Data Mining for Smart Cities

Need congestion control
Supply and Demand (A Dating Website [China])

Males — Females

Need congestion control
Internet

Need congestion control
Internet = Largest Optimization Problem in History

- Internet Congestion Control Optimization
  - Largest Optimization Problem in History
  - Daily nearly 9 billion computers join the computation
Markets

- Markets defined by
  - Google
  - Amazon
  - Yahoo!
  - Ebay

- Massive computational power available for running these markets

- Important to find good models and algorithms for these markets
Internet Resource Allocation

- network: links \( \{l\} \), capacities \( \{c_l\} \)
- sources \( S \): \((L(s), U_s(x_s))\), \( s \in S \)
  - \( L(s) \) - links used by source \( s \)
  - \( U_s(x_s) \) – utility, strictly concave function of source rate \( x_s \)

User 0
\( U_0(x_0) \)

User 1
\( U_1(x_1) \)

User 2
\( U_2(x_2) \)

\( U_s(x_s) \)

\( x_s \)

Ack: Don Towsley
Frank Kelly’s System Problem

\[
\max_{x_0, x_1, x_2} \sum_{i=1}^{3} U_i(x_i)
\]

subject to

\[
\begin{align*}
  x_0 + x_1 & \leq c_A \\
  x_0 + x_2 & \leq c_B \\
  x_i & \geq 0
\end{align*}
\]

Ack: Don Towsley
Online Dating Resource Allocation

Max no. messages from user 1 (per day)

$x_1$  

$x_2$  

$x_3$  

$x_4$

Max messages to user 1 (per day)

$x_2 < c_1$

$x_1 + x_3 + \alpha x_4 < c_2$

$(1-\alpha) x_4 < c_3$

$0 < c_4$
Optimization Problem

\[
\max \sum_{s} U_s(x_s) \quad \text{“system” problem}
\]

s.t. \[ \sum_{s \in S(l)} x_s \leq c_l, \quad \forall l \in L \]

- maximize system utility (note: all sources “equal”)
- constraint: bandwidth used less than capacity
- centralized solution to optimization impractical
  - must know all utility functions
  - impractical for large number of sources
  - we’ll see: congestion control as distributed asynchronous algorithms to solve this problem
Issues

- will users truthfully reveal their utility functions?
- if not, can we design a pricing scheme (mechanism) to induce truth-telling?
- is there a distributed algorithm to compute the prices?
- what are good choices for utilities?
- fairness among users?
Max-min Fairness

rates \{x_r\} are max-min fair if for any other feasible rates \{y_r\}, if \(y_s > x_s\), then \(\exists\ p\), such that \(x_p \leq x_s\) and \(y_p < x_p\).

What is corresponding utility function?

\[
U_r(x_r) = \lim_{\alpha \to \infty} \frac{x_r^{1-\alpha}}{1 - \alpha}
\]
Proportional fairness

- rates \( \{x_r\} \) are proportionally fair if for any feasible \( \{y_r\} \),

\[
\sum_{r \in S} \frac{y_r - x_r}{x_r} \leq 0
\]

- corresponds to \( U_r (x_r) = \log x_r \)
- weighted proportional fairness if \( U_r (x_r) = w_r \log x_r \)

\[
\sum_{r \in S} w_r \frac{y_r - x_r}{x_r} \leq 0
\]
Primal-Dual Problem

Primal:

\[
\text{minimize } \quad z = \sum_{i=1}^{n} c_i x_i \\
\text{s.t. } \quad \sum_{i=1}^{n} a_{ij} x_i \geq b_j, \quad 1 \leq j \leq m \\
\text{and } \quad x_i \geq 0, \quad 1 \leq i \leq n
\]

Dual:

\[
\text{maximize } \quad z^* = \sum_{j=1}^{m} b_j y_j \\
\text{s.t. } \quad \sum_{j=1}^{m} a_{ij} y_j \leq c_i, \quad 1 \leq i \leq n \\
\text{and } \quad y_j \geq 0, \quad 1 \leq j \leq m
\]
The Duality Theorem (Gale, Kuhn, Tucker 1951)

- The primal has an optimal solution iff the dual has an optimal solution

- The objective functions of optimal solutions are equivalent

\[ z = \sum_{i=1}^{n} c_i x_i^\dagger = \sum_{j=1}^{m} b_j y_j^\dagger = z^* \]
Distributed algorithm

Optimization problem decouples into (Kelly):

- greedy optimization problem for every session
  \[ \max U_r(x_r) - q_r x_r \]
- price \( q_r \) given by network as function of rates, \( q_r = \sum p_l \) where \( p_l \) is (shadow) price for link \( l \)
- provides solution to system problem

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Computing Source Rates

\[ L(x, p) = \sum_r U_r(x_r) - \sum_l p_l(y_l - c_l) \]

\[ y_l = \sum_{r: l \in r} x_r \]

\[ \frac{\partial L}{\partial x_r} = 0 \]

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Remove constraints: Primal approach

- Consider following problem

\[ V(x) = \sum_r U_r(x_r) - \sum_{l \in L} g_l \left( \sum_s x_s \right) \]

- \( g_l(y) \) – penalty function
  - \( g_l(y) \) non decreasing, convex and
    \[ g_l(y) \to \infty \text{ as } y \to \infty \]
  - \( f_l(y) = \frac{dg_l(y)}{dy} \)

Ack: Don Towsley
\[
\text{max } V(x)
\]
\[
\frac{\partial V(x)}{\partial x_r} = 0, \quad r \in S
\]
\[
U'_r(x_r) - \sum_{l: l \in r} f_l(y_l) = 0, \quad r \in S
\]
\[
y_l = \sum_{s: l \in s} x_s, \quad l \in L
\]
\( p_l(t) \) price of link \( l \) at time \( t \)

\[
p_l(t) = f_l(y_l(t))
\]

\[
U'_r(x_r) - \sum_{l \in r} p_l = 0, \quad r \in S
\]

- optimization problem decouples into:
  - greedy optimization problem for every session
    \[
    \max U_s(x_s) - q_s x_s
    \]
  - price \( q_s \) given by network as function of rates
Controller design

- algorithm to maximize $h(x)$
- progressively change $x$ s.t. $h(x(t+\delta)) > h(x(t))$ by choosing direction of maximum ascent

- 1-D

  $x(t+\delta) = x(t) + k(t) \delta \frac{dh}{dx}$

- as $\delta \to 0$, described by

  $\dot{x} = k(t) \frac{dh}{dx}$
Source Algorithm

- source needs only its path price:
  \[ \dot{x}_r = k_r(x_r)(U_r'(x_r) - q_r) \]

- \( k_r() \) nonnegative nondecreasing function
- above algorithm converges to unique solution for any initial condition
- can show convergence using Lyapunov functions
- example: \( q_r \) – loss/marking probability

Ack: Don Towsley
Pricing

- can network choose pricing scheme to achieve fair resource allocation?
- suppose network charges price \( q_r \) ($/message) where \( q_r = \sum p_i \)
- user's strategy: spend \( w_r \) ($/sec.) to maximize

\[
U_r \left( \frac{w_r}{q_r} \right) - w_r
\]
Optimal User Strategy

\[
\frac{1}{q_r} U'_r \left( \frac{w_r}{q_r} \right) = 1
\]

- equivalently,

\[
w_r = x_r U'_r(x_r)
\]
Distributed Computation

- with optimal choice of $w_r$, controller becomes

$$\dot{x}_r = x_r \left( U'_r(x_r) - q_r \right)$$

- we have already seen that this solves

$$\max_{\{x_r\}} \sum_r U_r(x_r)$$
Price Takers vs. Strategic Users

\[
\max_{w_r} U_r \left( \frac{w_r}{q_r} \right) - w_r
\]

- Kelly Mechanism: users are price takers, i.e., user does not know the impact of its action on the price

- strategic users:

\[
q_r = g_r(w_r)
\]
Efficiency and Competition

- price takers: selfish users can maximize social welfare

- strategic users: competition leads to loss of efficiency, i.e., social welfare is not maximized
Proportionally-Fair Controller

If utility function is

\[ U_r(x_r) = w_r \log x_r \]

then a controller that implements it is given by

\[ \dot{x} = \kappa_r(w_r - x_rq_r) \]
Price functions

Mark packet when queue length above threshold \( B \)
- example - M/M/1 queue

\[
p_l = f_l(y_l) = \begin{cases} 
(y_l / c_l)^B, & y_l < c_l \\
1, & y_l \geq c_l 
\end{cases}
\]

Or mark them earlier
Price functions

Drop packets when queue is full

- example - M/M/1 queue

Probability of loss \[ \frac{1 - \rho}{1 - \rho^{B+1}} \rho^B \]

where \( \rho = \frac{y_l}{c_l} \)

for large \( B \),

\[ p_l = f_l(y_l) = \frac{(y_l - c_l)^+}{y_l} \]

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