Recursive Functions

#1.py

So far we have used functions that called other functions, but never did call themselves.

A Recursive function is a function that calls itself. But if it calls itself, then when does it stop calling itself and start to return (i.e., it has to return through the whole sequence of calls it makes to itself.

To step calling itself forever, a recursive function needs a statement called a BASE CASE. The BASE CASE tells it whether to stop (and start the chain of returns) or continue with another call to itself.

A function that uses a loop to do some computation (i.e., it does not call itself) is an ITERATIVE function (i.e., it is non-recursive).

Let's take a look at an iterative function --- the kind we already know.

"""Example: We will write an iterative function to print all the integers starting at integer low and stopping at integer high."

```python
def iRange(low, high):
    # iterative function to print range of integers
    
    while (low <= high):  # the loop will stop as soon as j increases past high
        print(low)
        low = low + 1

    """ Note these two points:

1. local variable "low" keeps getting incremented; local variable "high" does not change at all
```
2. loop continues as long as low <= high

Now let's try to do this by making the function call itself. Remember that it must have a BASE
# CASE so that it can stop, and start the sequence of returns.

def rRange(low, high):
    # recursive function to print range of integers
    '''print all integers, from low to high.'''
    if (low <= high):
        print(low)
        rRange(low+1, high)  # notice how we enter the recursive call with a "smaller" problem

# Note: Both iRange and rRange go through the exact same sequence of calls and produce
# the same values s output

def main():
    iRange(11, 22);
    print("\n\n")
    rRange(11, 22)

main()

#
#2.py

# In the previous example we only printed things recursively. Now let's try to recursively
# create values and return them.

# We used this example earlier, when we first defined functions.

#Problem: write a function sum(low, high) that computes and returns the sum of integers
# low + (low + 1) + (low + 2) + ......... + high; it returns 0 if called with low > high
# Recursion step: low + sum(low + 1, high)

# Observe how the recursion step replaces the original problem sum(low, high) with the subproblem
# low + sum(low + 1, high). This is the key idea in recursion, but do not forget the base step.

def sum(low, high):
    ''' Returns the sum of all integers from low to high.'''

    if (low > high):  # notice how this base case test gives us a way to stop recursing
        return 0

    else:
        return (low + sum(low + 1, high))  # farming off a subproblem to next call

...

Using recursion involves the "divide-and-conquer" strategy.

You are given a problem in terms of n

Try to reduce that problem to a combination of a partial solution and the same problem in terms of a smaller n (typically n-1, or perhaps n//2).

In the above example we added "low" to the result of a smaller problem with one less number in the sum.

Instead of going from low to high, the smaller subproblem went from (low + 1) to high.

...

# __________________ TRACING a recursive function _______________________
#3.py

# Let's trace (i.e., see how the calls are sequenced and how the returns are sequenced) a recursive function call

# To help see the pattern we must do some pretty-printing

# Every time we make a recursive call we will print the arguments to the call. But also, every time
# we go deeper into the recursion, we will move the printing to the right, so the pattern is clear.

# When the base case is hit and the recursive function starts to return, we will do the same
# printing but in reverse, but instead of printing the arguments, we will print the result that
# each particular call returns ..... until we return to the first call which gives the final result.

def sum(low, high, margin):
    #add all integers between low and high
    ''' Returns a sum of the numbers from low to high. 
    Print the recursive call parameters and the return results in such a way that the 
    recursive pattern is clear.''

    blanks = " " * margin  # margin tells us how many spaces to the right or left we must print

    print(blanks, low, high)  # print margin spaces, then print arguments to first call

    if (low > high):  # we have reached the bottom of the recursive calls if this is true
        print(blanks, 0)  # sum cannot compute anything if low > high. So print 0.
        return 0  # since nothing to compute, there is nothing to return. Return 0.

    else:

        result = low + sum(low + 1, high, margin + 4)  # please study this recursion

        print(blanks, result)  # low + new subproblem

        return (result)

    # Hint: tracing in this way can help you debug recursion because the pattern will not
    # confuse you. Make sure you understand that the prints going to the right make up the
    # sequence of recursive calls. The prints going to the left make up the returns up this chain
def main():
    sum(1, 7, 0)

# _____ Mapping a math recursion to a recursive Python function_____________

#4.py

# The famous Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, .......
# F(1) = 1, F(2) = 1, F(3) = 2, F(4) = 3, F(5) = 5, F(6) = 8, F(7) = 13, ....

# The first two numbers are 1, 1
# from then on, the next number is the sum of the previous two.

# Let's write an iterative version of this function first

def iFib(n):
    # returns the nth Fib number in the sequence

    if (n == 1):
        return 1
    elif (n == 2):
        return 1
    else:
        previous = 1 # save the previous and the
        current = 1 # current fib numbers

        index = 3
        while (index <= n):
            next = previous + current # now start to add the
                                      # previous two to get the current
                                      
            previous = current
            current = next # update the current
                            # and previous

            index = index + 1

        return next

''' Now let's look at the math (recursive) definition of Fibonacci
F(n) = 1 , when n = 1 or n = 2
F(n) = F(n-1) + F(n-2)     when n > 2

and observe how it easily maps into recursive Python code.

def rFib(n):
    # recursive definition of Fibonacci, using the math recursion
    if ( n < 3):
        return 1
    else:
        return (rFib(n-1) + rFib(n-2))

def main():
    print("Iteration \n")
    for j in range (1,12):
        print(iFib(j))
    print ("\n\n")
    print("Recursion \n")
    for j in range (1,12):
        print(rFib(j))

main()

#___________________ recursive factorials _________________________________

#5.py

# We know that n! = n(n-1)(n-2)(n-3)........ 3.2.1. Let's compute it recursively.

def rFact(n):
    # returns n!
    if (n ==1):
        return 1
    else:
        return(n*rFact(n-1))
# Home-work: write the iterative version to check results

# n! gives you the number of ways of permuting n objects.
# What if you wanted to (do combinations) select objects. How many ways are there of
# selecting k objects from n objects? This is nCk or (n,k) ...... and .......
# the answer is :  n! / (k! (n-k)!)  
# so you see that to compute this, you need 3 factorials.

def Comb(n,k):   # number of ways of choosing k objects from n objects

    denom = rFact(k) * rFact(n-k)
    num = rFact(n)
    return ((num//denom))

#________________________________________________________________
# Now supposing you had a string of n arbitrary objects .... let's say 'abcdefg'. How
# can we list out all arrangements of k objects at a time?

def combinations(string, k):   # want list of all k things from string

    outlist = [ ]      # start with empty list and
    build it recursively

    for j in range (len(string)):

        if (k ==1):
            outlist.append(string[j])   # if there's only one
            object, put it in the list

        # Next slice through the smaller sublist recursively, and
        # new element to each sub arrangement
        for element in combinations(string[j+1: ], k-1):
            outlist.append(string[j] + element)

    return outlist

#HW:  Also read the example on anagrams (page 437) .... it is similar.
def main():
    print("Factorial 5 = ", rFact(5))
    print("Factorial 10 = ", rFact(10))

    print("Factorial 27 = ", rFact(27))

    print(" ")
    print("Number of ways of choosing two objects from ten objects = ", Comb(10, 2))

    print(" ")
    print("Combinations: ", combinations("ABCDEFGH", 3))

main()

#_____________________ Recursive Binary Search_____________________________

#6.py

# When you trace the possible paths a binary search can take, you get a tree.
# It starts with a single node with n elements. After one step you are down by one level,
# with two nodes, each having roughly n/2 elements. As in the first step, when you go down
# further by one level, each node again splits into two nodes with roughly n/4 items each.
# In this way, in log(n) steps (log to the base 2) we arrive at n nodes, each containing
# a single list element, and by this step the search problem has been solved.

# Whenever you see a problem that exhibits this tree-pattern you know you have a
# candidate for recursion.

def rBinSearch(x, nlist, low, high):
    # search for x from nlist[low] to nlist[high]

    if (low > high):
        return -1  # no more work

    mid = (low + high) // 2
item = nlist[mid]  # is this x? or is it smaller? or is it larger?

if (item == x):
    return mid  # found it!

elif (x < item):
    # if x is in nlist, it's in the left sublist
    return rBinSearch(x, nlist, low, mid - 1)

else:  # or since its greater than item, it's in the right sublist
    return rBinSearch(x, nlist, mid + 1, high)

def main():
    nlist = [1, 5, 7, 19, 21, 27, 31, 45, 56, 63, 66, 71, 73, 79, 84, 85, 89, 93, 97, 99]

    index = rBinSearch(7, nlist, 0, 19)

    if (index > 0):
        print("Found the number in location ",index)

main()

#________________ Computing things with exponents very quickly

#7.py

# In your lab you will do this with matrices. The idea is the same. Here we will use numbers. In
# lab you replace the numbers by matrices. No conceptual difference when taking powers.

# Idea: you want to compute x raised to the power 10.

# One way is to multiply it out 10 times like this

def power(x, n):
    # x raised to the power n
    result = 1
    for i in range(n):
result = result * x  # if the power is n, you do n multiplications in this approach

return(result)

# Faster way: x^5 * x^5 = x^10, so we can get x^10 with one multiplication if we had x^5
# x^4 = x^2 * x^2, so things are looking good
#
# first get x^2 which is x*x
# then get x^4 which is x^2*x^2
# then get x^5 which is x^4*x  (so far we have done 3 multiplications in total)
# finally get x^10 which is x^5*x^5 ----> presto! in 4 multiplications we have the answer
#
# This is a lot faster than doing 10 multiplications to get x^10
#
# The only (small) problem is that the power can be odd or even. See the simple formula
# at the bottom of page 438 in your text.

def rpower(x, n):
    # c raised to the power n, recursively, fast

    if (n == 0):
        return 1
    else:

        factor = rpower(x, n//2)  # get half the power recursively

        if (n % 2 == 0):
            return (factor*factor)  # even number, no extra multiplication needed
        else:
            return(factor*factor*x)  # need an extra multiplication because n is odd

#_____________ Iterative and Recursive string reversal_____________________
#8.py

# Let's write a simple iterative function that takes in a string, reverses it and returns the reverse string. ** It uses a new string to do this. **
def iReverse_with_new(s):    # s is the input string
    new = ""
    j = len(s) - 1    # this is the position of last character in string

    while (j >= 0):
        new = new + s[j]    # built the new string starting from last
character, backwards

        j = j - 1

    return new

# Now let's try it recursively

def rReverse(s):
    # Idea: if there is only one character in the string, return it; there
    #          is no work to do.
    #          suppose the length of the string is n
    #          otherwise, put the first character in the n-th position, and
    #          recursively call the
    #          function to work on the first (n-1) characters

    if (len(s) <= 1):    # if string has only one element or is empty
        return s
    else:
        return (rReverse(s[1: ]) + s[0])    # put first char in last position
and call rReverse on the
        # rest of the
string, recursively

def main():

    print("iReverse_with_new")
    print(" ")
    print(iReverse_with_new("This is a small string"))
    print(iReverse_with_new("0123456789"))
    print(iReverse_with_new(" "))
    print(iReverse_with_new("0 1"))
    print(iReverse_with_new(""))

    print("\nrReverse")
    print(" ")
    print(rReverse("This is a small string"))
    print(rReverse("0123456789"))
print(rReverse(" "))
print(rReverse("0 1"))
print(rReverse(""))

main()