid [Bao et al. 2018].

Given two worlds  $W = (L_1, L_2, f)$  and  $W' = (L'_1, L'_2, f')$ , and two sets of locations  $L$  and  $L'$ , we define the relation of  $W$  and  $W'$  (written as  $W \sqsubseteq_{(L, L')} W'$ ) as follows:

$$\begin{aligned} W \sqsubseteq_{(L, L')} W' &\stackrel{\text{def}}{=} L \subseteq L_1 \wedge L \subseteq L'_1 \wedge L' \subseteq L_2 \wedge L' \subseteq L'_2 \wedge \\ &(\forall \ell_1, \ell_2. \ell_1 \in L_1 \wedge \ell_2 \in L_2 \wedge (\ell_1, \ell_2) \in f \Rightarrow (\ell_1, \ell_2) \in f') \wedge \\ &(\forall \ell_1, \ell_2. (\ell_1 \in L_1 \vee \ell_2 \in L_2) \wedge (\ell_1, \ell_2) \in f' \Rightarrow (\ell_1, \ell_2) \in f) \end{aligned}$$

We write  $W \sqsubseteq W'$  to mean  $W \sqsubseteq_{(\text{dom}_1(W), \text{dom}_2(W))} W'$ .

#### 4.4 Binary Logical Relations for $\lambda_\epsilon^*$

This section presents the definition of binary logical relations for  $\lambda_\epsilon^*$ . The relational value environment has to satisfy the context interpretation. We define the interpretation of typing contexts:

$$\begin{aligned} G[[\emptyset^\varphi]] &= \emptyset \\ G[[\Gamma, x : T^\varphi]] &= \{(W, \hat{H}; (x \mapsto (v_1, v_2))) \mid (W, \hat{H}) \in G[[\Gamma^\varphi]] \wedge \varphi \subseteq \text{dom}(\Gamma) \wedge q \subseteq \text{dom}(\Gamma) \wedge \\ & (W, v_1, v_2) \in \mathcal{V}[[T]]^{\hat{H}} \wedge \\ & (\forall q, q'. q \subseteq \varphi \wedge q' \subseteq \varphi \wedge \Rightarrow \\ & \quad (\text{locs}_{\hat{H}_1}((q^*)) \cap \text{locs}_{\hat{H}_1}(q'^*) \subseteq \text{locs}_{\hat{H}_1}((q^* \cap q'^*)) \wedge \\ & \quad \text{locs}_{\hat{H}_2}((q^*)) \cap \text{locs}_{\hat{H}_2}(q'^*) \subseteq \text{locs}_{\hat{H}_2}((q^* \cap q'^*)))\} \end{aligned}$$

In the above definition,  $\hat{H}$  ranges over relational value environment that are finite maps from variables  $x$  to pairs of values  $(v_1, v_2)$ . If  $\hat{H}(x) = (v_1, v_2)$ , then  $\hat{H}_1(x)$  denotes  $v_1$  and  $\hat{H}_2(x)$  denotes  $v_2$ .

*The Binary Value Interpretation.* The definition of binary value interpretation of types is shown in Fig. 8. The relational interpretation of type  $T$ , written as  $\mathcal{V}[[T]]^{\hat{H}}$ , is a set of tuples of form  $(W, v_1, v_2)$ , where  $v_1$  and  $v_2$  are values, and  $W$  is a world. We say  $v_1$  and  $v_2$  are related at type  $T$  with respect to  $W$ .

*Ground Types.* A pair of Boolean values are related if they are both true or false. A pair of locations  $(\ell_1, \ell_2)$  are related if they are in the domain of the relational store with respect to  $W$ , (written as  $(\sigma_1, \sigma_2) : W$ ), such that  $W(\ell_1, \ell_2)$ . It means that a pair of related locations store related values.

*Function Types.* Two closure records  $\langle H_1, (\lambda x.t_1)^{q_1} \rangle$  and  $\langle H_2, (\lambda x.t_2)^{q_2} \rangle$ , are related at type  $T^P \rightarrow^\epsilon U^r$  with respect to world  $W$ , meaning that it satisfies the following conditions:

<sup>2</sup>A complete approach would require a notion of allocation effects that specify store allocation occurs during a computation. As this report focuses on the proof the re-ordering rule (Section 4.8), allocation effects are omitted.

## Value Interpretation of Types and Terms

 $\lambda_{\varepsilon}^*$ 

$$\begin{aligned}
\mathcal{V}[[B]]^{\hat{H}} &= \{(W, v, v) \mid v = \text{true} \vee v = \text{false}\} \\
\mathcal{V}[[\text{Ref } B]]^{\hat{H}} &= \{(W, \ell_1, \ell_2) \mid \forall \sigma_1, \sigma_2. (\sigma_1, \sigma_2) : W \wedge \ell_1 \in \text{dom}(\sigma_1) \wedge \ell_2 \in \text{dom}(\sigma_2) \wedge W(\ell_1, \ell_2) \wedge \\
&\quad (W, \sigma_1(\ell_1), \sigma_2(\ell_2)) \in \mathcal{V}[[B]]^{\hat{H}}\} \\
\mathcal{V}[(x : T^p) \rightarrow^{\varepsilon} U^r]^{\hat{H}} &= \{(W, \langle H_1, (\lambda x.t_1)^{q_1} \rangle, \langle H_2, (\lambda x.t_2)^{q_2} \rangle) \mid \text{locs}(\langle H_1, (\lambda x.t_1)^{q_1} \rangle) \subseteq \text{dom}_1(W) \wedge \\
&\quad \text{locs}(\langle H_2, (\lambda x.t_2)^{q_2} \rangle) \subseteq \text{dom}_2(W) \wedge \\
&\quad (\forall \ell_1, \ell_2. W(\ell_1, \ell_2) \Rightarrow \ell_1 \in \text{locs}(\langle H_1, (\lambda x.t_1)^{q_1} \rangle) \iff \ell_2 \in \text{locs}(\langle H_2, (\lambda x.t_2)^{q_2} \rangle)) \wedge \\
&\quad (\forall v_1, v_2, W', \sigma_1, \sigma_2. W \sqsubseteq_{(\text{locs}(\langle H_1, (\lambda x.t_1)^{q_1} \rangle), \text{locs}(\langle H_2, (\lambda x.t_2)^{q_2} \rangle))} W' \wedge (\sigma_1, \sigma_2) : W' \Rightarrow \\
&\quad (W', v_1, v_2) \in \mathcal{V}[[T]]^{\hat{H}} \Rightarrow \text{locs}(\langle H_1, (\lambda x.t_1)^{q_1} \rangle) \cap \text{locs}(v_1) \subseteq \text{locs}_{\hat{H}_1}(p) \Rightarrow \\
&\quad \text{locs}(\langle H_2, (\lambda x.t_2)^{q_2} \rangle) \cap \text{locs}(v_2) \subseteq \text{locs}_{\hat{H}_2}(p) \Rightarrow \\
&\quad \exists W'', \sigma'_1, \sigma'_2, v'_1, v'_2, t_1, H_1; (x, v_1), \sigma_1 \Downarrow v'_1, \sigma'_1 \wedge t_2, H_2; (x, v_2), \sigma_2 \Downarrow v'_2, \sigma'_2 \wedge \\
&\quad W' \sqsubseteq W'' \wedge (\sigma'_1, \sigma'_2) : W'' \wedge (W'', v'_1, v'_2) \in \mathcal{V}[[U]]^{\hat{H}} \wedge \\
&\quad (x \in r \Rightarrow v'_1 \sim^{\sigma_1} (\text{locs}_{\hat{H}_1}(r) \cap \text{locs}(\langle H_1, (\lambda x.t_1)^{q_1} \rangle)) \cup \text{locs}(v_1)) \wedge \\
&\quad v'_2 \sim^{\sigma_2} (\text{locs}_{\hat{H}_2}(r) \cap \text{locs}(\langle H_2, (\lambda x.t_2)^{q_2} \rangle) \cup \text{locs}(v_2)) \wedge \\
&\quad (x \notin r \Rightarrow v'_1 \sim^{\sigma_1} (\text{locs}_{\hat{H}_1}(r) \cap \text{locs}(\langle H_1, (\lambda x.t_1)^{q_1} \rangle)) \wedge \\
&\quad v'_2 \sim^{\sigma_2} (\text{locs}_{\hat{H}_2}(r) \cap \text{locs}(\langle H_2, (\lambda x.t_2)^{q_2} \rangle)) \wedge \\
&\quad (x \in \varepsilon \Rightarrow \sigma_1 \hookrightarrow^{\text{locs}_{\hat{H}_1}(\varepsilon \triangleright p)} \sigma'_1 \wedge \sigma_2 \hookrightarrow^{\text{locs}_{\hat{H}_2}(\varepsilon \triangleright p)} \sigma'_2) \wedge \\
&\quad (x \notin \varepsilon \Rightarrow \sigma_1 \hookrightarrow^{\text{locs}(\hat{H}_1(\varepsilon))} \sigma'_1 \wedge \sigma_2 \hookrightarrow^{\text{locs}(\hat{H}_2(\varepsilon))} \sigma'_2)\} \\
\sigma \hookrightarrow^{\varepsilon} \sigma' &\stackrel{\text{def}}{=} \forall l \in \text{dom}(\sigma). \sigma(l) = \sigma'(l) \vee l \in \varepsilon \\
\mathcal{E}[[T^q \ \varepsilon]]_{\varphi}^{\hat{H}} &= \{(W, t_1, t_2) \mid \forall \sigma_1, \sigma_2. (\sigma_1, \sigma_2) : W \wedge \exists W', \sigma'_1, \sigma'_2, v_1, v_2, t_1, \hat{H}_1, \sigma_1 \Downarrow v_1, \sigma'_1 \wedge \\
&\quad t_2, \hat{H}_2, \sigma_2 \Downarrow v_2, \sigma'_2 \wedge W \sqsubseteq W' \wedge (\sigma'_1, \sigma'_2) : W' \wedge \\
&\quad (W', v_1, v_2) \in \mathcal{V}[[T]]^{\hat{H}} \wedge v_1 \sim^{\sigma_1} (\text{locs}_{\hat{H}_1}(\varphi \cap q)) \wedge v_2 \sim^{\sigma_2} (\text{locs}_{\hat{H}_2}(\varphi \cap q)) \wedge \\
&\quad \sigma_1 \hookrightarrow^{\text{locs}_{\hat{H}_1}(\varepsilon)} \sigma'_1 \wedge \sigma_2 \hookrightarrow^{\text{locs}_{\hat{H}_2}(\varepsilon)} \sigma'_2\}
\end{aligned}$$

Fig. 8. Binary value and term interpretation for the  $\lambda_{\varepsilon}^*$ -calculus.

- The set of locations reachable from the two closure records are well-formed with respect to the world, *i.e.*,  $\text{locs}(\langle H_1, (\lambda x.t_1)^{q_1} \rangle) \subseteq \text{dom}_1(W)$  and  $\text{locs}(\langle H_2, (\lambda x.t_2)^{q_2} \rangle) \subseteq \text{dom}_2(W)$ .
- If a pair of locations  $(\ell_1, \ell_2)$  are related at world  $W$ , then  $\ell_1$  is reachable from its closure record (*i.e.*,  $\text{locs}(\langle H_1, (\lambda x.t_1)^{q_1} \rangle)$ ) if and only if  $\ell_2$  is reachable from its closure record (*i.e.*,  $\text{locs}(\langle H_2, (\lambda x.t_2)^{q_2} \rangle)$ ).
- The arguments are allowed if
  - $W \sqsubseteq_{(\text{locs}(\langle H_1, (\lambda x.t_1)^{q_1} \rangle), \text{locs}(\langle H_2, (\lambda x.t_2)^{q_2} \rangle))} W'$ ; and
  - the arguments  $v_1$  and  $v_2$  are related at type  $T$  with respect  $W'$ ; and
  - the overlapping locations reachable from the functions and their arguments are permissible by the argument's qualifier  $p$ , *i.e.*,  $\text{locs}(\langle H_1, (\lambda x.t_1)^{q_1} \rangle) \cap \text{locs}(v_1) \subseteq \text{locs}_{\hat{H}_1}(p)$  and  $\text{locs}(\langle H_2, (\lambda x.t_2)^{q_2} \rangle) \cap \text{locs}(v_2) \subseteq \text{locs}_{\hat{H}_2}(p)$ .
- Under their extended value environments  $H_1; (x, v_1)$  and  $H_2; (x, v_2)$ ,  $t_1$  and  $t_2$  are reduced to some values  $v'_1$  and  $v'_2$  with some final stores  $\sigma'_1, \sigma'_2$  and world  $W''$ , such that
  - the world  $W''$  are extended from the world  $W'$ , such that  $W' \sqsubseteq W''$ ; and
  - $\sigma'_1$  and  $\sigma'_2$  are related with respect to world  $W''$ , *i.e.*,  $(\sigma'_1, \sigma'_2) : W''$ .
  - $v'_1$  and  $v'_2$  are related at type  $U$  with respect to world  $W''$ ; and
  - If the return value's qualifier  $r$  depends on the argument (*i.e.*,  $x \in r$ ), then the locations reachable from  $v'_1$  and  $v'_2$  are subsets of those reachable both from the function and  $r$ ,

- plus those reachable from the arguments; otherwise (*i.e.*,  $x \notin r$ ), they are just subset of those reachable both from the function and  $r$ ; and
- If a bound variable  $x$  appears in the effect  $\varepsilon$ , meaning the function body may modify the argument, then the effect will include the qualifier that may reach the value of function argument  $p$ ; otherwise it is just  $\varepsilon$ .

*The Binary Term Interpretation.* Two related terms,  $t_1$  and  $t_2$ , are defined based on the relation of their computational behaviors, *i.e.*, returned values, reachability qualifiers and effects, which is defined by  $\mathcal{E}[[T \ \varepsilon]]_{\varphi}^{\hat{H}}$ . It means for all related stores with respect to world,  $(\sigma_1, \sigma_2) : W$ , if

- $t_1$  is evaluated to some value  $v_1$  with some final store  $\sigma'_1$ ; and
- $t_2$  is evaluated to some value  $v_2$  with some final store  $\sigma'_2$ ; and
- there exists a world  $W'$ , such that  $W \sqsubseteq W'$ ; and
- $v_1$  and  $v_2$  are related at type  $T$  with respect to world  $W'$ ; and
- $\sigma'_1$  and  $\sigma'_2$  are related with respect to  $W'$ ; and
- The locations reachable from the values in the domain of pre-stores are subset of those reachable from  $\text{locs}_{\hat{H}_1}((\varphi \cap q))$  and  $\text{locs}_{\hat{H}_2}((\varphi \cap q))$  for each of the term; and
- The effect captures what may be read/modified in the pre-state store.

Note that we interpret the function body (after substitution) and other terms separately, which allows us to provide more precise reasoning in the logical relations of function types.

## 4.5 Metatheory

This section discusses several key lemmas used in the proof of compatibility lemmas (Section 4.6) and soundness of the re-ordering rules (Section 4.8).

### 4.5.1 Well-formedness.

LEMMA 4.3 (WELL-FORMED VALUE INTERPRETATION). *Let  $(W, \hat{H}) \in G[[\Gamma^\varphi]]$ . If  $(W, v_1, v_2) \in \mathcal{V}[[T]]^{\hat{H}}$ , then  $\text{locs}(v_1) \subseteq \text{dom}_1(W)$  and  $\text{locs}(v_2) \subseteq \text{dom}_2(W)$ .*

PROOF. By induction on type  $T$  and the constructs of value  $v_1$  and  $v_2$ . □

LEMMA 4.4 (WELL-FORMED TYPING CONTEXT INTERPRETATION). *Let  $(W, \hat{H}) \in G[[\Gamma^\varphi]]$ , then for all  $q \subseteq \varphi$ ,  $\text{locs}_{\hat{H}_1}(q) \subseteq \text{dom}_1(W)$  and  $\text{locs}_{\hat{H}_2}(q) \subseteq \text{dom}_2(W)$ .*

PROOF. By definition of the typing context interpretation and Lemma 4.3. □

LEMMA 4.5. *Let  $(W, \hat{H}) \in G[[\Gamma^\varphi]]$ , then  $\text{dom}(\hat{H}_1) = \text{dom}(\hat{H}_2) = \text{dom}(\Gamma)$ , and  $\text{dom}(\Gamma)^*$ .*

PROOF. Immediately by the definition of typing context interpretation and the definition of saturation in Fig. 2. □

### 4.5.2 World Extension and Relational Stores.

LEMMA 4.6 (RELATIONAL STORE UPDATE). *If  $(\sigma_1, \sigma_2) : W$ , and  $(W, \ell_1, \ell_2) \in \mathcal{V}[[\text{Ref B}]]^{\hat{H}}$ , and  $(W, v_1, v_2) \in \mathcal{V}[[B]]^{\hat{H}}$ , then  $(\sigma_1[\ell_1 \mapsto v_1], \sigma_2[\ell_2 \mapsto v_2]) : W$ .*

PROOF. By definition of relational stores. □

LEMMA 4.7 (RELATIONAL STORE EXTENSION). *If  $(\sigma_1, \sigma_2) : W$ , and  $(W, v_1, v_2) \in \mathcal{V}[[B]]^{\hat{H}}$ , then  $(\sigma_1; (\ell_1 : v_1), \sigma_2; \ell_2 : v_2) : W; (\ell_1, \ell_2, (\ell_1, \ell_2) \in f)$ , where  $\ell_1 \notin \text{dom}(\sigma_1)$  and  $\ell_2 \notin \text{dom}(\sigma_2)$ .*

PROOF. By definition of relational stores. □

LEMMA 4.8 (LOGICAL RELATION CLOSED UNDER RELATIONAL VALUE SUBSTITUTION EXTENSION). *If  $T$  is closed under  $\Gamma^\varphi$ , and  $(W, \hat{H}) \in G[[\Gamma^\varphi]]$ , then  $(W, v_1, v_2) \in \mathcal{V}[[T]]^{\hat{H}}$  if and only if  $(W, v_1, v_2) \in \mathcal{V}[[T]]^{\hat{H}; \hat{H}'}$ , for all  $\hat{H}'$ .*

PROOF. By induction on type  $T$  and the constructs of values  $v_1$  and  $v_2$ .  $\square$

LEMMA 4.9 (LOGICAL RELATION LOCALIZATION). *If  $(W, v_1, v_2) \in \mathcal{V}[[T]]^{\hat{H}}$ , and for all  $W'$ , such that  $W \sqsubseteq_{(\text{locs}(v_1), \text{locs}(v_2))} W'$ , then  $(W', v_1, v_2) \in \mathcal{V}[[T]]^{\hat{H}}$ .*

PROOF. By induction on type  $T$  and the constructs of values  $v_1$  and  $v_2$ .  $\square$

LEMMA 4.10 (LOGICAL RELATION CLOSED UNDER WORLD EXTENSION). *If  $(W, v_1, v_2) \in \mathcal{V}[[T]]^{\hat{H}}$ , and for all  $W'$ , such that  $W \sqsubseteq W'$ , then  $(W', v_1, v_2) \in \mathcal{V}[[T]]^{\hat{H}}$ .*

PROOF. By the definition of logical relation, Lemma 4.3 and Lemma 4.9.  $\square$

#### 4.5.3 Semantic Typing Context.

LEMMA 4.11 (SEMANTIC TYPING CONTEXT TIGHTEN). *If  $(W, \hat{H}, ) \in G[[\Gamma^\varphi]]$ , then for all  $p \subseteq \varphi$ ,  $(W, \hat{H}) \in G[[\Gamma^p]]$ .*

PROOF. By the definition of typing context interpretation.  $\square$

LEMMA 4.12 (SEMANTIC TYPING CONTEXT EXTENSION 1). *If  $(W, \hat{H}) \in G[[\Gamma^\varphi]]$ , and  $q \subseteq \text{dom}(\Gamma)$ , and  $(W, v_1, v_2) \in \mathcal{V}[[T]]^{\hat{H}}$ , and  $\text{locs}_{\hat{H}_1}(\varphi) \cap \text{locs}(v_1) \subseteq \text{locs}_{\hat{H}_1}(q)$ , and  $\text{locs}_{\hat{H}_2}(\varphi) \cap \text{locs}(v_2) \subseteq \text{locs}_{\hat{H}_2}(q)$ , then  $(W, \hat{H}; (x \mapsto (v_1, v_2))) \in G[[\Gamma, x : T^q]^{\varphi, x}]$*

PROOF. By typing context interpretation and Lemma 4.8.  $\square$

LEMMA 4.13 (SEMANTIC TYPING CONTEXT EXTENSION 2). *If  $(W, \hat{H}) \in G[[\Gamma^\varphi]]$ , and  $W \sqsubseteq W'$ , and  $(W', v_1, v_2) \in \mathcal{V}[[T]]^{\hat{H}}$ , and  $\text{locs}_{\hat{H}_1}(q) \cap \text{locs}(v_1) \subseteq \text{locs}_{\hat{H}_1}(p)$ , and  $\text{locs}_{\hat{H}_2}(q) \cap \text{locs}(v_2) \subseteq \text{locs}_{\hat{H}_2}(p)$ , and  $q \subseteq \varphi$ , then  $(W', \hat{H}; (x \mapsto (v_1, v_2))) \in G[[\Gamma, x : T^p]^{\varphi, x}]$ .*

PROOF. By typing context interpretation, Lemma 4.8, Lemma 4.10 and Lemma 4.11.  $\square$

LEMMA 4.14 (SEMANTIC TYPING CONTEXT LOCALIZATION). *If  $(W, \hat{H}) \in G[[\Gamma^\varphi]]$ , and  $W \sqsubseteq_{(\text{locs}(\hat{H}_1(\varphi)), \text{locs}(\hat{H}_2(\varphi)))} W'$ , then  $(W', \hat{H}) \in G[[\Gamma^\varphi]]$ .*

PROOF. By definition of typing context interpretation and Lemma 4.9.  $\square$

#### 4.5.4 Reachability Qualifiers.

LEMMA 4.15. *For all  $\sigma, b, p$  and  $q, b \rightsquigarrow^\sigma \text{locs}(p \cap q)$ , where  $b$  is true or false.*

PROOF. Immediate by the definition in Fig. 4.  $\square$

LEMMA 4.16. *For all  $\sigma, \ell, p$  and  $q, \ell \rightsquigarrow^\sigma \text{locs}(p \cap q)$ , where  $\ell \notin \text{dom}(\sigma)$ .*

PROOF. Immediate by the definition in Fig. 4.  $\square$

LEMMA 4.17.  $\langle H_1, (\lambda x. t_1)^{p_1^* \cap q_1^*} \rightsquigarrow_{\text{dom}_1(W)} \text{locs}_{\hat{H}_1}(p_1^* \cap q_1^*)$  and  $\langle H_2, (\lambda x. t_2)^{p_2^* \cap q_2^*} \rightsquigarrow_{\text{dom}_2(W)} \text{locs}_{\hat{H}_2}(p_2^* \cap q_2^*)$ .

PROOF. Immediate by the definition in Fig. 4.  $\square$

4.5.5 *Effects*. To streamline the presentation, we introduce the following notation. We write  $(\sigma \downarrow L)$  to mean retroving a partial store with respect to  $L$ , meaning  $\text{dom}((\sigma \downarrow L)) = \text{dom}(\sigma) \cap L \wedge \forall \ell \in \text{dom}((\sigma \downarrow L)).(\sigma \downarrow L)(\ell) = \sigma(\ell)$ .

LEMMA 4.18 (READ/WRITE EFFECTS). *If  $\ell \in \text{dom}(\sigma)$ , and  $\ell \rightsquigarrow^\sigma \text{locs}(p \cap q)$ , then  $\sigma \hookrightarrow^{\text{locs}(q)} \sigma[\ell \mapsto v]$ .*

PROOF. By Lemma 4.16 and interpretation of effects.  $\square$

LEMMA 4.19 (NO EFFECTS).  $\sigma \hookrightarrow^\emptyset \sigma$ .

PROOF. Immediate by the definition of effects.  $\square$

LEMMA 4.20 (SUBEFFECTS). *If  $\text{locs}(\varepsilon_1) \subseteq \text{locs}(\varepsilon_2)$ , and  $\sigma \hookrightarrow^{\text{locs}(\varepsilon_1)} \sigma'$ , then  $\sigma \hookrightarrow^{\text{locs}(\varepsilon_2)} \sigma'$ .*

PROOF. By the interpretation of effects.  $\square$

LEMMA 4.21 (EFFECTS COMPOSITION). *If  $\sigma \hookrightarrow^{\text{locs}(\varepsilon_1^*)} \sigma'$ , and  $\sigma' \hookrightarrow^{\text{locs}((\varepsilon_2 \triangleright \varepsilon_3)^*)} \sigma''$ , and  $\varepsilon_2^* \cap \varepsilon_3^* = \emptyset$ , and  $\text{locs}(\varepsilon_2^*) \subseteq \text{dom}(\sigma)$ , and  $\text{locs}(\varepsilon_3^*) \cap \text{dom}(\sigma) = \emptyset$ . then  $\sigma \hookrightarrow^{\text{locs}((\varepsilon_1 \triangleright \varepsilon_2)^*)} \sigma''$*

PROOF. By the interpretation of effects.  $\square$

LEMMA 4.22 (FRAMING). *If  $\sigma \hookrightarrow^{\text{locs}(\varepsilon)} \sigma'$ , then  $\sigma \downarrow (\text{dom}(\sigma) - \text{locs}(\varepsilon^*)) = \sigma' \downarrow (\text{dom}(\sigma) - \text{locs}(\varepsilon^*))$*

PROOF. By the interpretation of observable effects: the set of locations that may be written in the reduction of  $t$  must be in  $\varepsilon$ . Thus, the values stored in the locations  $\sigma$ , but are separate from  $\varepsilon^*$  must be preserved.  $\square$

#### 4.5.6 Other auxiliary lemmas.

LEMMA 4.23 (QUALIFIER INTERSECTION DISTRIBUTES OVER LOCATIONS). *Let  $(W, \hat{H}) \in G[[\Gamma^\varphi]]$ , and  $(\sigma_1, \sigma_2) : W$ . For all  $\sigma'_1, \sigma'_2$  and  $W'$ , such that  $W' \sqsubseteq W$  and  $(\sigma'_1, \sigma'_2) : W'$  if  $v_{f_1} \rightsquigarrow^{\sigma_1} \text{locs}_{\hat{H}_1}(q_f)$ , and  $v_{f_2} \rightsquigarrow^{\sigma_2} \text{locs}_{\hat{H}_2}(q_f)$ , and  $v_1 \rightsquigarrow^{\sigma'_1} \text{locs}_{\hat{H}_1}(p)$ , and  $v_2 \rightsquigarrow^{\sigma'_2} \text{locs}_{\hat{H}_2}(p)$ , and  $\text{locs}(v_{f_1}) \subseteq \text{dom}(\sigma'_1)$ , and  $\text{locs}(v_{f_2}) \subseteq \text{dom}(\sigma'_2)$ , then  $(\text{locs}(v_{f_1}) \cap \text{locs}(v_1)) \subseteq \text{locs}_{\hat{H}_1}((p^* \cap q_f^*))$  and  $(\text{locs}(v_{f_2}) \cap \text{locs}(v_2)) \subseteq \text{locs}_{\hat{H}_2}((p^* \cap q_f^*))$ .*

PROOF. By typing context interpretation, Lemma 4.4 and set theory.  $\square$

LEMMA 4.24 (SEMANTIC FUNCTION ABSTRACTION). *Let  $(W, \hat{H}) \in G[[\Gamma^\varphi]]$ ,  $(\sigma_1, \sigma_2) : W$ , and  $\text{dom}(\Gamma^*)$ . For all  $W'$ , such that  $W \sqsubseteq_{(\text{locs}(\langle H_1, (\lambda x.t_1)^{q_1} \rangle), \text{locs}(\langle H_1, (\lambda x.t_1)^{q_1} \rangle))} W'$ , if  $(W', v_1, v_2) \in \mathcal{V}[[T]^{\hat{H}}]$ , and  $\text{locs}(\langle H_1, (\lambda x.t_1)^{q_1} \rangle) \cap \text{locs}(v_1) \subseteq \text{locs}_{\hat{H}_1}(p)$ , and  $\text{locs}(\langle H_2, (\lambda x.t_2)^{q_2} \rangle) \cap \text{locs}(v_2) \subseteq \text{locs}_{\hat{H}_2}(p)$ , and  $(W', \hat{H}; x \mapsto (v_1, v_2)) \in G[[\langle \Gamma, x : T^p \rangle^{q,x}]]$  implies that there exists  $W''$ , such that  $W' \sqsubseteq W''$ ,  $(W'', t_1, t_2) \in \mathcal{E}[[U^r \ \varepsilon]]_{q,x}^{(H_1, H_2); (x, (v_1, v_2))}$ . and  $p \subseteq q$ , then exists  $W'', v'_1, v'_2$ , such that*

- (1)  $W' \sqsubseteq W''$
- (2)  $t_1, H_1; (x, v_1), \sigma_1 \Downarrow v_1, \sigma'_1$
- (3)  $t_1, H_2; (x, v_2), \sigma_2 \Downarrow v_2, \sigma'_2$
- (4)  $(\sigma'_1, \sigma'_2) : W''$
- (5)  $(W'', v_3, v_4) \in \mathcal{V}[[U]^{\hat{H}}]$
- (6)  $(x \in r \Rightarrow v'_1 \rightsquigarrow^{\sigma'_1} (\text{locs}_{\hat{H}_1}(r) \cap \text{locs}(\langle H_1, (\lambda x.t_1)^{q_1} \rangle) \cup \text{locs}(v_1)) \wedge v'_2 \rightsquigarrow^{\sigma'_2} (\text{locs}_{\hat{H}_2}(r) \cap \text{locs}(\langle H_1, (\lambda x.t_2)^{q_1} \rangle) \cup \text{locs}(v_2)))$
- (7)  $(x \notin r \Rightarrow v'_1 \rightsquigarrow^{\sigma'_1} (\text{locs}_{\hat{H}_1}(r) \cap \text{locs}(\langle H_1, (\lambda x.t_1)^{q_1} \rangle)) \wedge v'_2 \rightsquigarrow^{\sigma'_2} (\text{locs}_{\hat{H}_2}(r) \cap \text{locs}(\langle H_2, (\lambda x.t_2)^{q_2} \rangle)))$

PROOF. By Lemma 4.13,  $(W', (H_1, H_2); (x \mapsto (v_1, v_2))) \in G[[\Gamma, x : T^p]^{q,x}]$ . Thus, there exists  $W''$ , such that  $(W'', t_1, t_2) \in \mathcal{E}[[U^r \ \varepsilon]]_{q,x}^{(H_1, H_2); (x, (v_1, v_2))}$ , which can be used to prove (2) - (4). (6) and (7) can be proved by inspecting  $x \in r$ , Lemma 4.3 and Lemma 4.17.  $\square$

LEMMA 4.25 (SEMANTIC APPLICATION). *Let  $(W, \hat{H}) \in G[[\Gamma^\varphi]]$ .*

*If  $W \sqsubseteq_{(\text{locs}(\langle H_1, (\lambda x.t_1)^{q_1} \rangle), \text{locs}(\langle H_2, (\lambda x.t_2)^{q_2} \rangle))} W'$  and  $(W', \langle H_1, (\lambda x.t_1)^{q_1} \rangle, \langle H_2, (\lambda x.t_2)^{q_2} \rangle) \in \mathcal{V}[[T^{p^* \cap q^*} \rightarrow^\varepsilon U^r]]^{\hat{H}}$ , and  $\langle H_1, (\lambda x.t_1)^{q_1} \rangle \rightsquigarrow^{\sigma_1} \text{locs}_{\hat{H}_1}(q)$ , and  $\langle H_2, (\lambda x.t_2)^{q_2} \rangle \rightsquigarrow^{\sigma_2} \text{locs}_{\hat{H}_2}(q)$ , and  $W''' \sqsubseteq W$  and  $(W'', v_1, v_2) \in \mathcal{V}[[T]]^{\hat{H}}$ , and  $v_1 \rightsquigarrow^{\text{dom}_1(W')} \text{locs}_{\hat{H}_1}(p)$ , and  $v_2 \rightsquigarrow^{\text{dom}_2(W')} \text{locs}_{\hat{H}_2}(p)$ , and  $r \sqsubseteq \varphi, x$ , and  $\varepsilon \sqsubseteq q, x$ , and  $(\sigma_1, \sigma_2) : W''$  then there exists  $v_2, v'_2, \sigma'_1, \sigma'_2, W'''$ , such that*

- (1)  $W'' \sqsubseteq W'''$
- (2)  $t_1, H_1; (x, v_1), \sigma_1 \Downarrow v_1, \sigma'_1$
- (3)  $t_2, H_2; (x, v_2), \sigma_2 \Downarrow v_2, \sigma'_2$
- (4)  $(\sigma'_1, \sigma'_2) : W'''$ ;
- (5)  $(W''', v'_1, v'_2) \in \mathcal{V}[[U]]^{\hat{H}}$ ;
- (6)  $x \in r \Rightarrow v'_1 \rightsquigarrow^{\sigma_1} (\text{locs}_{\hat{H}_1}(r) \cap \text{locs}(\langle H_1, (\lambda x.t_1)^{q_1} \rangle) \cup \text{locs}(v_1)) \wedge v'_2 \rightsquigarrow^{\sigma_2} (\text{locs}_{\hat{H}_2}(r) \cap \text{locs}(\langle H_2, (\lambda x.t_2)^{q_2} \rangle) \cup \text{locs}(v_2))$
- (7)  $x \notin r \Rightarrow v'_1 \rightsquigarrow^{\sigma_1} (\text{locs}_{\hat{H}_1}(r) \cap \text{locs}(\langle H_1, (\lambda x.t_1)^{q_1} \rangle)) \wedge v'_2 \rightsquigarrow^{\sigma_2} (\text{locs}_{\hat{H}_2}(r) \cap \text{locs}(\langle H_2, (\lambda x.t_2)^{q_2} \rangle))$

PROOF. By Lemma 4.3, we know the following:

- $\text{locs}(\langle H_1, (\lambda x.t_1)^{q_1} \rangle) \subseteq \text{dom}_1(W')$ ;
- $\text{locs}(\langle H_2, (\lambda x.t_2)^{q_2} \rangle) \subseteq \text{dom}_2(W')$ ;
- $\text{locs}(v_1) \subseteq \text{dom}_1(W'')$ ;
- $\text{locs}(v_2) \subseteq \text{dom}_2(W'')$ ;

Then (2) - (5) can be proved by the assumption:

$(W', \langle H_1, (\lambda x.t_1)^{q_1} \rangle, \langle H_2, (\lambda x.t_2)^{q_2} \rangle) \in \mathcal{V}[[T^{p^* \cap q^*} \rightarrow^\varepsilon U^r]]^{\hat{H}}$ , and  $(W'', v_1, v_2) \in \mathcal{V}[[T]]^{\hat{H}}$ , and Lemma 4.23. (6) - (7) can be proved by inspecting  $x \in r$ .  $\square$

#### 4.6 Compatibility Lemmas

The following compatibility lemmas show that the logical relations is *compatible* with all the constructs of the language [Pierce 2004].

LEMMA 4.26 (COMPATIBILITY: B).  $\Gamma^\varphi \models \text{true} \approx_{\log} \text{true} : B^\varnothing \ \emptyset$

PROOF. By the typing context interpretation, value interpretation in Fig. 8 and Lemma 4.15.  $\square$

LEMMA 4.27 (COMPATIBILITY: B).  $\Gamma^\varphi \models \text{false} \approx_{\log} \text{false} : B^\varnothing \ \emptyset$

PROOF. By the typing context interpretation, value interpretation in Fig. 8 and Lemma 4.15.  $\square$

LEMMA 4.28 (COMPATIBILITY: VARIABLES). *If  $x : T^q \in \Gamma$  and  $x \subseteq \varphi$ , then  $\Gamma^\varphi \models x \approx_{\log} x : T^x \ \emptyset$*

PROOF. Immediate by the typing context interpretation in Fig. 8.  $\square$

LEMMA 4.29 (COMPATIBILITY:  $\lambda$ ). *If  $(\Gamma, x : T^p]^{q,x} \models t_1 \approx_{\log} t_2 : U^r \ \varepsilon, q \subseteq \varphi$ , then  $\Gamma^\varphi \models (\lambda x.t_1)^{q_1} \approx_{\log} (\lambda x.t_2)^{q_2} : (x : T^p \rightarrow^\varepsilon U^r)^q \ \emptyset$ .*

PROOF. Let  $(W, \hat{H}) \in G[[\Gamma]]$  and  $(\sigma_1, \sigma_2) : W$ , and  $(\forall \ell_1, \ell_2. W(\ell_1, \ell_2) \Rightarrow \ell_1 \in \text{locs}(\langle H_1, (\lambda x.t_1)^{q_1} \rangle) \iff \ell_2 \in \text{locs}(\langle H_2, (\lambda x.t_2)^{q_2} \rangle))$ . By definition of term interpretation, we need to show there exists  $W', \sigma', v_1$  and  $v_2$  such that:



- (1)  $W \sqsubseteq (\text{locs}(\langle H_1, (\lambda x.t_1)^{q_1} \rangle), \text{locs}(\langle H_2, (\lambda x.t_2)^{q_2} \rangle)) W'$
- (2)  $\langle \lambda x.t_1 \rangle^{q_1}, \hat{H}_1, \sigma_1 \Downarrow \langle \hat{H}_1, (\lambda x.t_1)^{q_1} \rangle, \sigma'_1$
- (3)  $\langle \lambda x.t_1 \rangle^{q_2}, \hat{H}_2, \sigma_2 \Downarrow \langle \hat{H}_2, (\lambda x.t_2)^{q_2} \rangle, \sigma'_2$
- (4)  $(\sigma'_1, \sigma'_2) : W'$
- (5)  $(W', v_1, v_2) \in \mathcal{V}[\![x : T^p \rightarrow^\varepsilon U^r]\!]^{\hat{H}}$
- (6)  $v_1 \rightsquigarrow^{\sigma_1} \text{locs}_{\hat{H}_1}(\varphi \cap q)$
- (7)  $v_2 \rightsquigarrow^{\sigma_2} \text{locs}_{\hat{H}_2}(\varphi \cap q)$
- (8)  $\sigma_1 \hookrightarrow^\varnothing \sigma'_1$
- (9)  $\sigma_2 \hookrightarrow^\varnothing \sigma'_2$

By reduction semantics, we pick  $W' = W$ ,  $v_1 = \langle VE_1, (\lambda x.t_1)^{q_1} \rangle$ ,  $v_2 = \langle \hat{H}_2, (\lambda x.t_2)^{q_2} \rangle$ ,  $\sigma'_1 = \sigma_1$  and  $\sigma'_2 = \sigma_2$ . Thus, (1)- (4) are discharged. (5) can be proved by Lemma 4.5 and Lemma 4.24. (6) and (7) can be proved by Lemma 4.17. (8) and (9) can be proved by Lemma 4.19.  $\square$

LEMMA 4.30 (COMPATIBILITY : ALLOCATION). *If  $\Gamma^\varphi \models t_1 \approx_{\log} t_2 : B^q \varepsilon$ , then  $\Gamma^\varphi \models \mathbf{ref} t_1 \approx_{\log} \mathbf{ref} t_2 : (\mathbf{Ref} B)^q \varepsilon$ .*

PROOF. Let  $(W, \hat{H}) \in G[\![\Gamma]\!]$  and  $(\sigma_1, \sigma_2) : W$ . By the assumption, we know that there exists  $\sigma'_1, \sigma'_2, W', v_1$  and  $v_2$ , such that

- $W \sqsubseteq W'$
- $t_1, \hat{H}_1, \sigma_1 \Downarrow v_1, \sigma'_1$
- $t_2, \hat{H}_2, \sigma_2 \Downarrow v_2, \sigma'_2$
- $(\sigma'_1, \sigma'_2) : W'$
- $(W', v_1, v_2) \in \mathcal{V}[\![B]\!]^{\hat{H}}$
- $v_1 \rightsquigarrow^{\sigma_1} \text{locs}_{\hat{H}_1}(\varphi \cap q)$
- $v_2 \rightsquigarrow^{\sigma_2} \text{locs}_{\hat{H}_2}(\varphi \cap q)$
- $\sigma_1 \hookrightarrow^{\text{locs}_{\hat{H}_1}(\varepsilon_1)} \sigma'_1$
- $\sigma_2 \hookrightarrow^{\text{locs}_{\hat{H}_2}(\varepsilon_1)} \sigma'_2$

By reduction semantics, we know

- $\mathbf{ref} t_1, \hat{H}_1, \sigma_1 \Downarrow \ell_1, \sigma'_1; (\ell_1, v_1)$ , where  $\ell_1 \notin \text{dom}(\sigma'_1)$
- $\mathbf{ref} t_2, \hat{H}_2, \sigma_2 \Downarrow \ell_2, \sigma'_2; (\ell_2, v_1)$ , where  $\ell_2 \notin \text{dom}(\sigma'_2)$

By Lemma 4.7, we know  $(\sigma'_1; (\ell_1 \mapsto v_1), \sigma'_2; (\ell_2 \mapsto v_2)) : W'; ((\ell_1 \mapsto v_1), (\ell_2 \mapsto v_2), \{(\ell_1, \ell_2)\})$ . The rest of the proof can be done by the definition of value interpretation, Lemma 4.21 and Lemma 4.16.  $\square$

LEMMA 4.31 (COMPATIBILITY: DEREFERENCE (!)). *If  $\Gamma^\varphi \models t_1 \approx_{\log} t_2 : (\mathbf{Ref} B)^q \varepsilon$ , then  $\Gamma^\varphi \models !t_1 \approx_{\log} !t_2 : B^\varnothing \varepsilon \triangleright q$ .*

PROOF. Let  $(W, \hat{H}) \in G[\![\Gamma^\varphi]\!]$  and  $(\sigma_1, \sigma_2) : W$ . By the assumption,  $(W, t_1, t_2) \in \mathcal{E}[\![\mathbf{Ref} B^q \varepsilon]\!]^{\hat{H}}_\varphi$ , and reduction semantics, we know there exists  $\sigma'_1, \sigma'_2, \ell_1$  and  $\ell_2$  such that

- $W \sqsubseteq W'$
- $t_1, \hat{H}_1, \sigma_1 \Downarrow \ell_1, \sigma'_1$ , where  $\sigma'_1(\ell_1) = v_1$
- $t_2, \hat{H}_2, \sigma_2 \Downarrow \ell_2, \sigma'_2$ , where  $\sigma'_2(\ell_2) = v_2$
- $(\sigma'_1, \sigma'_2) : W'$
- $(W', \ell_1, \ell_2) \in \mathcal{V}[\![\mathbf{Ref} B]\!]^{\hat{H}}$
- $\ell_1 \rightsquigarrow^{\sigma_1} \text{locs}_{\hat{H}_1}(\varphi \cap q)$
- $\ell_2 \rightsquigarrow^{\sigma_2} \text{locs}_{\hat{H}_2}(\varphi \cap q)$

- $\sigma_1 \hookrightarrow^{\text{locs}_{\hat{H}_1}(\varepsilon)} \sigma'_1$
- $\sigma_2 \hookrightarrow^{\text{locs}_{\hat{H}_2}(\varepsilon)} \sigma'_2$

We can finish the proof by reduction semantics, value interpretation, Lemma 4.15, Lemma 4.20, where we pick  $\sigma'_1$  to be  $\sigma'_1$ ,  $\sigma'_2$  to be  $\sigma'$ , and  $W''$  to be  $W'$ .  $\square$

LEMMA 4.32 (COMPATIBILITY: ASSIGNMENTS (:=)). *If  $\Gamma^\varphi \models t_1 \approx_{\log} t_2 : (\text{Ref B})^q \varepsilon_1$ ,  $\Gamma^\varphi \models t_3 \approx_{\log} t_4 : B^\emptyset \varepsilon_2$ , then  $\Gamma^\varphi \models t_1 := t_3 \approx_{\log} t_2 := t_4 : B^\emptyset \varepsilon_1 \triangleright \varepsilon_2 \triangleright q$ .*

PROOF. Let  $(W, \hat{H}) \in G[[\Gamma^\varphi]]$  and  $(\sigma_1, \sigma_2) : W$ . By the first assumption, we know that there exists  $\sigma'_1, \sigma'_2, W', \ell_1$  and  $\ell_2$  such that

- $W \sqsubseteq W'$
- $t_1, \hat{H}_1, \sigma_1 \Downarrow \ell_1, \sigma'_1$
- $t_2, \hat{H}_2, \sigma_2 \Downarrow \ell_2, \sigma'_2$
- $(\sigma'_1, \sigma'_2) : W'$
- $(W', \ell_1, \ell_2) \in \mathcal{V}[[\text{Ref B}]]^{\hat{H}}$
- $\ell_1 \rightsquigarrow^{\sigma_1} \text{locs}_{\hat{H}_1}(\varphi \cap q)$
- $\ell_2 \rightsquigarrow^{\sigma_2} \text{locs}_{\hat{H}_2}(\varphi \cap q)$
- $\sigma_1 \hookrightarrow^{\text{locs}_{\hat{H}_1}(\varepsilon_1)} \sigma'_1$
- $\sigma_2 \hookrightarrow^{\text{locs}_{\hat{H}_2}(\varepsilon_1)} \sigma'_2$

By the second assumption, we know that there exists  $\sigma''_1, \sigma''_2, W'', v_1$  and  $v_2$ , such that

- $W' \sqsubseteq W''$
- $t_3, \hat{H}_1, \sigma'_1 \Downarrow v_1, \sigma''_1$
- $t_4, \hat{H}_2, \sigma'_2 \Downarrow v_2, \sigma''_2$
- $(\sigma''_1, \sigma''_2) : W''$
- $(W'', v_1, v_2) \in \mathcal{V}[[\text{B}]]^{\hat{H}}$
- $v_1 \rightsquigarrow^{\sigma'_1} \text{locs}_{\hat{H}_1}(\varphi \cap \emptyset)$
- $v_2 \rightsquigarrow^{\sigma'_2} \text{locs}_{\hat{H}_2}(\varphi \cap \emptyset)$
- $\sigma'_1 \hookrightarrow^{\text{locs}_{\hat{H}_1}(\varepsilon_2)} \sigma''_1$
- $\sigma'_2 \hookrightarrow^{\text{locs}_{\hat{H}_2}(\varepsilon_2)} \sigma''_2$

Then the proof can be done by the reduction semantics, Lemma 4.6, value interpretation, Lemma 4.15, Lemma 4.18 and Lemma 4.21.  $\square$

LEMMA 4.33 (COMPATIBILITY: APPLICATIONS ( $\beta$ )). *If  $\Gamma^\varphi \models t_1 \approx_{\log} t_2 : (x : T^{p^* \cap q^*} \rightarrow_{\varepsilon_3} U^r)^q \varepsilon_2$ , and  $\Gamma^\varphi \models t_3 \approx_{\log} t_4 : T^p \varepsilon_1$ , and  $x \notin \text{fv}(U)$ , and  $r \subseteq \varphi, x$ , and  $\varepsilon_3 \subseteq \varphi, x$ , and  $\theta = [p/x]$ , then  $\Gamma^\varphi \models t_1 t_3 \approx_{\log} t_2 t_4 : (U^r \varepsilon_1 \triangleright \varepsilon_2 \triangleright \varepsilon_3)\theta$ .*

PROOF. The proof is done by the definition of term interpretation, Lemma 4.25 and Lemma 4.21.  $\square$

LEMMA 4.34 (COMPATIBILITY: SEQ). *If  $\Gamma^{\varphi_1} \models t_1 \approx_{\log} t_2 : B^q \varepsilon_1$ , and  $\Gamma^{\varphi_2} \models t_3 \approx_{\log} t_4 : B^p \varepsilon_2$ , and  $\varphi_1 \subseteq \varphi$  and  $\varphi_2 \subseteq \varphi$ , then  $\Gamma^\varphi \models t_1; t_3 \approx_{\log} t_2; t_4 : B^\emptyset \varepsilon_1 \triangleright \varepsilon_2 \triangleright q$*

PROOF. Let  $(W, \hat{H}) \in G[[\Gamma^\varphi]]$  and  $(\sigma_1, \sigma_2) : W$ . By the first assumption, we know that there exists  $\sigma'_1, \sigma'_2, W', b_1$  and  $b_2$  such that

- $W \sqsubseteq W'$

- $t_1, \hat{H}_1, \sigma_1 \Downarrow b_1, \sigma'_1$
- $t_2, \hat{H}_2, \sigma_2 \Downarrow b_2, \sigma'_2$
- $(\sigma'_1, \sigma'_2) : W'$
- $(W', b_1, b_2) \in \mathcal{V}[[\mathbf{B}]]^{\hat{H}}$
- $b_1 \rightsquigarrow^{\sigma_1} \text{locs}_{\hat{H}_1}(\varphi \cap q)$
- $b_2 \rightsquigarrow^{\sigma_2} \text{locs}_{\hat{H}_2}(\varphi \cap q)$
- $\sigma_1 \hookrightarrow^{\text{locs}_{\hat{H}_1}(\varepsilon_1)} \sigma'_1$
- $\sigma_2 \hookrightarrow^{\text{locs}_{\hat{H}_2}(\varepsilon_1)} \sigma'_2$

By the second assumption, we know that there exists  $\sigma''_1, \sigma''_2, W'', b_3$  and  $b_4$ , such that

- $W' \sqsubseteq W''$
- $t_3, \hat{H}_1, \sigma'_1 \Downarrow b_3, \sigma''_1$
- $t_3, \hat{H}_2, \sigma'_2 \Downarrow b_4, \sigma''_2$
- $(\sigma''_1, \sigma''_2) : W''$
- $(W'', b_3, b_4) \in \mathcal{V}[[\mathbf{B}]]^{\hat{H}}$
- $b_3 \rightsquigarrow^{\sigma'_1} \text{locs}_{\hat{H}_1}(\varphi \cap \emptyset)$
- $b_4 \rightsquigarrow^{\sigma'_2} \text{locs}_{\hat{H}_2}(\varphi \cap \emptyset)$
- $\sigma'_1 \hookrightarrow^{\text{locs}_{\hat{H}_1}(\varepsilon_2)} \sigma''_1$
- $\sigma'_2 \hookrightarrow^{\text{locs}_{\hat{H}_2}(\varepsilon_2)} \sigma''_2$

Then the proof can be done by the reduction semantics, value interpretation, Lemma 4.15, and Lemma 4.21.  $\square$

LEMMA 4.35 (COMPATIBILITY: SUBTYPING). *If  $\Gamma^\varphi \models t_1 \approx_{\log} t_2 : S^P \varepsilon_1$  and  $\Gamma \vdash S^P \varepsilon_1 <: T^Q \varepsilon_2$  and  $q, \varepsilon_2 \subseteq \varphi$ , then  $\Gamma^\varphi \models t_1 \approx_{\log} t_2 : T^Q \varepsilon_2$ .*

PROOF. By induction on the subtyping derivation.  $\square$

#### 4.7 The Fundamental Theorem and Soundness

THEOREM 4.36 (FUNDAMENTAL PROPERTY). *If  $\Gamma^\varphi \vdash t : T^Q \varepsilon$ , then  $\Gamma^\varphi \models t \approx_{\log} t : T^Q \varepsilon$ .*

PROOF. By induction on the derivation of  $\Gamma^\varphi \vdash t : T^Q \varepsilon$ . Each case follows from the corresponding compatibility lemma.  $\square$

LEMMA 4.37 (CONGRUENCY OF BINARY LOGICAL RELATIONS). *The binary logical relation is closed under well-typed program contexts, i.e., if  $\Gamma^\varphi \models t_1 \approx_{\log} t_2 : T^P \varepsilon$ , and  $C : (\Gamma^\varphi; T^P \varepsilon) \Rightarrow (\Gamma' \varphi'; T' P' \varepsilon')$ , then  $\Gamma' \varphi' \models C[t_1] \approx_{\log} C[t_2] : T' P' \varepsilon'$ .*

PROOF. By induction on the derivation of context  $C$ . Each case follows from the corresponding compatibility lemma and may use the fundamental theorem (Theorem 4.36) if necessary.  $\square$

LEMMA 4.38 (ADEQUACY OF THE BINARY LOGICAL RELATIONS). *The binary logical relation preserves termination, i.e., if  $\emptyset \models t_1 \approx_{\log} t_2 : T^\emptyset \emptyset$ , then  $\exists \sigma, \sigma', v, t_1, \emptyset, \sigma \Downarrow v_1, \sigma'_1 \wedge t_2, \emptyset, \sigma_2 \Downarrow v, \sigma'_2$ .*

PROOF. We know  $(\emptyset, \emptyset) \in G[[\emptyset]]$  by the interpretation of typing context. Then we can prove the result by the binary term interpretation (Fig. 8).  $\square$

THEOREM 4.39 (SOUNDNESS OF BINARY LOGICAL RELATIONS). *The binary logical relation is sound w.r.t. contextually equivalence, i.e., if  $\Gamma^\varphi \vdash t_1 : T^P \varepsilon$  and  $\Gamma^\varphi \vdash t_2 : T^P \varepsilon$ , then  $\Gamma^\varphi \models t_1 \approx_{\log} t_2 : T^P \varepsilon$  implies  $\Gamma^\varphi \models t_1 \approx_{ctx} t_2 : T^P \varepsilon$ .*

$$\text{(RE-ORDER)} \frac{\Gamma^{\varphi_1} \models t_1 : B^q \varepsilon_1 \quad \Gamma^{\varphi_2} \models t_2 : B^p \varepsilon_2 \quad \varphi_1 \subseteq \varphi \quad \varphi_2 \subseteq \varphi \quad \varphi_1 * \cap \varphi_2 * = \emptyset}{\Gamma^\varphi \models t_1; t_2 \approx_{\log} t_2; t_1 : B^\emptyset \varepsilon_1 \triangleright \varepsilon_2 \triangleright q}$$

Fig. 9. The re-ordering rule for the  $\lambda_\varepsilon^*$ -calculus.

PROOF. By the refined definition of contextual equivalence, to prove the result, we are given a well-typed context  $C : (\Gamma^\varphi; T^p \varepsilon) \Rightarrow (\emptyset; B^\emptyset \emptyset)$ , and we need to show  $\exists \sigma, \sigma', v. \emptyset \mid C[t_1] \longrightarrow_v^* \sigma \mid v \wedge \emptyset \mid C[t_2] \longrightarrow_v^* \sigma' \mid v$ . By the assumption, and the congruency lemma (Lemma 4.37), we have  $\emptyset \models C[t_1] \approx_{\log} C[t_2] : B^\emptyset \emptyset$ , which leads to  $\exists \sigma, \sigma', v. \emptyset \mid C[t_1] \longrightarrow_v^* \sigma \mid v \wedge \emptyset \mid C[t_2] \longrightarrow_v^* \sigma' \mid v$  by the adequacy lemma (Lemma 4.38).  $\square$

#### 4.8 Re-ordering

Fig. 9 shows the re-ordering rule for  $\lambda_\varepsilon^*$ -calculus. It permits re-ordering of two terms if they observe disjoint set of store locations specified by reachability qualifiers. This section shows the proof of the re-ordering rule by using our logical relations.

To streamline the presentation, we introduce the following notations. Let  $W = (\sigma_1, \sigma_2, f)$  be a world, we write  $W_f$  to mean the partial bijection defined in  $M$ , i.e.,  $f$ .

We identify important store invariants entailed by our logical relations.

LEMMA 4.40 (STORE INVARIANCE 1). *If  $\Gamma^\varphi \vdash t : T^q \varepsilon$ , and  $(W, \hat{H}) \in G[[\Gamma^\varphi]]$ , and  $(\sigma_1, \sigma_2) : W$ , and  $\forall \ell_1, \ell_2. W(\ell_1, \ell_2). \ell_1 \notin L$  and  $\sigma_1 \hookrightarrow^L \sigma'_1$ , and  $\text{dom}(\sigma'_1) \subseteq \text{dom}(\sigma_1)$ , then we can construct a world  $W'$ , such that  $W' = (\text{dom}(\sigma'_1), \text{dom}(\sigma_2), W_f)$  and  $(W', t, t) \in \mathcal{E}[[T^q \varepsilon]]_{\hat{H}}^{\hat{H}}$ .*

PROOF. The first proof obligation can be discharged by the definition of relational store and Lemma 4.4. The second proof obligation can be discharged by Theorem 4.36 and the definition of logical relations on terms.  $\square$

LEMMA 4.41 (STORE INVARIANCE 2). *If  $\Gamma^\varphi \vdash t : T^q \varepsilon$ , and  $(W, \hat{H}) \in G[[\Gamma^\varphi]]$ , and  $(\sigma_1, \sigma_2) : W$ , and  $\sigma_1 \hookrightarrow^L \sigma'_1$ , and  $\text{dom}(\sigma'_1) \subseteq \text{dom}(\sigma_1)$ , and  $\text{locs}_{\hat{H}_1}(\varphi) \cap L = \emptyset$  then there exists  $W'$ , such that  $W \sqsubseteq (\text{locs}_{\hat{H}_1}(\varphi), \text{locs}_{\hat{H}_2}(\varphi)) W'$  and  $(W', t, t) \in \mathcal{E}[[T^q \varepsilon]]_{\hat{H}}^{\hat{H}}$ .*

PROOF. The proof uses Lemma 4.4 and Lemma 4.40, Theorem 4.36 and the definition of logical relations on terms.  $\square$

There are two other store invariances regarding the second store, which are similar to the above two, thus are omitted.

LEMMA 4.42 (RE-ORDERING). *If  $\Gamma^{\varphi_1} \models t_1 : B^q \varepsilon_1$ , and  $\Gamma^{\varphi_2} \models t_2 : B^p \varepsilon_2$ , and  $\varphi_1 \subseteq \varphi$ , and  $\varphi_2 \subseteq \varphi$ , and  $\varphi_1 * \cap \varphi_2 * = \emptyset$  then  $\Gamma^\varphi \models t_1; t_2 \approx_{\log} t_2; t_1 : B^\emptyset \varepsilon_1 \triangleright \varepsilon_2 \triangleright q$ .*

PROOF. The proof uses Theorem 4.36, Lemma 4.40, Lemma 4.41 and the definition of logical relations on terms.  $\square$

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