Characterizing Continuous-time Random Walks on Dynamic Networks

Bruno Ribeiro, Don Towsley
Department of Computer Science
University of Massachusetts Amherst
{ribeiro,towsley}@cs.umass.edu

Daniel Figueiredo, Edmundo de Souza e Silva
COPPE/PESC
Federal University of Rio de Janeiro
{daniel,edmundo}@land.ufrj.br

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1. INTRODUCTION

Most of the networks that pervade our lives are dynamic in nature in the sense that their structural configuration (i.e., topology) is constantly changing over time. Some of these networks are both very large and very dynamic, changing at timescales that are relatively small. Thus, analyzing and measuring such networks represents a challenge, as traditional techniques designed for static networks are rendered unsuitable.

A commonly used technique for measuring and characterizing *static networks* are random walks (RW), which have been extensively studied and applied in the literature. For example, the simple closed form solution of the distribution of the number of visits that stationary RW makes to a node (which is proportional to the node degree) is the basis for many principled algorithms [3, 1]. In contrast, little is known even about the stationary distribution of random walks on dynamic networks.

In this paper we study the steady state behavior of continuous-time random walks (CTRW) on Markov dynamic networks. We consider two types of CTRWs: one that walks at a constant rate (CTRW-C) and another that walks with a rate proportional to the vertex degree (CTRW-D). We derive closed-form analytical expressions for the steady state (SS) distribution of these walkers. For CTRW-C we obtain the approximate SS distribution for either a very fast or very slow walker. We show that the behavior of CTRW-C and CTRW-D is strikingly different. Surprisingly, the steady state distribution of the fixed rate walker depends on the walker rate, which is not the case for the degree proportional walker. Such findings have direct implication on the design of algorithms to measure dynamic networks.

2. DYNAMIC GRAPHS & RANDOM WALKS

The dynamic network models we consider assume that nodes are always present but that edges can come and go over time, as follows. A Markov dynamic graph is a set of graphs having the same vertex set and a Markov process over them. In particular, let $S = \{G_1, ..., G_m\}$ be a set of undirected graphs all having the same vertex set V, thus, $G_k = (V, E_k)$, where E_k denotes the set of edges of graph G_k, for $k = 1, \ldots, m$. Let $n = |V|$ denote the number of vertices in a graph. Moreover, let A_k denote the adjacency matrix of graph G_k. Thus, for all $i, j \in V$, $A_k(i, j) = 1$ if $(i, j) \in E_k$ and 0 otherwise. Finally, let $\deg(i, k)$ denote the degree of vertex $i \in V$ in graph G_k.

We introduce the dynamic graph process $\{G(t)\}$ as a continuous-time stationary Markov process with state space S. Let $\lambda_{kl} \geq 0$ denote the transition rate from state (graph) G_k to G_l. Note that the graph dynamics are fully determined by the transition rates.

2.1 Random walks on dynamic graphs

Consider a continuous time random walk (CTRW) on a general Markov dynamic graph. Intuitively, the walker can only traverse edges that are incident to the vertex at which the walker resides, choosing uniformly at random among them. Let $\{W(t)\}$ be the continuous time process representing the vertex where the walker resides, thus $W(t) = i \in V$, for any t. The time between two consecutive steps of the random walk is exponentially distributed with rate γ. We refer to γ as the *walking rate* and to this random walk as CTRW-C (for constant walking rate).

Without loss of generality, assume that when the random walk takes a step at time t, the Markov dynamic graph process is in state G_k, that is $G(t) = G_k$ and that $W(t) = i$. Let $N_{i,k}$ denote the set of neighbors of vertex i in the graph G_k. Thus, the probability the walker steps to vertex $j \in N_{i,k}$ is given by $1/|N_{i,k}|$. Finally, if $N_{i,k}$ is empty, the walker stays at vertex i during that step.

2.2 Degree-dependent random walks

We now introduce a random walk where its walking rate is not constant, but instead depends on the degree of the vertex where the walker is located. Intuitively, our walker will move faster when it finds itself at vertices with large degrees and slower when located at vertices with small degrees. More precisely, the inter-step time of the random walk is exponentially distributed with rate $\deg(i, k)\gamma$, where $\deg(i, k)$ is the degree of vertex $i \in V$ in graph $G_k \in S$ and $\gamma > 0$. We refer to this random walk as CTRW-D (for degree-proportional walking rates).

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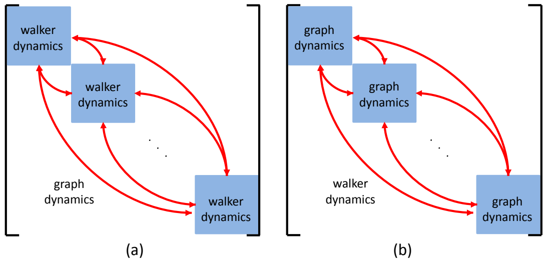

Figure 1: Two different matrix arrangements for CTRW on Markov dynamic graphs.

2.3 Joint graph and walker dynamics

The dynamic graph process and the random walk process can be represented as a continuous-time Markov process $\{R(t)\}$ where $R(t) \in V \times S$. Let $s_{i,k}$ represent a state of this process where i represents a vertex of V ($i \in V$) and k represents a graph in S ($k \in [1, \ldots, m]$). Let $U = \{s_{i,k} : \forall i \in V, k = 1, \ldots, m\}$ denote the set of all vertex snapshots ($s_{i,k}$ represents vertex i at graph G_k). The infinitesimal generator matrix of $\{R(t)\}$, denoted by Q, is determined by combining the graph dynamics with the random walk.

Two specific arrangements of the state space into matrix Q are of particular interest, as they have intuitive interpretation and block representations. Moreover, such representations are fundamental in establishing the proof of the next two theorems. Let $P^w = \{P_1^w, \ldots, P_m^w\}$ be a partition of the state space U, where each subset P_k^w consists of all random walk states corresponding to the graph G_k. In particular, $P_k^w = \{s_{i,k} | i \in V\}$. Thus, each partition corresponds to a square matrix block of size $n = |V|$. Moreover, transitions within a partition correspond to random walk dynamics and transitions among blocks correspond to graph dynamics, as illustrated in Figure 1(a).

We consider a second partition where states $s_{i,k} \in U$ are first grouped by i. We denote this second partition as $P^g = \{P_1^g, \ldots, P_n^g\}$, where each subset P_i^g consists of all graph states corresponding to vertex $i \in V$. In particular, $P_i^g = \{s_{i,k} | k \in [1, \ldots, m]\}$. Thus, each partition corresponds to a square matrix block of size m . Thus, within a block we have the graph dynamics and transitions among blocks represent the walker dynamics, as illustrated in Figure 1(b).

We note that matrix Q representing each random walker (CTRW-C and CTRW-D) can be written in block-form using closed-form equations in each of the two partitions described above. Due to space constraints, we omit such equations (see [2]).

3. STEADY STATE DISTRIBUTIONS

We investigate the steady state distribution of CTRW on Markov dynamic graphs. In particular, we are interested in the fraction of time that the random walk spends at each vertex $i \in V$. Clearly, this metric can be obtained from the steady state probability distribution of $\{R(t)\}$. Let $\pi(s_{i,k})$ denote the fraction of time in state $s_{i,k}$ where $i \in V$ and $k \in [1, \ldots, m]$. In particular, assuming $\{R(t)\}$ is ergodic, we have $\pi(s_{i,k}) = \lim_{t\to\infty} P[R(t) = s_{i,k}]$. We denote vector $\pi = (\pi_1, \ldots, \pi_n)$ in which the i-th component is the fraction of time the random walk spends in vertice i of set V, where $n = |V|$. Therefore, $\pi_i = \sum_k \pi(s_{i,k})$.

3.1 CTRW-C steady state distribution

We investigate the SS distribution of either a very fast or a very slow CTRW-C. Consider a very fast walker, in particular, much faster than the timescale at which the graphs change. Thus, every time the graph changes, the random walk quickly steps through all vertices in this graph. Intuitively, the random walk converges to the SS distribution of the static graph before the dynamic graph process changes to a new graph. This intuition leads to the following theorem (see [2] for proof).

Let Π_k denote the steady state fraction of time spent in graph G_k. Let $\pi(G_k) = (\pi_1(G_k), \ldots, \pi_n(G_k))$ denote the steady state distribution of a random walk on the static graph G_k.

THEOREM 3.1. *For a sufficiently large γ,*

$$\pi = \sum_{k=1}^{m} \Pi_k \pi(G_k)$$

Now consider a very slow walker, in particular, much slower than the timescale at which the graphs change. Every time the walker steps onto a vertex, the incident edges of this vertex change many times before the walkers steps off. Intuitively, when the walker steps off it will observe the incident edges in "steady state". Thus, the probability the walker steps from vertex i to vertex j is proportional to the steady state probability that edge (i, j) is present. The SS distribution of the random walk is the solution of this new random walk on a weighted static graph. This intuition leads to the following theorem (see [2] for proof).

THEOREM 3.2. *For a sufficiently small γ, the steady state distribution π is equivalent to the steady state of a continuous time Markov chain with infinitesimal generator matrix $R = [r_{i,j}]$, $\forall i, j \in V$, where*

$$r_{i,j} = \begin{cases} \gamma \sum_{k=1}^{m} \mathbf{1}((i,j) \in E_k)\, \Pi_k \, 1/\deg(i,k) & \textit{if } i \neq j \\ 0 & \textit{otherwise} \end{cases}$$

and $\mathbf{1}(\cdot)$ is the indicator function and $\deg(i, k)$ is the degree of vertex i in graph G_k.

3.2 CTRW-D steady state distribution

In a CTRW-D the walker spends a fraction of time that is constant across the vertices of the graph. This observation is quite remarkable, since it means we can traverse the graph at arbitrary speeds, independently of graph dynamics, while being able to characterize the steady state distribution that the walker will observe. This observation leads to the following theorem (see [2] for proof).

Let $s_{i,k} \in U$ be a vertex snapshot.

THEOREM 3.3. *In CTRW-D, $\pi(s_{i,k}) = \Pi_k/n$, $i \in V$, $k = 1, \ldots, m$,*

As a consequence, we have that $\pi_i = 1/n$, for all $i \in V$.

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