Randomized Linear Algebra for Interior Point Methods

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Randomized Numerical Linear Algebra

RandNLA Highlight (work over the past 20+ years):

In problems that involve matrices, using a sketch of the matrix instead of the original matrix returns provably accurate results theoretically and works well empirically.

(1) The sketch can be just a few rows/columns/elements of the matrix, selected carefully (or not).

(2) The sketch can simply be the product of a matrix with a few random (Gaussian) vectors.

(3) Better sketches (in terms of the accuracy vs. running time tradeoff to construct the sketch) have been heavily researched.

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(3) Better sketches (in terms of the accuracy vs. running time tradeoff to construct the sketch) have been heavily researched.

Sketching works! In theory and in practice.

This is a major oversimplification that both helps and hurts the field.

Highlights of 20+ years of RandNLA

<u>Lesson learned:</u>

- Sketches can be used as a proxy of the matrix in the original problem (e.g., in the streaming or pass-efficient model), **<u>BUT:</u>**

Highlights of 20+ years of RandNLA

Lesson learned:

- Sketches can be used as a proxy of the matrix in the original problem (e.g., in the streaming or pass-efficient model), **BUT:**
- A much better use of a sketch is as a preconditioner or to compute a starting point for an iterative process.

(1) As a preconditioner in iterative methods for regression problems, (pioneered by Blendenpik).

(2) To compute a "seed" vector in subspace iteration for SVD/PCA, or to compute a Block Krylov subspace.

Neither (1) nor (2) are novel in Numerical Analysis, but the introduction of randomization to construct the sketch was/is/will be ground-breaking.

(Re (2): Drineas, Ipsen, Kontopoulou, & Magdon-Ismail SIMAX 2018; Drineas & Ipsen SIMAX 2019; Bose et al. Bioinformatics 2019; building on ideas from Musco & Musco NeurIPS 2015.)

Using Haim Avron's slide:

Sketch-and-Solve

1. High success rate
2. Polynomial accuracy dependence (e.g. ϵ^{-2})
3. No iterations

Pros:

1. **Very** fast
2. Deterministic running time

Cons:

1. Only crude accuracy
2. "Monte-Carlo" algorithm

Sketch-to-Precondition

1. High success rate
2. Exponential accuracy dependence (e.g. $\log(1/\epsilon)$)
3. Iterations

Pros:

1. Very high accuracy possible
2. Success = good solution

Cons:

1. Slower than sketch-and-solve
2. Iterations (no streaming)

Add an outer iteration

Goal: Combine iterative sketching-based solvers (think Blendenpik) with iterative algorithms, such as:

- Interior Point Methods (IPM) for Linear Programming

- Iterative Re-Weighted Least-Squares (IRWLS) for Generalized Linear Models

- Etc.

Thus, there is an outer iteration (say, from the IPM for LPs) and an inner iteration (from the solver).

RandNLA and Linear Programming

- Primal-dual interior point methods necessitate solving least-squares problems (projecting the gradient on the null space of the constraint matrix in order to remain feasible).

  (Dating back to the mid/late 1980's and work by Karmarkar, Ye, Freund)

- **Modern approaches:** path-following interior point methods iterate using the Newton direction. A system of linear equations must be solved at each iteration.

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- **Well-known by practitioners:** the number of iterations in interior point methods is **not** the bottleneck, but the computational cost of solving a linear system is.
- **Goal:** Use RandNLA approaches to design efficient preconditioners to **approximately** solve systems of linear equations that arise in IPMs faster.

Path-Following IPMs

A broad classification of **Interior Point Methods (IPM) for Linear Programming (LP):**

IPM: Path Following Methods

- Long step methods (worse theoretically, fast in practice)
- Short step methods (better in theory, slow in practice)
- Predictor-Corrector (good in theory and practice)
- Can be further divided to feasible and infeasible methods (depending on starting point).

Especially relevant in practice for long step and predictor corrector methods.

IPM: Potential-Reduction algorithms

Not explored in our work.

Standard Form Linear Programs

Consider the standard form of the primal LP problem:

$$\min\ \mathbf{c}^{\mathsf{T}}\mathbf{x}\,,\ \text{subject to}\ \mathbf{A}\mathbf{x} = \mathbf{b}\,, \mathbf{x} \geq \mathbf{0}$$

The associated dual problem is

$$\max\ \mathbf{b}^{\mathsf{T}}\mathbf{y}\,,\ \text{subject to}\ \mathbf{A}^{\mathsf{T}}\mathbf{y} + \mathbf{s} = \mathbf{c}\,, \mathbf{s} \geq \mathbf{0}$$

$\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^{m}$, and $\mathbf{c} \in \mathbb{R}^{n}$ are inputs

$\mathbf{x} \in \mathbb{R}^{n}$, $\mathbf{y} \in \mathbb{R}^{m}$, and $\mathbf{s} \in \mathbb{R}^{n}$ are variables

Interior Point Methods (IPMs)

- **Duality measure**:

$$\mu = \frac{\mathbf{x}^\mathsf{T}\mathbf{s}}{n} = \frac{\mathbf{x}^\mathsf{T}(\mathbf{c} - \mathbf{A}^\mathsf{T}\mathbf{y})}{n} = \frac{\mathbf{c}^\mathsf{T}\mathbf{x} - \mathbf{b}^\mathsf{T}\mathbf{y}}{n} \downarrow 0$$

- **Path-following methods:**
  - Let $\mathcal{F}^0 = \{(\mathbf{x}, \mathbf{y}, \mathbf{s}) : (\mathbf{x}, \mathbf{s}) > \mathbf{0},\ \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{A}^\mathsf{T}\mathbf{y} + \mathbf{s} = \mathbf{c}\}$.
  - Central path: $\mathcal{C} = \{(\mathbf{x}, \mathbf{y}, \mathbf{s}) \in \mathcal{F}^0 : \mathbf{x} \circ \mathbf{s} = \sigma\mu\mathbf{1}_n\}$, $\sigma \in (0, 1)$ is the centering parameter.
  - Neighborhood: $\mathcal{N}(\gamma) = \left\{(\mathbf{x}, \mathbf{y}, \mathbf{s}) \in \mathcal{F}^0 : \mathbf{x} \circ \mathbf{s} \geq (1 - \gamma)\mu\mathbf{1}_n\right\}$, $\gamma \in (0, 1)$
  - Given the step size $\alpha \in [0, 1]$ and $(\mathbf{x}, \mathbf{y}, \mathbf{s}) \in \mathcal{N}(\gamma)$, it computes the Newton search direction $(\Delta\mathbf{x}, \Delta\mathbf{y}, \Delta\mathbf{s})$ and update the current iterate

$$\boxed{(\mathbf{x}(\alpha), \mathbf{y}(\alpha), \mathbf{s}(\alpha)) = (\mathbf{x}, \mathbf{y}, \mathbf{s}) + \alpha(\Delta\mathbf{x}, \Delta\mathbf{y}, \Delta\mathbf{s}) \in \mathcal{N}(\gamma)}$$

Interior Point Methods (IPMs)

(long-step, feasible)

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$$\mu = \frac{\mathbf{x}^\top \mathbf{s}}{n} = \frac{\mathbf{x}^\top(\mathbf{c} - \mathbf{A}^\top \mathbf{y})}{n} = \frac{\mathbf{c}^\top \mathbf{x} - \mathbf{b}^\top \mathbf{y}}{n} \downarrow 0$$

- After $k = \mathcal{O}\left(n \log \frac{1}{\epsilon}\right)$ iterations, $\mu_k \leq \epsilon\, \mu_0$.

- **Path-following methods:**
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Interior Point Methods (IPMs)

(long-step, feasible/infeasible)

Path-following IPMs, at every iteration, solve a system of linear equations :

$$\begin{pmatrix} \mathbf{A} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^\mathsf{T} & \mathbf{I}_n \\ \mathbf{S} & \mathbf{0} & \mathbf{X} \end{pmatrix} \begin{pmatrix} \Delta\mathbf{x} \\ \Delta\mathbf{y} \\ \Delta\mathbf{s} \end{pmatrix} = \begin{pmatrix} -\mathbf{r}_p \\ -\mathbf{r}_d \\ -\mathbf{XS1}_n + \sigma\mu\mathbf{1}_n \end{pmatrix}$$

$\mathbf{D} = \mathbf{X}^{1/2}\mathbf{S}^{-1/2}$ is a diagonal matrix.

normal equations

$$\mathbf{AD}^2\mathbf{A}^\mathsf{T}\Delta\mathbf{y} = \underbrace{-\mathbf{r}_p - \sigma\mu\mathbf{AS}^{-1}\mathbf{1}_n + \mathbf{Ax} - \mathbf{AD}^2\mathbf{r}_d}_{\mathbf{p}},$$

$$\Delta\mathbf{s} = -\mathbf{r}_d - \mathbf{A}^\mathsf{T}\Delta\mathbf{y},$$

$$\Delta\mathbf{x} = -\mathbf{x} + \sigma\mu\mathbf{S}^{-1}\mathbf{1}_n - \mathbf{D}^2\Delta\mathbf{s}.$$

RandNLA & IPMs for LPs

Research Agenda: Explore how approximate, iterative solvers for the normal equations affect the convergence of

(1) long-step (feasible and infeasible) IPMs,

(2) feasible predictor-corrector IPMs.

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- We seek to investigate **standard, practical solvers**, such as Preconditioned Conjugate Gradients, Preconditioned Steepest Descent, Preconditioned Richardson's iteration, etc.
- The preconditioner is constructed using RandNLA sketching-based approaches.

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- We seek to investigate **standard, practical solvers**, such as Preconditioned Conjugate Gradients, Preconditioned Steepest Descent, Preconditioned Richardson's iteration, etc.
- The preconditioner is constructed using RandNLA sketching-based approaches.
- **Remark:** For feasible path-following IPMs, an additional design choice is whether we want the final solution to be feasible or approximately feasible.

Preconditioning in Interior Point Methods

(joint with H. Avron, A. Chowdhuri, G. Dexter, and P. London, NeurIPS 2020)

Standard form of primal LP:

$$\mathbf{x} \in \mathbb{R}^n$$

$$\min \mathbf{c}^{\mathsf{T}}\mathbf{x}\,, \text{ subject to } \mathbf{A}\mathbf{x} = \mathbf{b}\,, \mathbf{x} \geq \mathbf{0}$$

$$\mathbf{A} \in \mathbb{R}^{m \times n},\ \mathbf{b} \in \mathbb{R}^m, \text{ and } \mathbf{c} \in \mathbb{R}^n$$

Path-following, long-step IPMs: compute the Newton search direction; update the current iterate by following a (long) step towards the search direction.

A standard approach involves solving the normal equations:

$$\mathbf{A}\mathbf{D}^2\mathbf{A}^{\mathsf{T}}\Delta\mathbf{y} = \mathbf{p} \quad \text{where } \mathbf{D} \in \mathbb{R}^{n \times n},\ \mathbf{p} \in \mathbb{R}^m$$

Vector of m unknowns

Use a preconditioned method to solve the above system: we analyzed preconditioned Conjugate Gradient solvers; preconditioned Richardson's; and preconditioned Steepest Descent, all with randomized preconditioners.

Challenges

Immediate problem: even assuming a feasible starting point, approximate solutions do not lead to feasible updates.

- As a result, standard analyses of the convergence of IPMs are not applicable.
- We use RandNLA approaches to efficiently and provably accurately correct the error induced by the approximate solution and guarantee convergence.

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Details: the approximate solution violates critical equalities:

$$\mathbf{A}\mathbf{D}^2\mathbf{A}^\top\hat{\Delta}\mathbf{y} \neq \mathbf{p} \quad \text{and} \quad \mathbf{A}\hat{\Delta}\mathbf{x} \neq -\mathbf{r}_p$$

- The vector r_p is the primal residual; for feasible long-step IPMs, it is the all-zero vector.
- Standard analyses of long-step (infeasible/feasible) IPMs critically need the second inequality to be an equality.
- Without the above equalities, in the case of feasible IPMs, we can not terminate with a feasible solution; we will end up with an approximately feasible solution.

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- Without the above equalities, in the case of feasible IPMs, we can not terminate with a feasible solution; we will end up with an approximately feasible solution.

Results

(correction vector idea also in O'Neal and Monteiro 2003)

We construct a "correction" vector $v \in R^n$ s.t.:

$$
\begin{aligned}
\mathbf{A}\mathbf{D}^2\mathbf{A}^\mathsf{T}\hat{\Delta}\mathbf{y} &= \mathbf{p} + \mathbf{A}\mathbf{S}^{-1}\mathbf{v}\,, \\
\hat{\Delta}\mathbf{s} &= -\cancel{\mathbf{r}_d} - \mathbf{A}^\mathsf{T}\hat{\Delta}\mathbf{y}\,, \\
\hat{\Delta}\mathbf{x} &= -\mathbf{x} + \sigma\mu\mathbf{S}^{-1}\mathbf{1}_n - \mathbf{D}^2\hat{\Delta}\mathbf{s} - \mathbf{S}^{-1}\mathbf{v}
\end{aligned}
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Then $\mathbf{A}\hat{\Delta}\mathbf{x} = -\cancel{\mathbf{r}_p}\,\mathbf{0}_m$

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Then $\mathbf{A}\hat{\Delta \mathbf{x}} = -\cancel{\mathbf{r}_p}\, \mathbf{0}_m$

- The vector r_p is the primal residual; the vector r_d is the dual residual. For feasible long-step IPMs, they are both all-zero vectors.
- Our (sketching-based) "correction" vector $v \in R^n$ works with probability $1-\delta$ and can be constructed in time

$$\mathcal{O}\Big(\mathrm{nnz}(\mathbf{A}) \cdot \log(m/\delta) + m^3 \log(m/\delta)\Big)$$

- If sketching-based, randomized preconditioned solvers are used, then **we only need mat-vecs to construct v.**
- Using this "correction" vector $v \in R^n$, analyses of long-step (infeasible/feasible) IPMs work!

Results: feasible, long-step IPMs

If the constraint matrix $A \in R^{m \times n}$ is short-and-fat $(m \ll n)$, then

- Run $O\left(n \cdot log\left(\frac{1}{\epsilon}\right)\right)$ outer iterations of the IPM solver.
- In each outer iteration, the normal equations are solved by $O(\log n)$ inner iterations of the PCG solver.
- Then, the feasible, long-step IPM converges.
- Can be generalized to (exact) low-rank matrices A with rank $k \ll \min\{m, n\}$.

Thus, approximate solutions suffice; ignoring failure probabilities, each inner iteration needs time

$$\boxed{\mathcal{O}((\text{nnz}(\mathbf{A}) + m^3) \log n)}$$

Results: infeasible, long-step IPMs

If the constraint matrix $A \in R^{m \times n}$ is short-and-fat $(m \ll n)$, then

- Run $O\left(n^2 \cdot log\left(\frac{1}{\epsilon}\right)\right)$ outer iterations of the IPM solver.
- In each outer iteration, the normal equations are solved by $O(\log n)$ inner iterations of the PCG solver.
- Then, the infeasible, long-step IPM converges.
- Can be generalized to (exact) low-rank matrices A with rank $k \ll \min\{m, n\}$.

Thus, approximate solutions suffice; ignoring failure probabilities, each inner iteration needs time

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Feasible Predictor-Corrector IPMs

(joint work with H. Avron, A. Chowdhuri, G. Dexter)

- By oscillating between the following two types of steps at each iteration, *Predictor-Corrector (PC)* IPMs achieve twofold objective of **(i) reducing duality measure μ** and **(ii) improving centrality :**

  - Predictor step ($\sigma = 0$) to reduce the duality measure **μ**.
  - Corrector steps ($\sigma = 1$) to improve centrality.

- PC obtains the best of both worlds: **(i) the practical flexibility of long-step IPMs** and **(ii) the convergence rate of short-step IPMs**.

Feasible Predictor-Corrector IPMs

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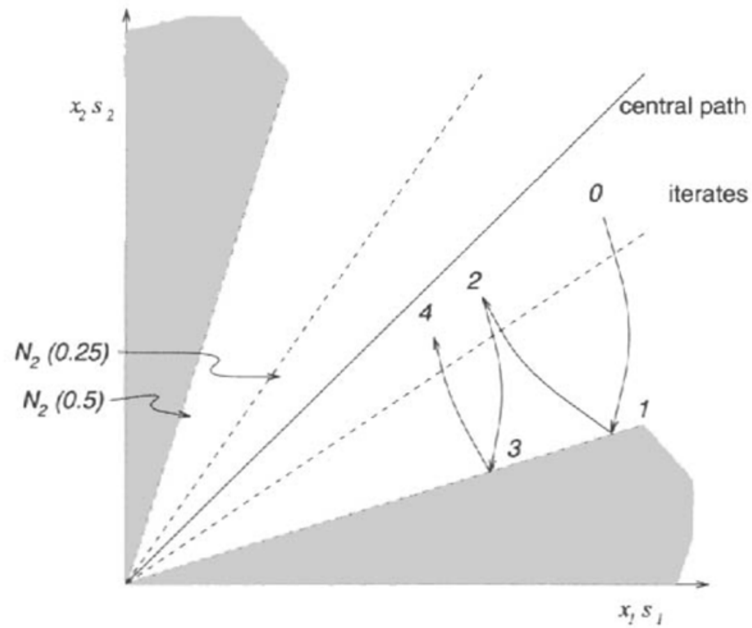
  - Predictor step ($\sigma = 0$) to reduce the duality measure **μ**.
  - Corrector steps ($\sigma = 1$) to improve centrality.
- PC obtains the best of both worlds: **(i) the practical flexibility of long-step IPMs** and **(ii) the convergence rate of short-step IPMs**.
- Our work combines the prototypical PC algorithm (e.g., see Wright (1997)) with (preconditioned) inexact solvers.
- **Major challenge:** analyze inexact PC is to guarantee that the duality measure after each corrector step of the PC iteration decreases.

(Standard analysis breaks; the (feasible) long-step proof was easier; we had to come up with new inequalities for an approximate version of the duality measure.)

Predictor-corrector Algorithm Overview

Alternates between predictor and corrector steps

- Predictor step greatly decreases duality measure, while deviating from the central path (centering parameter $\sigma = 1$).
- Corrector step keeps duality measure constant but returns iterate to near central path (centering parameter $\sigma = 0$).
- Alternates between two neighborhoods of the central path $N_2(0.25)$ and $N_2(0.5)$.



$$\mathcal{N}_2(\theta) = \left\{ (\mathbf{x}, \mathbf{y}, \mathbf{s}) \in \mathbb{R}^{2n+m} : \|\mathbf{x} \circ \mathbf{s} - \mu \mathbf{1}_n\|_2 \leq \theta\mu,\ (\mathbf{x}, \mathbf{s}) > 0 \right\}.$$

Solving the linear system

At each iteration of the Predictor-Corrector IPM, we need to solve the following linear system:

$$\mathbf{A}\mathbf{D}^2\mathbf{A}^\top \Delta \mathbf{y} = \underbrace{-\sigma\mu \mathbf{A}\mathbf{S}^{-1}\mathbf{1}_n + \mathbf{A}\mathbf{x}}_{\mathbf{p}}$$
$$\Delta \mathbf{s} = -\mathbf{A}^\top \Delta \mathbf{y}$$
$$\Delta \mathbf{x} = -\mathbf{x} + \sigma\mu \mathbf{S}^{-1}\mathbf{1}_n - \mathbf{D}^2 \Delta \mathbf{s}.$$

Note that the last two equations only involve matrix-vector products. Therefore, we only focus on solving the first equation efficiently.

Structural Conditions for Inexact PC

- Let $\Delta\tilde{y}$ be an approximate solution to the normal equations $(AD^2A^T)\cdot\Delta y = p$.
- If $\Delta\tilde{y}$ satisfies (sufficient conditions):

$$\|\Delta\tilde{\mathbf{y}} - \Delta\mathbf{y}\|_{\mathbf{AD^2A}^T} \leq \Theta\left(\frac{\epsilon}{\sqrt{n}\log 1/\epsilon}\right)$$

$$\|\mathbf{AD}^2\mathbf{A}^T\Delta\tilde{\mathbf{y}} - \mathbf{p}\|_2 \leq \Theta\left(\frac{\epsilon}{\sqrt{n}\log 1/\epsilon}\right)$$

- Then, we prove that the Inexact PC method converges in $O\left(\sqrt{n}\cdot\log\left(\frac{1}{\epsilon}\right)\right)$ iterations, as expected.
- The final solution (and all intermediate iterates) are only *approximately* feasible.

Structural Conditions for Inexact PC using a correction vector v

- We modified the PC method **using a correction vector v** to make iterates *exactly* feasible.
- Let $\Delta\tilde{y}$ be an approximate solution to the normal equations $(AD^2A^T) \cdot \Delta y = p$.
- If $\Delta\tilde{y}$ and v satisfy (sufficient conditions):

$$\mathbf{AS}^{-1}\mathbf{v} = \mathbf{AD}^2\mathbf{A}^T\Delta\tilde{\mathbf{y}} - \mathbf{p}$$

$$\|\mathbf{v}\|_2 < \Theta(\epsilon)$$

- Then, we prove that this modified Inexact PC method converges in $O\left(\sqrt{n} \cdot \log\left(\frac{1}{\epsilon}\right)\right)$ iterations, as expected.
- The final solution (and all intermediate iterates) are *exactly* feasible.

Preconditioned equation

Corresponding preconditioned equation is given by

$$\mathbf{Q}^{-1/2}\,\mathbf{A}\mathbf{D}^2\,\mathbf{A}^\top \underbrace{\mathbf{Q}^{-1/2}\mathbf{z}}_{\Delta\mathbf{y}} = \mathbf{Q}^{-1/2}\mathbf{p}$$

Here $Q \in R^{m\times m}$ is the preconditioner.

Clearly, we need a matrix Q which is “easily” invertible.

(Will come back to this later.)

Satisfying the sufficient conditions for Inexact Predictor-Corrector IPMs with/without the correction vector

For an accuracy parameter $\zeta \in (0,1)$, it can be shown that the following two conditions on the preconditioner $Q^{-1/2}$ of the iterative solver can be used to derive the sufficient conditions:

(C1) All singular values $\sigma_i, i = 1 \dots m$ of the preconditioned matrix $Q^{-1/2}\, AD$ satisfy:

$$\frac{2}{2+\zeta} \leq \sigma_i^2\left(\mathbf{Q}^{-\frac{1}{2}}\mathbf{AD}\right) \leq \frac{2}{2-\zeta}$$

(C2) As the number of iterations t of the iterative solver increase, the residual norm w.r.t the preconditioned system decreases monotonically:

$$\left\|\mathbf{Q}^{-1/2}\mathbf{AD}^2\mathbf{A}^\top\mathbf{Q}^{-1/2}\tilde{\mathbf{z}}^t - \mathbf{Q}^{-1/2}\mathbf{p}\right\|_2 \leq \zeta^t\left\|\mathbf{Q}^{-1/2}\mathbf{p}\right\|_2$$

Satisfying the structural conditions

- We analyzed Preconditioned Conjugate Gradients (PCG) solvers with randomized preconditioners for constraint matrices $A \in R^{n \times n}$ that are: short-and-fat $(m \ll n)$, tall-and-thin $(m \gg n)$ or have exact low-rank $\mathrm{k} \ll \min\{m, n\}$.

- **<u>Satisfying the structural conditions for "standard" Inexact PC:</u>** the PCG solver needs $O\left(\log\left(\frac{n \cdot \sigma_1(AD)}{\epsilon}\right)\right)$ iterations (inner iterations).

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- **Satisfying the structural conditions for the "modified" Inexact PC:** the PCG solver needs $O\left(\log\left(\frac{n}{\epsilon}\right)\right)$ iterations (inner iterations).

- Notice that using the error-adjustment vector v in the modified Inexact PC *eliminates the dependency on the largest singular value of the matrix* AD.

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- **<u>Satisfying the structural conditions for the "modified" Inexact PC:</u>** the PCG solver needs $O\left(\log\left(\frac{n}{\epsilon}\right)\right)$ iterations (inner iterations).

- Notice that using the error-adjustment vector v in the modified Inexact PC *eliminates the dependency on the largest singular value of the matrix* AD.

- Computing the error-adjustment vector v is fast and can be done (combined with randomized preconditioners and PCG) in $O(nnz(A) \log n)$ time (just mat-vecs).

- Similar results can be derived for preconditioned steepest descent, preconditioned Chebyschev, and preconditioned Richardson solvers.

Constructing our preconditioner

- For a suitable sketching matrix $W \in R^{n\times w}$ with $w \ll n$ let $Q = ADWW^TDA^T$.
- To invert Q, it is sufficient to compute the SVD of ADW, which takes $O(m^2w)$ time.
- **Choice of the sketching matrix W**:
  - W could be the CountSketch matrix with $w = O(m \log m)$ and $\log m$ non-zero entries per row.
  - Many, many other choices exist (random Gaussians, fast randomized transforms, etc.)
  - ADW can be computed in $O(\log m \cdot nnz(A))$ time.
- We can compute $Q^{-1/2}$ in time:

$$\mathcal{O}\Big(\mathrm{nnz}(\mathbf{A}) \cdot \log m + m^3 \log m\Big)$$

Iterative solver: summary

Input: $\mathbf{AD} \in \mathbb{R}^{m \times n}$ with $m \ll n, \mathbf{p} \in \mathbb{R}^m$, sketching matrix $\mathbf{W} \in \mathbb{R}^{n \times w}$, iteration count t;

Step 1. Compute $\mathbf{ADW}$ and its SVD. Let $\mathbf{U_Q} \in \mathbb{R}^{m \times m}$ be the matrix of its left singular vectors and let $\mathbf{\Sigma_Q^{1/2}} \in \mathbb{R}^{m \times m}$ be the matrix of its singular values;

Step 2. Compute $\mathbf{Q}^{-1/2} = \mathbf{U_Q \Sigma_Q^{-1/2} U_Q^\top}$;

Step 3. Initialize $\tilde{\mathbf{z}}^0 \leftarrow \mathbf{0}_m$ and run standard CG on $\mathbf{Q}^{-1/2}\mathbf{AD}^2\mathbf{A}^T\mathbf{Q}^{-1/2}\tilde{\mathbf{z}} = \mathbf{Q}^{-1/2}\mathbf{p}$ for t iterations;

Output: return $\Delta\tilde{\mathbf{y}} = \mathbf{Q}^{-1/2}\tilde{\mathbf{z}}^t$.

- Approximate solution $\Delta\tilde{y}$ can be found by pre-multiplying the solution by the preconditioner.
- Instead of Conjugate Gradients (CG), one can use other iterative solvers, namely, Chebyshev iteration, Steepest descent etc.

Satisfying condition **C1**: Bounding the condition number of the preconditioned matrix

Let $\mathbf{Q} = \mathbf{ADWW}^\top\mathbf{DA}^\top$ and if the sketching matrix $\mathbf{W}$ satisfies $\left\|\mathbf{V}^\top\mathbf{WW}^\top\mathbf{V} - \mathbf{I}_m\right\|_2 \leq \zeta/2$, then, for all $i = 1 \ldots m$

$$(1+\zeta/2)^{-1} \leq \sigma_i^2\left(\mathbf{Q}^{-1/2}\mathbf{AD}\right) \leq (1-\zeta/2)^{-1}$$

- Here V is the matrix of the right singular vectors of A (thin SVD, containing only the singular vectors corresponding to non-zero singular values.
- This is the so-called ℓ_2-subspace embedding condition and implies that the condition number of $Q^{-1/2}AD$ remains small.
- Our Wsatisfies the ℓ_2-subspace embedding condition with high probability.

Satisfying condition **C2**: the residual norm w.r.t the preconditioned system decreases monotonically

Given our preconditioner $\mathbf{Q}^{-1/2} = \left(\mathbf{ADWW}^\top\mathbf{DA}^\top\right)^{-1/2}$ satisfying condition **(C1)** for an accuracy parameter $\zeta \in (0,1)$ and all $\mathbf{i} = 1, 2 \ldots \mathbf{m}$, our iterative solver satisfies

$$\left\|\mathbf{Q}^{-1/2}\mathbf{AD}^2\mathbf{A}^\top\mathbf{Q}^{-1/2}\tilde{\mathbf{z}}^t - \mathbf{Q}^{-1/2}\mathbf{p}\right\|_2 \leq \zeta^t \left\|\mathbf{Q}^{-1/2}\mathbf{p}\right\|_2$$

Here $\tilde{\mathbf{z}}^t$ is the approximate solution returned by the CG iterative solver after t iterations.

- Residual drops exponentially fast as the number of iterations t increases.
- The above guarantee holds for various iterative solvers including CG, Chebyshev iteration, Steepest descent etc.

Satisfying condition **C2** using conjugate gradient

Result (Theorem 8 of Bouyouli et al. (2009)):

Let $\tilde{\mathbf{f}}^{(j)} = \mathbf{Q}^{-1/2}\mathbf{A}\mathbf{D}^2\mathbf{A}^\top\mathbf{Q}^{-1/2}\tilde{\mathbf{z}}^j - \mathbf{Q}^{-1/2}\mathbf{p}$ be the residual of by the CG solver at steps j. Then,

$$\|\tilde{\mathbf{f}}^{(j)}\|_2 \leq \frac{\kappa^2(\mathbf{Q}^{-1/2}\mathbf{A}\mathbf{D}) - 1}{2}\|\tilde{\mathbf{f}}^{(j-1)}\|_2$$

- Note that the above recursive relation, in general, does not ensure that the residual norms decrease monotonically.
- But, due to **(C1)**, we already have a bound on the condition number of $Q^{-1/2}AD$.
- If we combine the above inequality with the recursion, we get **(C2)**.
- Therefore, our preconditioner ensures the CG residual decreases monotonically, which otherwise might fluctuate.

Satisfying condition **C2** using Chebyshev iteration

Result (Theorem 1.6.2 of Gutknecht (2008)):

The residual norm reduction of the Chebyshev iteration, when applied to an SPD system whose condition number is upper bounded by $\mathcal{U}$, is bounded according to

$$\frac{\|\tilde{\mathbf{f}}^{(t)}\|_2}{\|\tilde{\mathbf{f}}^{(0)}\|_2} \leq 2\left[\left(\frac{\sqrt{\mathcal{U}}+1}{\sqrt{\mathcal{U}}-1}\right)^t + \left(\frac{\sqrt{\mathcal{U}}-1}{\sqrt{\mathcal{U}}+1}\right)^t\right]^{-1}$$

- Chebyshev iteration avoids the computation of the communication intensive inner products which is typically needed for CG or other non-stationary methods.
- Therefore, this solver is convenient in parallel or distributed settings.
- Due to **(C1)**, we already have a bound for U. Using this, we establish **(C2)**.

Other solvers

- Similarly, our preconditioner also satisfies **(C2)** with respect to other two popular iterative solvers, namely Steepest descent and Richardson iteration.

- The proofs for both the solvers rely on the fact that due to the efficient preconditioning the residuals of the preconditioned system decrease monotonically.

Constructing the vector v

- Any iterative solver solves the system approximately. Therefore, due to the approximation error caused by the solver, the iterates of our predictor-corrector IPM lose feasibility right after the first iteration.
- As already discussed, for our inexact *corrected* predictor-corrector, we introduce a correction vector v in order to maintain feasibility at each iteration of the IPM.
- v must satisfy the following invariant at each iteration:

$$\mathbf{A}\mathbf{S}^{-1}\mathbf{v} = \mathbf{A}\mathbf{D}^2\mathbf{A}^{\mathbf{T}}\Delta\tilde{\mathbf{y}} - \mathbf{p}$$

  Recall that $\Delta\tilde{y}$ is the solution returned by the iterative solver.

(*A* solution originally proposed by **Monteiro & O'Neal (2003)** is expensive.)

Constructing the vector v

- **Our solution:**

$$\mathbf{v} = (\mathbf{XS})^{1/2}\mathbf{W}(\mathbf{ADW})^{\dagger}\left(\mathbf{AD}^2\mathbf{A}^{\top}\Delta\tilde{\mathbf{y}} - \mathbf{p}\right)$$

- We use the same sketching matrix **W** that we used for constructing our preconditioner.
- Due to the "good" preconditioner we used, we can show that the norm of v is nicely bounded, which ensures that the sufficient conditions are satisfied.
- Proof is not too long; other constructions might be possible and perhaps better in theory and/or practice.

Time to compute the correction vector

- **Recall our solution:**

$$\mathbf{v} = (\mathbf{XS})^{1/2}\mathbf{W}(\mathbf{ADW})^{\dagger}\left(\mathbf{AD}^2\mathbf{A}^{\top}\Delta\tilde{\mathbf{y}} - \mathbf{p}\right)$$

- We have already computed the pseudoinverse of ADW when constructing our preconditioner.
- Pre-multiplying by W takes $O(nnz(A) \cdot \log m)$ time, assuming $nnz(A) \geq n$.
- $\boldsymbol{X}, \boldsymbol{S}$ are diagonal matrices.
- Therefore, computing v takes $O(nnz(A) \cdot \log m)$ time.

Overall running time (per iteration)

Accounting for the number of iterations of the solver, as well as the failure probability $\eta \in (0,1)$, the per-iteration cost of our approaches is given by:

- ***Without*** a correction vector:

$$\mathcal{O}\left(\mathrm{nnz}(\mathbf{A}) \cdot \log m/\eta + m^3 \log m/\eta + m \log \frac{\sigma_{\max}(\mathbf{AD}) n\mu}{\delta} + \mathrm{nnz}(\mathbf{A}) \cdot \log \frac{\sigma_{\max}(\mathbf{AD}) n\mu}{\delta}\right)$$

- ***With*** a correction vector:

$$\mathcal{O}\left(\mathrm{nnz}(\mathbf{A}) \cdot \log m/\eta + m^3 \log m/\eta + m \log \frac{n\mu}{\delta} + \mathrm{nnz}(\mathbf{A}) \cdot \log \frac{n\mu}{\delta}\right)$$

Open problems

- Can we prove similar results for infeasible predictor-corrector IPMs? Recall that such methods need $O(n)$ outer iterations (Yang & Namashita 2018).

- Are our structural conditions necessary? Can we derive simpler conditions?

- Could our structural conditions change from one iteration to the next? Could we use dynamic preconditioning or reuse preconditioners from one iteration to the next (e.g., low-rank updates of the preconditioners)?

- Connections with similar results in the TCS community (starting with Daitch & Spielman (2008)).
  - Analyzed a short-step (dual) path-following IPM (LP *not* in standard form).
  - No "correction" vector; an approximately feasible solution was returned.
  - Dependency on $\log(\kappa(S))$ for the outer iteration -- can it be removed?

Relevant literature

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