



Tensor sparsification via a bound on the spectral norm of random tensors

NAM NGUYEN*,

Department of Mathematics, Massachusetts Institute of Technology

*namn@mit.edu

PETROS DRINEAS

Department of Computer Science, Rensselaer Polytechnic Institute

drinep@cs.rpi.edu

AND

TRAC TRAN

Department of Electrical and Computer Engineering, The Johns Hopkins University

trac@jhu.edu

[Received on 16 January 2015]

Given an order- d tensor $\mathcal{A} \in \mathbb{R}^{n \times n \times \dots \times n}$, we present a simple, element-wise sparsification algorithm that zeroes out all sufficiently small elements of \mathcal{A} , keeps all sufficiently large elements of \mathcal{A} , and retains some of the remaining elements with probabilities proportional to the square of their magnitudes. We analyze the approximation accuracy of the proposed algorithm using a powerful inequality that we derive. This inequality bounds the spectral norm of a random tensor and is of independent interest. As a result, we obtain novel bounds for the tensor sparsification problem.

Keywords: tensor; tensor norm; tensor sparsification; fast tensor computation; random tensor

1. Introduction

Technological developments over the last two decades (in both scientific and internet domains) permit the automatic generation of very large data sets. Such data are often modeled as matrices, since an $m \times n$ real-valued matrix A provides a natural structure to encode information about m objects, each of which is described by n features. A generalization of this framework permits the modeling of the data by higher-order arrays or tensors (e.g., arrays with more than two modes). A natural example is time-evolving data, where the third mode of the tensor represents time [18]. Numerous other examples exist, including tensor applications in higher-order statistics, where tensor-based methods have been leveraged in the context of, for example, Independent Components Analysis (ICA), in order to exploit the statistical independence of the sources [26, 27, 25].

A large body of recent work has focused on the design and analysis of algorithms that efficiently create small “sketches” of matrices and tensors. By sketches, we mean a new matrix or tensor with significantly smaller size than the original ones. Such sketches are subsequently used in eigenvalue and eigenvector computations [19, 1], in data mining applications [31, 32, 16, 30], or even to solve combinatorial optimization problems [5, 13, 14]. Existing approaches include, for example, the selection of a small number of rows and columns of a matrix in order to form the so-called CUR matrix/tensor decomposition [15, 31, 32], as well as random-projection-based methods that employ fast randomized variants of the Hadamard-Walsh transform [36] or the Discrete Cosine Transform [33].

An alternative approach was pioneered by Achlioptas and McSherry in 2001 [1, 2] and leveraged the selection of a small number of elements in order to form a sketch of the input matrix. A rather straight-forward extension of their work to tensors was described by Tsourakakis in [40]. Another remarkable direction was pioneered in the work of Spielman, Teng, Srivastava, and collaborators [9, 37], who proposed algorithms for graph sparsification in order to create preconditioners for systems of linear equations with Laplacian input matrices. Partly motivated by their work, we define the following matrix/tensor sparsification problem:

DEFINITION 1.1 [MATRIX/TENSOR SPARSIFICATION] Given an order- d tensor $\mathcal{A} \in \mathbb{R}^{n \times n \times \dots \times n}$ and an error parameter $\varepsilon \geq 0$, construct a sketch $\tilde{\mathcal{A}} \in \mathbb{R}^{n \times n \times \dots \times n}$ such that

$$\|\mathcal{A} - \tilde{\mathcal{A}}\|_2 \leq \varepsilon \|\mathcal{A}\|_2 \tag{1.1}$$



and the number of non-zero entries in $\tilde{\mathcal{A}}$ is minimized. Here, the $\|\mathcal{A}\|_2$ norm is called the spectral norm of the tensor \mathcal{A} (see Section 2.1 for the definition).

A few comments are necessary to better understand the above definition. First, an order- d tensor is simply a d -way array (obviously, a matrix is an order-2 tensor). We let $\|\cdot\|_2$ denote the spectral norm of a tensor (see Section 2.1 for notation), which is a natural extension of the matrix spectral norm. It is worth noting that exactly computing the tensor spectral norm is computationally hard. Second, a similar problem could be formulated by seeking a bound for the Frobenius norm of $\mathcal{A} - \tilde{\mathcal{A}}$. Third, this definition places no constraints on the form of the entries of $\tilde{\mathcal{A}}$. However, in this work, we will focus on methods that return matrices and tensors $\tilde{\mathcal{A}}$ whose entries are either zeros or (rescaled) entries of \mathcal{A} . Prior work has investigated quantization as an alternative construction for the entries of $\tilde{\mathcal{A}}$, while the theoretical properties of more general methods remain vastly unexplored. Fourth, the running time needed to construct a sketch is not restricted. All prior work has focused on the construction of sketches in one or two sequential passes over the input matrix or tensor. Thus, we are particularly interested in sketching algorithms that can be implemented within the same framework (a small number of sequential passes).

We conclude this section by discussing applications of the sparse sketches of Definition 1.1. In the case of matrices, there are at least three important applications: approximate eigenvector computations, semi-definite programming (SDP) solvers, and matrix completion. The first two applications are based on the fact that, given a vector $x \in \mathbb{R}^n$, the product $\mathcal{A}x$ can be approximated by $\tilde{\mathcal{A}}x$ with a bounded loss in accuracy. The running time of the latter matrix-vector product is proportional to the number of non-zeros in $\tilde{\mathcal{A}}$, thus leading to immediate computational savings. This fast matrix-vector product operation can then be used to approximate eigenvectors and eigenvalues of matrices [1, 2, 7] via subspace iteration methods; yet another application would be a quick estimate of the Krylov subspace of a matrix. Additionally [6, 12] argue that fast matrix-vector products are useful in SDP solvers. The third application domain of sparse sketches is the so-called matrix completion problem, an active research area of growing interest, where the user only has access to $\tilde{\mathcal{A}}$ (typically formed by sampling a small number of elements of \mathcal{A} uniformly at random) and the goal is to reconstruct the entries of \mathcal{A} as accurately as possible. The motivation underlying the matrix completion problem stems from recommender systems and collaborative filtering and was initially discussed in [8]. More recently, methods using bounds on $\mathcal{A} - \tilde{\mathcal{A}}$ and trace minimization algorithms have demonstrated exact reconstruction of \mathcal{A} under – rather restrictive – assumptions [10, 11]. We expect that our work here will stimulate research towards generalizing matrix completion to tensor completion. More specifically, our tensor spectral norm bound could be a key ingredient in analyzing tensor completion algorithms, just like similar bounds for matrix sparsification were critical in matrix completion [10, 11]. Finally, similar applications in recommendation systems, collaborative filtering, monitoring IP traffic patterns over time, etc. exist for the $d > 2$ case in Definition 1.1; see [40, 31, 32] for details.

1.1 Our algorithm and our main theorem

Our main algorithm (Algorithm 1) zeroes out “small” elements of the tensor \mathcal{A} , keeps “large” elements of the tensor \mathcal{A} , and randomly samples the remaining elements of the tensor \mathcal{A} with a probability that depends on their magnitude. The following theorem is our main quality-of-approximation result for Algorithm 1.

Theorem 1 Let $\mathcal{A} \in \mathbb{R}^{n \times \dots \times n}$ be an order- d tensor and let $\tilde{\mathcal{A}}$ be constructed as described in Algorithm 1. Assume that $n \geq 320$. For $d \geq 3$, if the sampling parameter s satisfies

$$s = \Omega \left(\frac{d^3 20^{2d} n^{d/2} \ln^d n}{\varepsilon^2} \max \left\{ 1, \frac{\ln^{d+1} n}{n^{d/2-1}} \right\} \|\mathcal{A}\|_F^2 \right), \quad (1.2)$$

then, with probability at least $1 - n^{-2d}$,

$$\|\mathcal{A} - \tilde{\mathcal{A}}\|_2 \leq \varepsilon,$$

where the tensor spectral norm $\|\cdot\|_2$ is defined in (2.1). For $d = 2$, the same spectral norm bound holds whenever the sampling parameter s satisfies

$$s = \Omega \left(\frac{n \ln^5 n}{\varepsilon^2} \|\mathcal{A}\|_F^2 \right). \quad (1.3)$$

The number of samples s in Theorem 1 involves the tensor Frobenius norm. In the following corollary, we restate the theorem by using the stable rank of a tensor, denoted by $\text{sr}(\mathcal{A})$. The stable rank of a tensor is defined analogously to the stable rank

of a matrix, namely the ratio

$$\text{sr}(\mathcal{A}) \triangleq \frac{\|\mathcal{A}\|_F^2}{\|\mathcal{A}\|_2^2}.$$

Corollary 1 Let $\mathcal{A} \in \mathbb{R}^{n \times \dots \times n}$ (assume $n \geq 320$) be an order- d tensor and let $\tilde{\mathcal{A}}$ be constructed as described in Algorithm 1. If $n \geq \ln^8 n$ and the sampling parameter s is set to

$$s = \Omega \left(\frac{d^2 20^{2d} n^{d/2} \ln^d n}{\varepsilon^2} \text{sr}(\mathcal{A}) \right),$$

then, with probability at least $1 - n^{-2d}$,

$$\|\mathcal{A} - \tilde{\mathcal{A}}\|_2 \leq \varepsilon \|\mathcal{A}\|_2.$$

For $d = 2$, the sampling parameter s is simplified to $s = \Omega \left(\frac{n \ln^5 n}{\varepsilon^2} \text{sr}(A) \right)$.

In both Theorem 1 and Corollary 1, $\tilde{\mathcal{A}}$ has, in expectation, at most $2s$ non-zero entries and the construction of $\tilde{\mathcal{A}}$ can be implemented in one pass over the input tensor/matrix \mathcal{A} . Towards that end, we need to combine Algorithm 1 with the SAMPLE algorithm presented in Section 4.1 of [2]. Finally, in the context of Definition 1.1, our result essentially shows that we can get a sparse sketch $\tilde{\mathcal{A}}$ with $2s$ non-zero entries. In Theorem 1 and Corollary 1, we have not made any attempt to optimize the constants which could potentially be reduced. In addition, when $n \geq \ln^8 n$, the maximum value in (1.2) is at most one and the sampling parameter can be simplified to $s = \Omega \left(\frac{n^{d/2} \ln^d n}{\varepsilon^2} \text{sr}(\mathcal{A}) \right)$. Ignoring the polylog factor, the theorem implies that out of the n^d entries of the tensor, the algorithm only needs to selectively keep $\Omega(n^{d/2} \text{sr}(\mathcal{A}))$ entries and zero out the rest, while accurately approximating the spectral norm of the original tensor.

Finally, we discuss our bound in light of the so-called Kruskal and Tucker rank of a tensor. Let $\text{kr}(\mathcal{A})$ be the Kruskal rank of the d -mode tensor \mathcal{A} ; see [23] for the definition of the Kruskal rank and notice that the Kruskal rank is equal to the matrix rank when d is equal to two. It is known that the number of degrees of freedom of a tensor is of the order $n \text{kr}(\mathcal{A})$. While, in general, the inequality $\text{sr}(\mathcal{A}) \leq \text{kr}(\mathcal{A})$ does not hold, it does hold for the $d = 2$ case as well as for some tensors that can be orthogonally decomposed [22]. Another better way to bound the stable rank of a tensor is via the Tucker decomposition, which is similar to singular value decomposition of a matrix (see [23] for the definition). Decompose the order- d tensor \mathcal{A} via

$$\mathcal{A} = \sum_{i_1=1}^{k_1} \cdots \sum_{i_d=1}^{k_d} g_{i_1 \dots i_d} u_{i_1} \times_1 \cdots \times_d v_{i_d} = \mathcal{G} \times_1 U \cdots \times_d V$$

where U, \dots, V are orthogonal matrices of size $n \times k_1, \dots, n \times k_d$, respectively; \mathcal{G} is the core tensor of size $k_1 \times \dots \times k_d$. Here, the tensor-vector product is defined later in Section 2.1. The tuple (k_1, \dots, k_d) is called the Tucker rank of the tensor \mathcal{A} where each k_i is the column rank of the matrix $A_{(i)}$ constructed by unfolding \mathcal{A} along the i th direction. It can be easily seen that the degree of freedom of \mathcal{A} is roughly $n \sum_{i=1}^d k_i + \prod_{i=1}^d k_i$. In addition, the tensor Frobenius norm is

$$\|\mathcal{A}\|_F^2 = \|\mathcal{G}\|_F^2 \leq \left(\prod_{i=1}^d k_i \right) \max_{i_1, \dots, i_d} g_{i_1 \dots i_d}^2,$$

and the spectral norm of \mathcal{A} (see Section 2.1 for the definition) is crudely lower bounded by $\max_{i_1, \dots, i_d} g_{i_1 \dots i_d}$. Combining these two bounds and the fact that $\|\mathcal{A}\|_F \geq \|\mathcal{A}\|$ yield

$$1 \leq \text{sr}(\mathcal{A}) \leq \prod_{i=1}^d k_i.$$

In these situations, Corollary 1 essentially implies that in order for the sampled tensor to be close to the original one, the number of samples required is at most on the order of $\Omega(n^{d/2} \prod_{i=1}^d k_i)$, which is proportional to $\Omega(n^{d/2})$ for low Tucker rank tensor. This bound is substantially larger than the tensor's degree of freedom $n \sum_{i=1}^d k_i + \prod_{i=1}^d k_i$. An open question is whether the $d/2$ power in the number of samples can be removed?



1: **Input:** order- d tensor $\mathcal{A} \in \mathbb{R}^{n \times n \times \dots \times n}$, sampling parameter s .
For all $i_1, \dots, i_d \in [n] \times \dots \times [n]$ **do**

- **If** $\mathcal{A}_{i_1 \dots i_d}^2 \leq \frac{\ln^d n}{n^{d/2}} \frac{\|\mathcal{A}\|_F^2}{s}$ **then**

$$\widetilde{\mathcal{A}}_{i_1 \dots i_d} = 0,$$
- **ElseIf** $\mathcal{A}_{i_1 \dots i_d}^2 \geq \frac{\|\mathcal{A}\|_F^2}{s}$ **then**

$$\widetilde{\mathcal{A}}_{i_1 \dots i_d} = \mathcal{A}_{i_1 \dots i_d},$$
- **Else**

$$\widetilde{\mathcal{A}}_{i_1 \dots i_d} = \begin{cases} \frac{\mathcal{A}_{i_1 \dots i_d}}{p_{i_1 \dots i_d}} & , \text{with probability } p_{i_1 \dots i_d} = \frac{s \mathcal{A}_{i_1 \dots i_d}^2}{\|\mathcal{A}\|_F^2} \\ 0 & , \text{with probability } 1 - p_{i_1 \dots i_d} \end{cases}$$

2: **Output:** Tensor $\widetilde{\mathcal{A}} \in \mathbb{R}^{n \times n \times \dots \times n}$.

Algorithm 1: Tensor Sparsification Algorithm

1.2 Comparison with prior work

To the best of our knowledge, for $d > 2$, there exists no prior work on element-wise tensor sparsification that provides results comparable to Theorem 1. It is worth noting that the work of [40] deals with the Frobenius norm of the tensor, which is much easier to manipulate, and its main theorem is focused on approximating the so-called HOSVD of a tensor, as opposed to decomposing the tensor as a sum of rank-one components.

For the $d = 2$ case, prior work does exist and we will briefly compare our results in Corollary 1 with current state-of-the-art. In summary, our result in Corollary 1 outperforms prior work, in the sense that, using the same accuracy parameter ϵ in Definition 1.1, the resulting matrix $\widetilde{\mathcal{A}}$ has fewer non-zero elements. In [1, 2] the authors presented a sampling method that requires at least $O(\text{st}(\mathcal{A}) n \ln^4 n / \epsilon^2)$ non-zero entries in $\widetilde{\mathcal{A}}$ in order to achieve the proposed accuracy guarantee. (Here $\text{st}(\mathcal{A})$ denotes the stable rank of the matrix \mathcal{A} that is always upper bounded by the rank of \mathcal{A} .) Our result increases the sampling complexity by a $\ln n$ factor. This increment is due to the more general model (tensor) we consider. In [37, 9] the authors proposed sparsification schemes for structural Laplacian matrix and thus required smaller amount of non-zero entries, while our method can apply for any matrix \mathcal{A} with no restriction on its structure. It is harder to compare our method to the work of [7], which depends on the $\sum_{i,j=1}^n |\mathcal{A}_{ij}|$. The latter quantity is, in general, upper bounded only by $n \|\mathcal{A}\|_F$, in which case the sampling complexity of [7] is much worse, namely $O(\text{st}(\mathcal{A}) n^{3/2} / \epsilon)$. However, it is worth noting that the result of [7] is appropriate for matrices whose “energy” is focused only on a small number of entries, as well as that their bound holds with much higher probability than ours.

In parallel with our work, two related results appeared in ArXiv. First, [21] studied the $\|\cdot\|_{\infty \rightarrow 2}$ and $\|\cdot\|_{\infty \rightarrow 1}$ norms in the matrix sparsification context. The authors also presented a sampling scheme for the problem of Definition 1.1. Additionally, [17] leveraged a powerful matrix Bernstein inequality and improved the sampling complexity of Corollary 1 by an $O(\ln^2 n)$ factor. Subsequently to our work, [3] presented an alternative approach to [17] that is based on ℓ_1 sampling, e.g., sampling with respect to the absolute values of the entries of a matrix as opposed to their squares. However, neither of the aforementioned results generalizes to tensors. Indeed, establishing analogous bounds for d -mode tensors is a major open problem.

1.3 Bounding the spectral norm of random tensors

An important contribution of our work is the technical analysis and, in particular, the proof of a bound for the spectral norm of random tensors that is necessary in order to prove Theorem 1. It is worth noting that all known results for the $d = 2$ case of Theorem 1 are either combinatorial in nature (e.g., the proofs of [1, 2] are based on the result of [20], whose proof is fundamentally combinatorial) or use simple ϵ -net arguments [7]. The only exceptions are the recent results in [17, 21] which leverage powerful Bernstein and Chernoff-type inequalities for matrices [38]. It is also important to emphasize that over the last few years, there are active research in establish sharp bound for the sum of random matrices [4, 34, 38] (see the tutorial paper [39] of Tropp for more references). As stated above, none of these approaches can be extended to the $d > 2$

case; indeed, the $d > 2$ case seems to require novel tools and methods. In our work, we are only able to prove the following theorem using the so-called *entropy-concentration tradeoff*, an analysis technique that was originally developed by Latala [24] and has been recently investigated by Mark Rudelson and Roman Vershynin [35, 41]. The following theorem presents a spectral norm bound for random tensors and is fundamental in proving Theorem 1.

Theorem 2 Let $\widehat{\mathcal{A}} \in \mathbb{R}^{n \times \dots \times n}$ be an order- d tensor and let \mathcal{A} be a random tensor of the same dimensions whose entries are independent and $\mathbb{E}\mathcal{A} = \widehat{\mathcal{A}}$. For any $\lambda \leq \frac{1}{64}$, assume that $1 \leq q \leq 2d\lambda n \ln \frac{5e}{\lambda}$. Then,

$$\left(\mathbb{E} \left\| \mathcal{A} - \widehat{\mathcal{A}} \right\|_2^q \right)^{\frac{1}{q}} \leq c 8^d \sqrt{2d \ln \left(\frac{5e}{\lambda} \right)} \left(\left[\log_2 \left(\frac{1}{\lambda} \right) \right]^{d-1} \left(\sum_{j=1}^d \mathbb{E}_{\mathcal{A}} \alpha_j^q \right)^{\frac{1}{q}} + \sqrt{\lambda n} (\mathbb{E}_{\mathcal{A}} \beta^q)^{\frac{1}{q}} \right),$$

where

$$\alpha_j^2 \triangleq \max_{i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_d} \left(\sum_{i_j=1}^n \mathcal{A}_{i_1 \dots i_{j-1} i_j i_{j+1} \dots i_d}^2 \right) \quad \text{and} \quad \beta = \max_{i_1, \dots, i_d} |\mathcal{A}_{i_1 \dots i_d}|.$$

In the above inequality, c is a small constant and $\|\cdot\|_2$ refers to the tensor spectral norm defined in Section 2.1.

An immediate corollary of the above theorem emerges by setting tensor $\widehat{\mathcal{A}}$ to zero.

Corollary 2 Let $\mathcal{B} \in \mathbb{R}^{n \times \dots \times n}$ be a random order- d tensor, whose entries are independent, zero-mean, random variables. For any $\lambda \leq \frac{1}{64}$, assume that $1 \leq q \leq 2d\lambda n \ln \frac{5e}{\lambda}$. Then,

$$\left(\mathbb{E} \|\mathcal{B}\|_2^q \right)^{\frac{1}{q}} \leq c 8^d \sqrt{2d \ln \left(\frac{5e}{\lambda} \right)} \left(\left[\log_2 \left(\frac{1}{\lambda} \right) \right]^{d-1} \left(\sum_{j=1}^d \mathbb{E}_{\mathcal{B}} \alpha_j^q \right)^{\frac{1}{q}} + \sqrt{\lambda n} (\mathbb{E}_{\mathcal{B}} \beta^q)^{\frac{1}{q}} \right),$$

where

$$\alpha_j^2 \triangleq \max_{i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_d} \left(\sum_{i_j=1}^n \mathcal{B}_{i_1 \dots i_{j-1} i_j i_{j+1} \dots i_d}^2 \right) \quad \text{and} \quad \beta = \max_{i_1, \dots, i_d} |\mathcal{B}_{i_1 \dots i_d}|.$$

In the above inequality, c is a small constant and $\|\cdot\|_2$ refers to the tensor spectral norm defined in Section 2.1.

As will be clear in the proof, the parameter λ defines the entropy-concentration tradeoff. Depending on particular properties of the random tensor \mathcal{B} , one can set the parameter λ so that the bound on the right-hand side is optimized. In particular, when the entries of \mathcal{B} are of similar magnitudes (formally, $\max_j \alpha_j^2 = c_1 n \beta^2$), we can choose λ to be a small constant. (Note that we always have $\max_j \alpha_j^2 \leq n \beta^2$.) In this case, we have a simplified result.

Corollary 3 Let $\mathcal{B} \in \mathbb{R}^{n \times \dots \times n}$ be a random order- d tensor, whose entries are independent, zero-mean, random variables. Assume that $1 \leq q \leq Cdn$. Also, assume that $c_1 n \beta^2 \leq \max_j \alpha_j^2 \leq C_1 n \beta^2$. Then,

$$\left(\mathbb{E} \|\mathcal{B}\|_2^q \right)^{\frac{1}{q}} \leq c_d 8^d \sqrt{d} \left(\sum_{j=1}^d \mathbb{E}_{\mathcal{B}} \max_{i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_d} \left(\sum_{i_j=1}^n \mathcal{B}_{i_1 \dots i_{j-1} i_j i_{j+1} \dots i_d}^2 \right) \right)^{\frac{q}{2}}^{\frac{1}{q}}.$$

In the above inequality, c_d is a small constant depending on d and $\|\cdot\|_2$ refers to the tensor spectral norm defined in Section 2.1.

We note that this bound is optimal since $\|\mathcal{B}\|_2$ is always lower bounded by the $\max_{i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_d} \left(\sum_{i_j=1}^n \mathcal{B}_{i_1 \dots i_{j-1} i_j i_{j+1} \dots i_d}^2 \right)^{\frac{1}{2}}$. We also note that for the matrix case ($d = 2$), the result of Corollary 3 has a very similar structure with the result of [24]. In fact, our proof strategy is borrowed from [24], with significant modifications in order to adapt it to higher-order tensors. For a general random tensor, we can use the crude bound $\beta \leq \max_j \alpha_j$ and also set $\lambda = \frac{(\ln n)^{2(d-1)}}{n}$. Then, the following corollary provides a bound for the spectral norm of the random tensor.



Corollary 4 Let $\mathcal{B} \in \mathbb{R}^{n \times \dots \times n}$ be a random order- d tensor, whose entries are independent, zero-mean, random variables. Assume that $1 \leq q \leq Cd \ln n$. Then,

$$\left(\mathbb{E} \|\mathcal{B}\|_2^q\right)^{\frac{1}{q}} \leq c_d 8^d (\ln n)^{d-1/2} \left(\sum_{j=1}^d \mathbb{E}_{\mathcal{B}} \max_{i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_d} \left(\sum_{i_j=1}^n \mathcal{B}_{i_1 \dots i_{j-1} i_j i_{j+1} \dots i_d}^2 \right)^{\frac{q}{2}} \right)^{\frac{1}{q}}.$$

In the above inequality, c_d is a small constant depending on d and $\|\cdot\|_2$ refers to the tensor spectral norm defined in Section 2.1.

2. Preliminaries

2.1 Notation

We will use $[n]$ to denote the set $\{1, 2, \dots, n\}$. c_0, c_1, c_2 , etc. will denote small numerical constants, whose values change from one section to the next. $\mathbb{E}X$ will denote the expectation of a random variable X . When X is a matrix, then $\mathbb{E}X$ denotes the element-wise expectation of each entry of X . Similarly, $\mathbf{Var}(X)$ denotes the variance of the random variable X and $\mathbb{P}(\mathcal{E})$ denotes the probability of event \mathcal{E} . Finally, $\ln x$ denotes the natural logarithm of x and $\log_2 x$ denotes the base two logarithm of x .

We briefly remind the reader of vector norm definitions. Given a vector $x \in \mathbb{R}^n$ the ℓ_2 norm of x is denoted by $\|x\|_2$ and is equal to the square root of the sum of the squares of the elements of x . Also, the ℓ_0 norm of the vector x is equal to the number of non-zero elements in x . Finally, given a Lipschitz function $f: \mathbb{R}^n \mapsto \mathbb{R}$ we define the Lipschitz norm of f to be

$$\|f\|_L = \sup_{x, y \in \mathbb{R}^n} \frac{|f(x) - f(y)|}{\|x - y\|_2}.$$

For any d -mode or order- d tensor $\mathcal{A} \in \mathbb{R}^{n \times \dots \times n}$, its Frobenius norm $\|\mathcal{A}\|_F$ is defined as the square root of the sum of the squares of its elements. We now define tensor-vector products as follows: let x, y be vectors in \mathbb{R}^n . Then,

$$\begin{aligned} \mathcal{A} \times_1 x &= \sum_{i=1}^n \mathcal{A}_{ijk\dots l} x_i, \\ \mathcal{A} \times_2 x &= \sum_{j=1}^n \mathcal{A}_{ijk\dots l} x_j, \\ \mathcal{A} \times_3 x &= \sum_{k=1}^n \mathcal{A}_{ijk\dots l} x_k, \text{ etc.} \end{aligned}$$

Note that the outcome of the above operations is an order- $(d-1)$ tensor. The above definition may be extended to handle multiple tensor-vector products, e.g.,

$$\mathcal{A} \times_1 x \times_2 y = \sum_{i=1}^n \sum_{j=1}^n \mathcal{A}_{ijk\dots l} x_i y_j.$$

Note that the outcome of the above operation is an order- $(d-2)$ tensor. Using this definition, the spectral norm of a tensor is defined as

$$\|\mathcal{A}\|_2 = \sup_{x_1 \dots x_d \in \mathbb{S}^n} |\mathcal{A} \times_1 x_1 \dots \times_d x_d|, \quad (2.1)$$

where \mathbb{S}^n is the unit sphere in n -dimensional space. In words, the vectors $x_i \in \mathbb{R}^n$ are unit vectors, i.e., $\|x_i\|_2 = 1$ for all $i \in [d]$. It is worth noting that $\mathcal{A} \times_1 x_1 \dots \times_d x_d \in \mathbb{R}$ and also that our tensor norm definitions when restricted to matrices (order-2 tensors) coincide with the standard definitions of matrix norms.

We also present an inequality that will be useful in our work. For any two d -mode tensors \mathcal{A} and \mathcal{B} of the same dimensions and any scalar $q \geq 1$,

$$\|\mathcal{A} + \mathcal{B}\|_2^q \leq 2^{q-1} (\|\mathcal{A}\|_2^q + \|\mathcal{B}\|_2^q). \quad (2.2)$$

The proof is quite simple. Notice that for nonnegative scalars x and y , $(x+y)^q \leq 2^{q-1}(x^q + y^q)$ for $q \geq 1$ (see Lemma 11 for a more general proof). Thus, for any $x_1, \dots, x_d \in \mathbb{S}^n$,

$$|\mathcal{A} \times_1 x_1 \dots \times_d x_d + \mathcal{B} \times_1 x_1 \dots \times_d x_d|^q \leq 2^{q-1} |\mathcal{A} \times_1 x_1 \dots \times_d x_d|^q + 2^{q-1} |\mathcal{B} \times_1 x_1 \dots \times_d x_d|^q.$$

Taking the maximum of both sides completes the proof.



2.2 Measure concentration

We will need the following version of Bennett's inequality.

Lemma 1 Let X_1, X_2, \dots, X_n be independent, zero-mean, random variables with $|X_i| \leq 1$. For any $t \geq \frac{3}{2} \sum_{i=1}^n \mathbf{Var}(X_i) > 0$

$$\mathbb{P}\left(\sum_{i=1}^n X_i > t\right) \leq e^{-t/2}.$$

This version of Bennett's inequality can be derived from the standard one, stating that

$$\mathbb{P}\left(\sum_{i=1}^n X_i > t\right) \leq e^{-\sigma^2 h(t/\sigma^2)}.$$

Here $\sigma^2 = \sum_{i=1}^n \mathbf{Var}(X_i)$ and $h(u) = (1+u)\ln(1+u) - u$. Lemma 1 follows using the fact that $h(u) \geq u/2$ for $u \geq 3/2$. We also remind the reader of the following well-known result on measure concentration (see, for example, eqn. (1.4) of [29]).

Lemma 2 Let $f: \mathbb{R}^n \mapsto \mathbb{R}$ be a Lipschitz function and let $\|f\|_L$ be its Lipschitz norm. If $g \in \mathbb{R}^n$ is a standard Gaussian vector (i.e., a vector whose entries are independent standard Gaussian random variables), then for all $t > 0$

$$\mathbb{P}\left(f(g) \geq \mathbb{E}f(g) + t\sqrt{2}\|f\|_L\right) \leq e^{-t^2}.$$

The following lemma, whose proof may be found in the Appendix, converts a probabilistic bound for the random variable X to an expectation bound for X^q , for all $q \geq 1$, and might be of independent interest.

Lemma 3 Let X be a random variable assuming non-negative values. For all $t \geq 0$ and non-negative a, b , and h :

(a) If $\mathbb{P}(X \geq a + tb) \leq e^{-t+h}$, then, for all $q \geq 1$,

$$\mathbb{E}X^q \leq 2(a + bh + bq)^q.$$

(b) If $\mathbb{P}(X \geq a + tb) \leq e^{-t^2+h}$, then, for all $q \geq 1$,

$$\mathbb{E}X^q \leq 3\sqrt{q}\left(a + b\sqrt{h} + b\sqrt{q/2}\right)^q.$$

Finally, we present an ε -net argument that we will repeatedly use. Recall from Lemma 3.18 of [28] that the cardinality of an ε -net on the unit sphere is at most $(1 + 2/\varepsilon)^n$. The following lemma essentially generalizes the results of Lecture 6 of [42] to order- d tensors.

Lemma 4 Let \mathbb{N} be an ε -net for a set B associated with a norm $\|\cdot\|$. Then, the spectral norm of a d -mode tensor \mathcal{A} is bounded by

$$\sup_{x_1 \dots x_{d-1} \in B} \|\mathcal{A} \times_1 x_1 \dots \times_{d-1} x_{d-1}\|_2 \leq \left(\frac{1}{1-\varepsilon}\right)^{d-1} \sup_{x_1 \dots x_{d-1} \in \mathbb{N}} \|\mathcal{A} \times_1 x_1 \dots \times_{d-1} x_{d-1}\|_2.$$

Notice that, using our notation, $\mathcal{A} \times_1 x_1 \dots \times_{d-1} x_{d-1}$ is a vector in \mathbb{R}^n . The proof of the lemma may be found in the Appendix. An immediate implication of our result is that the spectral norm of a d -mode tensor \mathcal{A} is bounded by

$$\|\mathcal{A}\|_2 \leq \left(\frac{1}{1-\varepsilon}\right)^{d-1} \sup_{x_1 \dots x_{d-1} \in \mathbb{N}} \|\mathcal{A} \times_1 x_1 \dots \times_{d-1} x_{d-1}\|_2,$$

where \mathbb{N} is the ε -net for the unit sphere \mathbb{S}^{n-1} in \mathbb{R}^n .

3. Bounding the spectral norm of random tensors

This section will focus on proving Theorem 2, which essentially bounds the spectral norm of random tensors. Towards that end, we will first apply a symmetrization argument following the lines of [24]. This argument will allow us to reduce the



task-at-hand to bounding the spectral norm of a Gaussian random tensor. As a result, we will develop such an inequality by employing the so-called entropy-concentration technique, which has been developed by Mark Rudelson and Roman Vershynin [35, 41].

For simplicity of exposition and to avoid carrying multiple indices, we will focus on proving Theorem 2 for order-3 tensors (i.e., $d = 3$). Throughout the proof, we will carefully comment on derivations where d (the number of modes of the tensor) affects the bounds of the intermediate results. Notice that if $d = 3$, then a tensor $\mathcal{A} \in \mathbb{R}^{n \times n \times n}$ may be expressed as

$$\mathcal{A} = \sum_{i,j,k=1}^n \mathcal{A}_{ijk} \cdot e_i \otimes e_j \otimes e_k. \quad (3.1)$$

In the above, the vectors $e_i \in \mathbb{R}^n$ (for all $i \in [n]$) denote the standard basis for \mathbb{R}^n and \otimes denotes the outer product operation. Thus, for example, $e_i \otimes e_j \otimes e_k$ denotes an tensor in $\mathbb{R}^{n \times n \times n}$ whose (i, j, k) -th entry is equal to one, while all other entries are equal to zero.

3.1 A Gaussian symmetrization inequality

The main result of this section can be summarized in Lemma 5. In words, the lemma states that, by losing a factor of $\sqrt{2\pi}$, we can independently randomize each entry of \mathcal{A} via a Gaussian random variable. Thus, we essentially reduce the problem of finding a bound for the spectral norm of a tensor \mathcal{A} to finding a bound for the spectral norm of a Gaussian random tensor.

Lemma 5 Let $\widehat{\mathcal{A}} \in \mathbb{R}^{n \times n \times n}$ be any order-3 tensor and let \mathcal{A} be a random tensor of independent entries and of the same dimensions such that $\mathbb{E}_{\mathcal{A}} \mathcal{A} = \widehat{\mathcal{A}}$. Also let the g_{ijk} be Gaussian random variables for all triples $(i, j, k) \in [n] \times [n] \times [n]$. Then for any $q \geq 1$,

$$\mathbb{E}_{\mathcal{A}} \left\| \mathcal{A} - \widehat{\mathcal{A}} \right\|_2^q \leq (\sqrt{2\pi})^q \mathbb{E}_{\mathcal{A}} \mathbb{E}_g \left\| \sum_{i,j,k} g_{ijk} \mathcal{A}_{ijk} \cdot e_i \otimes e_j \otimes e_k \right\|_2^q. \quad (3.2)$$

Proof. Let \mathcal{A}' be an independent copy of the tensor \mathcal{A} . By applying a symmetrization argument and Jensen's inequality, we get

$$\mathbb{E}_{\mathcal{A}} \left\| \mathcal{A} - \widehat{\mathcal{A}} \right\|_2^q = \mathbb{E}_{\mathcal{A}} \left\| \mathcal{A} - \mathbb{E}_{\mathcal{A}} \mathcal{A} \right\|_2^q = \mathbb{E}_{\mathcal{A}} \left\| \mathcal{A} - \mathbb{E}_{\mathcal{A}'} \mathcal{A}' \right\|_2^q \leq \mathbb{E}_{\mathcal{A}} \mathbb{E}_{\mathcal{A}'} \left\| \mathcal{A} - \mathcal{A}' \right\|_2^q.$$

Note that the entries of the tensor $\mathcal{A} - \mathcal{A}'$ are independent symmetric random variables and thus their distribution is the same as the distribution of the random variables $\varepsilon_{ijk} (\mathcal{A}_{ijk} - \mathcal{A}'_{ijk})$, where the ε_{ijk} 's are independent, symmetric, Bernoulli random variables assuming the values $+1$ and -1 with equal probability. Hence,

$$\begin{aligned} \mathbb{E}_{\mathcal{A}} \mathbb{E}_{\mathcal{A}'} \left\| \mathcal{A} - \mathcal{A}' \right\|_2^q &= \mathbb{E}_{\mathcal{A}} \mathbb{E}_{\mathcal{A}'} \mathbb{E}_{\varepsilon} \left\| \sum_{i,j,k} \varepsilon_{ijk} (\mathcal{A}_{ijk} - \mathcal{A}'_{ijk}) e_i \otimes e_j \otimes e_k \right\|_2^q \\ &\leq 2^{q-1} \mathbb{E}_{\mathcal{A}} \mathbb{E}_{\varepsilon} \left\| \sum_{i,j,k} \varepsilon_{ijk} \mathcal{A}_{ijk} e_i \otimes e_j \otimes e_k \right\|_2^q \\ &\quad + 2^{q-1} \mathbb{E}_{\mathcal{A}'} \mathbb{E}_{\varepsilon} \left\| \sum_{i,j,k} \varepsilon_{ijk} \mathcal{A}'_{ijk} e_i \otimes e_j \otimes e_k \right\|_2^q. \end{aligned}$$

Here the inequality follows from eqn. (2.2). Now, since the entries of the tensors \mathcal{A} and \mathcal{A}' have the same distribution, we get

$$\mathbb{E}_{\mathcal{A}} \mathbb{E}_{\mathcal{A}'} \left\| \mathcal{A} - \mathcal{A}' \right\|_2^q \leq 2^q \mathbb{E}_{\mathcal{A}} \mathbb{E}_{\varepsilon} \left\| \sum_{i,j,k} \varepsilon_{ijk} \mathcal{A}_{ijk} e_i \otimes e_j \otimes e_k \right\|_2^q. \quad (3.3)$$

We now proceed with the Gaussian symmetrization argument. Let g_{ijk} for all i, j , and k be independent Gaussian random variables. It is well-known that $\mathbb{E} |g_{ijk}| = \sqrt{2/\pi}$. Using Jensen's inequality, we get

$$\mathbb{E}_{\mathcal{A}} \mathbb{E}_{\varepsilon} \left\| \sum_{i,j,k} \varepsilon_{ijk} \mathcal{A}_{ijk} e_i \otimes e_j \otimes e_k \right\|_2^q = \left(\frac{\pi}{2} \right)^{q/2} \mathbb{E}_{\mathcal{A}} \mathbb{E}_{\varepsilon} \left\| \sum_{i,j,k} \varepsilon_{ijk} \mathcal{A}_{ijk} (\mathbb{E}_g |g_{ijk}|) \cdot e_i \otimes e_j \otimes e_k \right\|_2^q \quad +$$

+

$$\begin{aligned}
&\leq \left(\frac{\pi}{2}\right)^{q/2} \mathbb{E}_{\mathcal{A}} \mathbb{E}_{\varepsilon} \mathbb{E}_g \left\| \sum_{i,j,k} \varepsilon_{ijk} \mathcal{A}_{ijk} |g_{ijk}| \cdot e_i \otimes e_j \otimes e_k \right\|_2^q \\
&= \left(\frac{\pi}{2}\right)^{q/2} \mathbb{E}_{\mathcal{A}} \mathbb{E}_g \left\| \sum_{i,j,k} g_{ijk} \mathcal{A}_{ijk} \cdot e_i \otimes e_j \otimes e_k \right\|_2^q.
\end{aligned}$$

The last equality holds since $\varepsilon_{ijk} |g_{ijk}|$ and g_{ijk} have the same distribution. Thus, combining the above with eqn. (3.3) we have finally obtained the Gaussian symmetrization inequality. \square

3.2 Bounding the spectral norm of a Gaussian random tensor

In this section we will seek a bound for the spectral norm of the tensor \mathcal{H} whose entries \mathcal{H}_{ijk} are equal to $g_{ijk} \mathcal{A}_{ijk}$ (we are using the notation of Lemma 5). Obviously, the entries of \mathcal{H} are independent, zero-mean Gaussian random variables. We would like to estimate

$$\mathbb{E}_g \|\mathcal{H}\|^q = \mathbb{E}_g \sup_{x,y} \|\mathcal{H} \times_1 x \times_2 y\|_2^q$$

over all unit vectors $x, y \in \mathbb{R}^n$. Our first lemma computes the expectation of the quantity $\|\mathcal{H} \times_1 x \times_2 y\|_2$ for a fixed pair of unit vectors x and y .

Lemma 6 Given a pair of unit vectors x and y

$$\mathbb{E}_g \|\mathcal{H} \times_1 x \times_2 y\|_2 \leq \sqrt{\max_{i,j} \sum_k \mathcal{A}_{ijk}^2}.$$

Proof. Let $s = \mathcal{H} \times_1 x \times_2 y \in \mathbb{R}^n$ and let $s_k = \sum_{i,j} \mathcal{H}_{ijk} x_i y_j$ for all $k \in [n]$. Thus,

$$\begin{aligned}
\|s\|_2^2 &= \sum_k \left(\sum_{i,j} \mathcal{H}_{ijk} x_i y_j \right)^2 \\
&= \sum_{i,j,k} \mathcal{H}_{ijk}^2 x_i^2 y_j^2 + \sum_k \sum_{i,j \neq p,q} \mathcal{H}_{ijk} \mathcal{H}_{pqk} x_i y_j x_p y_q.
\end{aligned}$$

Using $\mathbb{E}_g \mathcal{H}_{ijk} = 0$ and $\mathbb{E}_g \mathcal{H}_{ijk}^2 = \mathcal{A}_{ijk}^2 \mathbb{E}_g g_{ijk}^2 = \mathcal{A}_{ijk}^2$ we conclude that

$$\mathbb{E}_g \|s\|_2^2 = \sum_{i,j,k} \mathcal{A}_{ijk}^2 x_i^2 y_j^2 = \sum_i x_i^2 \sum_j y_j^2 \sum_k \mathcal{A}_{ijk}^2 \leq \max_{i,j} \sum_k \mathcal{A}_{ijk}^2.$$

The last inequality follows since $\|x\|_2 = \|y\|_2 = 1$. Using $\mathbb{E}_g \|s\|_2 \leq \sqrt{\mathbb{E}_g \|s\|_2^2}$ we obtain the claim of the lemma. \square

The next lemma argues that $\|\mathcal{H} \times_1 x \times_2 y\|_2$ is concentrated around its mean (which we just computed) with high probability.

Lemma 7 Given a pair of unit vectors x and y

$$\mathbb{P} \left(\|\mathcal{H} \times_1 x \times_2 y\|_2 \geq \sqrt{\max_{i,j} \sum_k \mathcal{A}_{ijk}^2} + t\sqrt{2} \max_k \sqrt{\sum_{i,j} \mathcal{A}_{ijk}^2 x_i^2 y_j^2} \right) \leq e^{-t^2}. \quad (3.4)$$

Proof. Consider the vector $s = \mathcal{H} \times_1 x \times_2 y \in \mathbb{R}^n$ and recall that $\mathcal{H}_{ijk} = g_{ijk} \mathcal{A}_{ijk}$ to get

$$\begin{aligned}
s &= \sum_{i,j,k} (\mathcal{H}_{ijk} x_i y_j) e_k \\
&= \sum_k \left(\sum_{i,j} \mathcal{H}_{ijk} x_i y_j \right) e_k \\
&= \sum_k \left(\sum_{i,j} g_{ijk} \mathcal{A}_{ijk} x_i y_j \right) e_k.
\end{aligned}$$

+



In the above the e_k for all $k \in [n]$ are the standard basis vectors for \mathbb{R}^n . Now observe that all $g_{ijk} \mathcal{A}_{ijk} x_i y_j$ are Gaussian random variables, which implies that their sum (over all i and j) is also a Gaussian random variable with zero mean and variance $\sum_{i,j} \mathcal{A}_{ijk}^2 x_i^2 y_j^2$. Let

$$q_k^2 = \sum_{i,j} \mathcal{A}_{ijk}^2 x_i^2 y_j^2 \quad \text{for all } k \in [n]$$

and rewrite the vector s as the sum of weighted standard Gaussian random variables:

$$s = \sum_k z_k q_k e_k.$$

In the above the z_k 's are standard Gaussian random variables for all $k \in [n]$. Let z be the vector in \mathbb{R}^n whose entries are the z_k 's and let

$$f(z) = \left\| \sum_k z_k q_k e_k \right\|_2.$$

We apply Lemma 2 to $f(z)$. It is clear that $f^2(z) = \sum_k z_k^2 q_k^2 \leq \|z\|_2^2 \max_k q_k^2$. Therefore, the Lipschitz norm of f is

$$\|f\|_L = \max_k |q_k| = \max_k \left(\sum_{i,j} \mathcal{A}_{ijk}^2 x_i^2 y_j^2 \right)^{1/2}.$$

Applying Lemma 2 and Lemma 6 completes the proof. \square

3.2.1 An ε -net construction: the entropy-concentration tradeoff argument. Given the measure concentration result of Lemma 7, one might be tempted to bound the quantity $\|\mathcal{H} \times_1 x \times_2 y\|_2$ for all unit vectors x and y by directly constructing an ε -net N on the unit sphere. Since the cardinality of N is well-known to be upper bounded by $(1 + \frac{2}{\varepsilon})^n$, it follows that by getting an estimate for the quantity $\|\mathcal{H} \times_1 x \times_2 y\|$ for a pair of vectors x and y in N and subsequently applying the union bound combined with Lemma 4, an upper bound for the norm of the tensor \mathcal{H} may be derived. Unfortunately, this simple technique does not yield a useful result: the failure probability of Lemma 7 is not sufficiently small in order to permit the application of a union bound over all vectors x and y in N .

In order to overcome this obstacle, we will apply a powerful and novel argument, the so-called *entropy-concentration tradeoff*, which was originally investigated by Latala [24] and has been recently developed by Mark Rudelson and Roman Vershynin [35, 41]. To begin with, we express a unit vector $x \in \mathbb{R}^n$ as a sum of two vectors $z, w \in \mathbb{R}^n$ satisfying certain bounds on the magnitude of their coordinates. Thus, $x = z + w$, where, for all $i \in [n]$,

$$z_i = \begin{cases} x_i & \text{if } |x_i| \geq \frac{1}{\sqrt{\lambda n}} \\ 0 & \text{, otherwise} \end{cases}$$

$$w_i = \begin{cases} x_i & \text{if } |x_i| < \frac{1}{\sqrt{\lambda n}} \\ 0 & \text{, otherwise} \end{cases}$$

In the above $\lambda \in (0, 1]$ is a small constant that will be specified later. It is easy to see that $\|z\|_2 \leq 1$, $\|w\|_2 \leq 1$, and that the number of non-zeros entries in z (i.e., the ℓ_0 norm of z) is bounded:

$$\|z\|_0 \leq \lambda n.$$

Essentially, we have ‘‘split’’ the entries of x in two vectors: a sparse vector z with a bounded number of non-zero entries and a spread vector w with entries whose magnitude is restricted. Thus, we can now divide the unit sphere into two sets:

$$B_{2,0} = \left\{ x \in \mathbb{R}^n : \|x\|_2 \leq 1, |x_i| \geq \frac{1}{\sqrt{\lambda n}} \text{ or } x_i = 0 \right\},$$

$$B_{2,\infty} = \left\{ x \in \mathbb{R}^n : \|x\|_2 \leq 1, \|x\|_\infty < \frac{1}{\sqrt{\lambda n}} \right\}.$$



Given the above two sets, we can apply an ε -net argument to each set separately. The advantage is that since vectors on $B_{2,0}$ only have a small number of non-zero entries, the size of the ε -net on $B_{2,0}$ is small. This counteracts the fact that the measure concentration bound that we get for vectors in $B_{2,0}$ is rather weak since the vectors in this set have arbitrarily large entries (upper bounded by one). On the other hand, vectors in $B_{2,\infty}$ have many non-zero coefficients of bounded magnitude. As a result, the cardinality of the ε -net on $B_{2,\infty}$ is large, but the measure concentration bound is much tighter. Combining the contribution of the sparse and the spread vectors results to a strong overall bound.

We conclude the section by noting that the above two sets are spanning the whole unit sphere \mathbb{S}^{n-1} in \mathbb{R}^n . Using the inequality $(\mathbb{E}(x+y)^q)^{1/q} \leq (\mathbb{E}x^q)^{1/q} + (\mathbb{E}y^q)^{1/q}$ we obtain

$$\left(\mathbb{E} \sup_{x,y \in \mathbb{S}^{n-1}} \|\mathcal{H} \times_1 x \times_2 y\|_2^q \right)^{1/q} \leq \left(\mathbb{E} \sup_{x,y \in B_{2,0}} \|\mathcal{H} \times_1 x \times_2 y\|_2^q \right)^{1/q} \quad (3.5)$$

$$+ \left(\mathbb{E} \sup_{x,y \in B_{2,\infty}} \|\mathcal{H} \times_1 x \times_2 y\|_2^q \right)^{1/q} \quad (3.6)$$

$$+ \left(\mathbb{E} \sup_{x \in B_{2,0}, y \in B_{2,\infty}} \|\mathcal{H} \times_1 x \times_2 y\|_2^q \right)^{1/q} \quad (3.7)$$

$$+ \left(\mathbb{E} \sup_{x \in B_{2,\infty}, y \in B_{2,0}} \|\mathcal{H} \times_1 x \times_2 y\|_2^q \right)^{1/q}. \quad (3.8)$$

3.2.2 Controlling sparse vectors. We now prove the following lemma bounding the contribution of the sparse vectors (term (3.5)) in our ε -net construction.

Lemma 8 Consider a d -mode tensor \mathcal{A} and let \mathcal{H} be the d -mode tensor after the Gaussian symmetrization argument as defined in Section 3.2. Let α and β be

$$\alpha^2 = \max \left\{ \max_{i,j} \sum_{k=1}^n \mathcal{A}_{ijk}^2, \max_{i,k} \sum_{j=1}^n \mathcal{A}_{ijk}^2, \max_{j,k} \sum_{i=1}^n \mathcal{A}_{ijk}^2 \right\}, \quad (3.9)$$

$$\beta = \max_{i,j,k} |\mathcal{A}_{ijk}|. \quad (3.10)$$

For all $q \geq 1$,

$$\left(\mathbb{E} \sup_{x,y \in B_{2,0}} \|\mathcal{H} \times_1 x \times_2 y\|_2^q \right)^{1/q} \leq (3\sqrt{q})^{1/q} 2^{(d-1)/q} \left(\alpha + \beta \sqrt{2d\lambda n \ln \frac{5e}{\lambda}} + \beta \sqrt{q} \right). \quad (3.11)$$

The expectation bound has two components: the first one relates to the maximum tensor row or column energy and the second one relates to largest entry of the tensor. While the first component involving α is fixed, the size of the set $B_{2,0}$ affects the second component which involves β . Roughly speaking, the above expectation bound is of the order of $\alpha + \beta \sqrt{\lambda n \ln 1/\lambda}$. It is also clear that λ control the size of the set $B_{2,0}$: smaller λ is associated with a smaller set $B_{2,0}$. If the entries of the tensor are spread out, then $\alpha \approx \beta \sqrt{n}$ and we can set λ to be a large constant and the expectation bound is optimal $O(\alpha)$. On the other hand, we can select a smaller value for $\lambda = c/n$ to get the bound $\alpha + \beta \sqrt{\ln n}$. We also emphasize that $\sup_{x,y \in B_{2,0}} \|\mathcal{H} \times_1 x \times_2 y\|_2$ is lower bounded by α , which can be seen by setting x and y to be basis vectors. Therefore, the above expectation bound is tight.

Proof. Let $K = \lambda n$ and let $B_{2,0,K}$ be the K -dimensional set defined by

$$B_{2,0,K} = \{x \in \mathbb{R}^K : \|x\|_2 \leq 1\}.$$

Then, the set $B_{2,0}$ corresponding to vectors with at most K non-zero entries can be expressed as a union of subsets of dimension K , i.e., $B_{2,0} = \bigcup B_{2,0,K}$. A simple counting argument indicates that there are at most $\binom{n}{K} \leq \left(\frac{en}{K}\right)^K$ such subsets. We now apply the ε -net technique to each of the subsets $B_{2,0,K}$ whose union is the set $B_{2,0}$. First, let us define $N_{B_{2,0,K}}$ to be the $1/2$ -net of a subset $B_{2,0,K}$. Lemma 3.18 of [28] bounds the cardinality of $N_{B_{2,0,K}}$ by 5^K . Applying Lemma 4 with $\varepsilon = 1/2$ we get

$$\sup_{x,y \in B_{2,0,K}} \|\mathcal{H} \times_1 x \times_2 y\|_2 \leq 2^{d-1} \sup_{x,y \in N_{B_{2,0,K}}} \|\mathcal{H} \times_1 x \times_2 y\|_2.$$

+

12 of 28

The right-hand side can be controlled by Lemma 7 which bounds the term $\mathcal{H} \times_1 x \times_2 y$ for a specific pair of unit vectors x and y . Noticing that

$$\max_k \left(\sum_{i,j} \mathcal{A}_{ijk}^2 x_i^2 y_j^2 \right)^{1/2} \leq \max_{i,j,k} |\mathcal{A}_{ijk}| \left(\sum_{i,j} x_i^2 y_j^2 \right)^{1/2} \leq \max_{i,j,k} |\mathcal{A}_{ijk}| = \beta,$$

we apply Lemma 7 and take the union bound over all $x, y \in N_{B_{2,0,K}}$ to yield

$$\mathbb{P} \left(\sup_{x,y \in B_{2,0,K}} \|\mathcal{H} \times_1 x \times_2 y\|_2 \geq 2^{d-1} (\alpha + t\sqrt{2}\beta) \right) \leq (5^K)^{d-1} e^{-t^2}.$$

In the above α and β are defined in eqns. (3.9) and (3.10) respectively. We now explain the $(5^K)^{d-1}$ term in the failure probability. In general, the product $\mathcal{H} \times_1 x \times_2 y \cdots$ should be evaluated on $d-1$ vectors x, y, \dots . Recall that the $1/2$ -net $N_{B_{2,0,K}}$ contains 5^K vectors and thus there is a total of $(5^K)^{d-1}$ possible vector combinations. A standard union bound now justifies the above formula. Finally, taking the union bound over all possible subsets $B_{2,0,K}$ that comprise the set $B_{2,0}$ and using $K = \lambda n$ yields

$$\begin{aligned} \mathbb{P} \left(\sup_{x,y \in B_{2,0}} \|\mathcal{H} \times_1 x \times_2 y\|_2 \geq 2^{d-1} (\alpha + t\sqrt{2}\beta) \right) &\leq \left(\left(\frac{en}{K} \right)^K \right)^{d-1} (5^K)^{d-1} e^{-t^2} \\ &= \left(\frac{5e}{\lambda} \right)^{\lambda n(d-1)} e^{-t^2} \\ &\leq \left(\frac{5e}{\lambda} \right)^{\lambda nd} e^{-t^2}. \end{aligned} \quad (3.12)$$

In the above, we again accounted for all $d-1$ modes of the tensor and also used $d-1 \leq d$. Using eqn. (3.12) and applying Lemma 3 (part (b)) with $a = 2^{d-1}\alpha$, $b = 2^{d-1}\beta\sqrt{2}$, and $h = d\lambda n \ln(5e/\lambda)$ we get

$$\mathbb{E} \sup_{x,y \in B_{2,0}} \|\mathcal{H} \times_1 x \times_2 y\|_2^q \leq 3\sqrt{q} \left(2^{d-1} (\alpha + \beta\sqrt{2d\lambda n \ln(5e/\lambda) + \beta\sqrt{q}}) \right)^q.$$

Raising both sides to $1/q$ completes the proof. \square

3.2.3 Controlling spread vectors. We now prove the following lemma bounding the contribution of the spread vectors (term (3.6)) in our ε -net construction.

Lemma 9 Consider a d -mode tensor \mathcal{A} and let \mathcal{H} be the d -mode tensor after the Gaussian symmetrization argument as defined in Section 3.2. Let α be defined as in eqn. (3.9). For all $q \geq 1$,

$$\left(\mathbb{E} \sup_{x,y \in B_{2,\infty}} \|\mathcal{H} \times_1 x \times_2 y\|_2^q \right)^{1/q} \leq (3\sqrt{q})^{1/q} 4^{d-1} \left(\log_2 \frac{1}{\lambda} \right)^{d-1} \alpha \left(1 + \sqrt{2d \ln \frac{2e}{\lambda}} + \sqrt{\frac{q}{\lambda n}} \right), \quad (3.13)$$

assuming that $\lambda \leq 1/64$.

It is worth noting that the particular choice of the upper bound for λ is an artifact of the analysis and that we could choose bigger values for λ by introducing a constant factor loss in the above inequality.

Proof. Our proof strategy is similar to the one used in Lemma 8. However, in this case, the construction of the ε -net for the set $B_{2,\infty}$ is considerably more involved. Recall the definition of $B_{2,\infty}$:

$$B_{2,\infty} = \left\{ x \in \mathbb{R}^n : \|x\|_2 \leq 1, \|x\|_\infty < \frac{1}{\sqrt{\lambda n}} \right\}.$$

We now define the following sets of vectors N_k with $k = 0, 1, \dots, 2M-1$ with $M \triangleq \lceil 2 + \log_2 1/\sqrt{\lambda} \rceil$, assuming that $\lambda \leq 1$:

$$N_k = \left\{ z \in B_{2,\infty} : \text{for all } i \in [n], z_i = \pm \frac{1}{2^{k/2}\sqrt{\lambda n}} \text{ or } z_i = 0 \right\}.$$

+



Our $\frac{1}{2}$ -net for $B_{2,\infty}$ will be the set

$$N_{B_{2,\infty}} = \left\{ z \in B_{2,\infty} : \text{for all } i \in [n], z_i = \pm \frac{1}{2^{k/2}\sqrt{\lambda n}} \text{ with either } k = 0, 1, \dots, 2M-1 \text{ or } z_i = 0 \right\},$$

Our first lemma argues that $N_{B_{2,\infty}}$ is indeed a $\frac{1}{2}$ -net for $B_{2,\infty}$.

Lemma 10 Assuming $\lambda \leq 1$. For all $x \in B_{2,\infty}$ there exists a vector $z \in N_{B_{2,\infty}}$ such that

$$\|x - z\|_\infty \leq \frac{1}{2\sqrt{\lambda n}} \quad \text{and} \quad \|x - z\|_2 \leq \frac{1}{2}.$$

Proof. Consider a vector $x \in B_{2,\infty}$ with coordinates x_i for all $i \in [n]$. If $\frac{1}{2^{(k+1)/2}\sqrt{\lambda n}} \leq |x_i| < \frac{1}{2^{k/2}\sqrt{\lambda n}}$ for some $k = 0, 1, \dots, 2M-1$, then we set $z_i = \mathbf{sign}(x_i) \frac{1}{2^{(k+1)/2}\sqrt{\lambda n}}$. It is clear from this construction that

$$|x_i - z_i| \leq \frac{1}{2^{k/2}\sqrt{\lambda n}} - \frac{1}{2^{(k+1)/2}\sqrt{\lambda n}} = \frac{\sqrt{2}-1}{2^{(k+1)/2}\sqrt{\lambda n}} \leq (\sqrt{2}-1)|x_i|.$$

On the other hand, if $|x_i| < \frac{1}{2^{2M}\sqrt{\lambda n}}$ then we set $z_i = 0$. It is also clear that

$$|x_i - z_i| < \frac{1}{2^{[2+\log_2 1/\sqrt{\lambda}]}\sqrt{\lambda n}} \leq \frac{1}{2^{2+\log_2 1/\sqrt{\lambda}}\sqrt{\lambda n}} = \frac{1}{4\sqrt{n}}.$$

This choice of z is clearly in $N_{B_{2,\infty}}$ and implies that for all $i \in [n]$,

$$|x_i - z_i| \leq \max \left\{ (\sqrt{2}-1)|x_i|, \frac{1}{4\sqrt{n}} \right\} \leq \frac{1}{2\sqrt{\lambda n}}.$$

In addition, $(x_i - z_i)^2 \leq \max \left\{ (\sqrt{2}-1)^2 x_i^2, \frac{1}{16n} \right\} \leq (\sqrt{2}-1)^2 x_i^2 + \frac{1}{16n}$ implies that

$$\|x - z\|_2^2 \leq \sum_{i=1}^n \left((\sqrt{2}-1)^2 x_i^2 + \frac{1}{16n} \right) = \frac{1}{16} + (\sqrt{2}-1)^2 \|x\|_2^2 < \frac{1}{4},$$

which concludes the lemma. \square

Given our definitions for N_k and $N_{B_{2,\infty}}$, it immediately follows that any vector in $N_{B_{2,\infty}}$ can be expressed as a sum of $2M$ vectors, each in N_k with $k = 0, 1, \dots, 2M-1$. Combining the above lemma with Lemma 4, we get

$$\begin{aligned} \sup_{x,y \in B_{2,\infty}} \|\mathcal{H} \times_1 x \times_2 y\|_2 &\leq 2^{d-1} \sup_{x,y \in N_{B_{2,\infty}}} \|\mathcal{H} \times_1 x \times_2 y\|_2 \\ &\leq 2^{d-1} \sum_{k=0}^{2M-1} \sum_{k'=0}^{2M-1} \sup_{x \in N_k, y \in N_{k'}} \|\mathcal{H} \times_1 x \times_2 y\|_2. \end{aligned}$$

We notice here that there are two summations associated with k and k' . However, for general order- d tensor, the total summations are $(d-1)$. We now raise both sides of the above inequality to the q -th power. In order to get a meaningful bound, we employ the following lemma, which is a direct consequence of the Hölder's inequality.

Lemma 11 Let $a_i, i = 1, \dots, n$ be nonnegative number. For any $q \geq 1$,

$$\left(\sum_{i=1}^n a_i \right)^q \leq n^{q-1} \left(\sum_{i=1}^n a_i^q \right).$$





14 of 28

Applying Lemma 11, we get

$$\sup_{x,y \in \mathcal{B}_{2,\infty}} \|\mathcal{H} \times_1 x \times_2 y\|_2^q \leq 2^{q(d-1)} (2M)^{2(q-1)} \left(\sum_{k=0}^{2M-1} \sum_{k'=0}^{2M-1} \sup_{x \in N_k, y \in N_{k'}} \|\mathcal{H} \times_1 x \times_2 y\|_2^q \right). \quad (3.14)$$

It is important to note that in the general case of order- d tensors we would have a total of $(2M)^{(d-1)(q-1)}$ terms involving $(d-1)$ summations (as opposed to $(2M)^{2(q-1)}$ in the case of order-3 tensors). Our final bound accounts for all these terms and we will return to this point later in this section. Our next lemma bounds the number of vectors in N_k .

Lemma 12 Given our definitions for N_k , $|N_k| \leq e^{2^k \lambda n \ln(2e/\lambda)}$.

Proof. For all $z \in N_k$, the number of non-zero entries in z is at most $2^k \lambda n$, since $\|z\|_2 \leq 1$. Let $\gamma = 2^k \lambda n$ and notice that the number of non-zero entries in z (the ‘‘sparsity’’ of z , denoted by s) can range from 1 up to $\min(\gamma, n)$. For each value of the sparsity parameter s , there exist $2^s \binom{n}{s}$ choices for the non-zero coordinates ($\binom{n}{s}$ positions times 2^s sign choices). Thus, for k such that $\gamma \leq n$, the cardinality of N_k is bounded by

$$|N_k| \leq \sum_{s=1}^{\gamma} \binom{n}{s} 2^s \leq \left(\frac{2en}{\gamma} \right)^\gamma = \left(\frac{2en}{2^k \lambda n} \right)^\gamma \leq \left(\frac{2e}{\lambda} \right)^\gamma \quad (3.15)$$

Similarly, for k such that $\gamma \geq n$, $|N_k| \leq \sum_{s=1}^n \binom{n}{s} 2^s = 3^n$ which is also less than $(\frac{2e}{\lambda})^\gamma$ for $\lambda \leq 1$. In both cases, we have $|N_k| \leq e^{\gamma \ln(2e/\lambda)} = e^{2^k \lambda n \ln(2e/\lambda)}$, as claimed. \square

We now proceed to estimate the quantity $\|\mathcal{H} \times_1 x \times_2 y\|_2$ over all vector combinations that appear in eqn. (3.14).

Lemma 13 Using our notation, for any fixed k and k' in $(0, 1, \dots, 2M-1)$

$$\mathbb{E} \sup_{(x,y) \in (N_k, N_{k'})} \|\mathcal{H} \times_1 x \times_2 y\|_2^q \leq 3\sqrt{q} \left(\alpha + \alpha \sqrt{2d \ln(2e/\lambda)} + \alpha \sqrt{\frac{q}{\lambda n}} \right)^q. \quad (3.16)$$

Proof. Without loss of generality, we assume $k \geq k'$. We first establish the probability bound via Lemma 7 and then apply Lemma 3 to obtain the expectation estimate. We have,

$$\begin{aligned} \max_l \left(\sum_{i,j} \mathcal{A}_{ijl}^2 x_i^2 y_j^2 \right) &= \max_l \left(\sum_i x_i^2 \sum_j y_j^2 \mathcal{A}_{ijl}^2 \right) \\ &\leq \max_i \frac{1}{2^k \lambda n} \left(\sum_j y_j^2 \sum_i \mathcal{A}_{ijl}^2 \right) \\ &\leq \frac{1}{2^k \lambda n} \max_{j,l} \sum_i \mathcal{A}_{ijl}^2. \end{aligned}$$

In the above we used the fact that $\|y\|_2 \leq 1$ and $\|x\|_\infty = \frac{1}{2^{k/2} \sqrt{\lambda n}}$. Applying Lemma 7, we get (recall the definition of α from eqn. (3.9)):

$$\mathbb{P} \left(\|\mathcal{H} \times_1 x \times_2 y\|_2 \geq \alpha + t\sqrt{2} \frac{1}{2^{k/2} \sqrt{\lambda n}} \alpha \right) \leq e^{-t^2}. \quad (3.17)$$

Taking the union bound over all possible combinations of vectors $x \in N_k$ and $y \in N_{k'}$ and using Lemma 12 and the fact that $|N_k| \leq e^{2^k \lambda n \ln(2e/\lambda)}$, we get

$$\mathbb{P} \left(\sup_{x,y \in N_k} \|\mathcal{H} \times_1 x \times_2 y\|_2 \geq \alpha + t\sqrt{2} \frac{1}{2^{k/2} \sqrt{\lambda n}} \alpha \right) \leq e^{-t^2 + (d-1)2^k \lambda n \ln(2e/\lambda)},$$

where the $(d-1)$ factor appears in the exponential because of a union bound over all $(d-1)$ vectors that could appear in the product $\mathcal{H} \times_1 x \times_2 y \times_3 \dots$.

To prove the expectation bound, we apply Lemma 3 with $a = \alpha$, $b = \frac{\sqrt{2}}{2^{k/2}\sqrt{\lambda n}}\alpha$ and $h = (d-1)2^k\lambda n \ln(2e/\lambda)$ to get

$$\begin{aligned} \mathbb{E} \sup_{(x,y) \in (N_k, N_{k'})} \|\mathcal{H} \times_1 x \times_2 y\|_2^q &\leq 3\sqrt{q} \left(a + b\sqrt{h} + b\sqrt{q/2} \right)^q \\ &= 3\sqrt{q} \left(\alpha + \alpha\sqrt{2(d-1)\ln(2e/\lambda)} + \alpha\sqrt{\frac{q}{2^k\lambda n}} \right)^q. \end{aligned}$$

Proving the lemma is now trivial using $d-1 \leq d$ and $2^k \geq 1$ for all $k \geq 0$. \square

Using the bounds of Lemma 13 and combining with eqn. (3.14), we get

$$\begin{aligned} \mathbb{E} \sup_{x,y \in B_{2,\infty}} \|\mathcal{H} \times_1 x \times_2 y\|_2^q &\leq 2^{q(d-1)}(2M)^{2(q-1)} \\ &\quad \times \left(\sum_{k=0}^{2M-1} \sum_{k'=0}^{2M-1} 3\sqrt{q} \left(\alpha + \alpha\sqrt{2d\ln(2e/\lambda)} + \alpha\sqrt{\frac{q}{\lambda n}} \right)^q \right) \\ &= 3 \times 2^{q(d-1)}(2M)^{2q} \sqrt{q} \left(\alpha + \alpha\sqrt{2d\ln(2e/\lambda)} + \alpha\sqrt{\frac{q}{\lambda n}} \right)^q. \end{aligned} \quad (3.18)$$

We note that in the last equation, the number two that appears in the exponent of the term $2M$ accounts for the two summations associated with $x \in N_k$ and $y \in N_{k'}$. In general, for order- d tensors, there are at most $(d-1)$ such summations. Therefore, after some rearranging of terms,

$$\mathbb{E} \sup_{x,y \in B_{2,\infty}} \|\mathcal{H} \times_1 x \times_2 y\|_2^q \leq 3\sqrt{q} \left(2^{d-1}(2M)^{d-1} \left(\alpha + \alpha\sqrt{2d\ln(2e/\lambda)} + \alpha\sqrt{\frac{q}{\lambda n}} \right) \right)^q.$$

To conclude the proof of Lemma 9 we use our assumption on λ and the following inequality:

$$M = \lceil 2 + \log_2 \frac{1}{\sqrt{\lambda}} \rceil \leq 3 + \log_2 \frac{1}{\sqrt{\lambda}} \leq 2 \log_2 \frac{1}{\sqrt{\lambda}} = \log_2 1/\lambda.$$

\square

3.2.4 Controlling combinations of sparse and spread vectors. We now prove the following lemma bounding the contribution of combinations of sparse and spread vectors (terms (3.7) and (3.8)) in our ε -net construction.

Lemma 14 Consider a d -mode tensor \mathcal{A} and let \mathcal{H} be the d -mode tensor after the Gaussian symmetrization argument as defined in Section 3.2. Let α be defined as in eqn. (3.9). For all $q \geq 1$,

$$\left(\mathbb{E} \sup_{x \in B_{2,0}, y \in B_{2,\infty}} \|\mathcal{H} \times_1 x \times_2 y\|_2^q \right)^{1/q} \leq (3\sqrt{q})^{1/q} 4^{d-1} \left(\log_2 \frac{1}{\lambda} \right)^{d-2} \alpha \left(1 + \sqrt{2d\ln \frac{5e}{\lambda}} + \sqrt{\frac{q}{\lambda n}} \right), \quad (3.19)$$

assuming that $\lambda \leq 1/64$.

It is worth noting that the particular choice of the upper bound for λn is an artifact of the analysis and that we could choose bigger values for λn by introducing a constant factor loss in the above inequality.

Proof. Let $x \in B_{2,0}$ and $y \in B_{2,\infty}$. In Sections 3.2.2 and 3.2.3 we defined $N_{B_{2,0}}$ (a $1/2$ -net for $B_{2,0}$) and $N_{B_{2,\infty}}$ (a $1/2$ -net for $B_{2,\infty}$). Recall that for $K = \lambda n$, $B_{2,0}$ was the union of $\binom{n}{K}$ K -dimensional subsets $B_{2,0,K}$. Consequently, the $1/2$ -net $N_{B_{2,0}}$ is the union of the $1/2$ -nets $N_{B_{2,0,K}}$ (each $N_{B_{2,0,K}}$ is the $1/2$ -net of $B_{2,0,K}$). Recall from Section 3.2.2 that the cardinality of $N_{B_{2,0}}$ is bounded by

$$|N_{B_{2,0}}| = \binom{n}{K} |N_{B_{2,0,K}}| \leq \left(\frac{en}{K} \right)^K 5^K = \left(\frac{5e}{\lambda} \right)^{\lambda n}. \quad (3.20)$$

We apply Lemma 4 to get

$$\sup_{x \in B_{2,0}, y \in B_{2,\infty}} \|\mathcal{H} \times_1 x \times_2 y\|_2 \leq 2^{d-1} \sup_{x \in N_{B_{2,0}}, y \in N_{B_{2,\infty}}} \|\mathcal{H} \times_1 x \times_2 y\|_2. \quad (3.21)$$



It is now important to note that for a general d -mode tensor \mathcal{H} the above product $\mathcal{H} \times_1 x \times_2 y \times_3 \cdots$ would be computed over $d-1$ vectors, with at least one those vectors (w.l.o.g. x) in $N_{B_{2,\infty}}$ and at least one of those vectors (w.l.o.g. y) in $N_{B_{2,0}}$. Each of the remaining $(d-3)$ vectors could belong either to $N_{B_{2,0}}$ or to $N_{B_{2,\infty}}$. In order to proceed with our analysis, we will need to further express the vectors belonging to $N_{B_{2,\infty}}$ as a sum of $2M$ vectors belonging to N_k with $k = 0, 1, \dots, 2M-1$ and $M = \lceil 2 + \log_2 1/\sqrt{\lambda} \rceil$, respectively. (The reader might want to recall our definition for N_k from Section 3.2.3). We note that the cardinality upper bound of the set $N_{B_{2,\infty}}$ is considerably larger than that of the set $N_{B_{2,0}}$. This can be easily seen by comparing the upper bound of $|N_k|$ in Lemma 12 with that of $|N_{B_{2,0}}|$ in (3.20). Therefore, we only need to consider the worse case scenario, in which all $(d-2)$ vectors in the product $\mathcal{H} \times_1 x \times_2 y \times_3 \cdots$ belong to $N_{B_{2,\infty}}$. The bound for other cases will be smaller than the bound under consideration. The product can be expressed as a sum of (at most) $(2M)^{d-2}$ terms as follows:

$$\mathcal{H} \times_1 x \times_2 y \cdots \times_d z = \sum_{k=1}^{2M-1} \cdots \sum_{k'=1}^{2M-1} \mathcal{H} \times_1 x \times_2 y_k \cdots \times_d z_{k'}$$

where $x \in N_{B_{2,0}}$, $y, z \in N_{B_{2,\infty}}$, $y_k \in N_k$, and $z_{k'} \in N_{k'}$. Therefore, applying Lemma 11 and taking the expectation, we get

$$\begin{aligned} & \mathbb{E} \sup_{x \in N_{B_{2,0}}, y \in N_{B_{2,\infty}}} \|\mathcal{H} \times_1 x \times_2 y \cdots\|_2^q \\ & \leq (2M)^{(d-2)(q-1)} \left(\sum_{k=1}^{2M-1} \cdots \sum_{k'=1}^{2M-1} \mathbb{E} \sup_{x \in N_{B_{2,0}}, y \in N_k, \dots, z \in N_{k'}} \|\mathcal{H} \times_1 x \times_2 y \cdots \times_d z\|_2^q \right). \end{aligned} \quad (3.22)$$

We now need a bound, in expectation, for the q -th power of the ℓ_2 norm for each of the $(2M)^{d-2}$ terms. Fortunately, this bound has essentially already been derived in Section 3.2.3. We start by noting that the bound of eqn. (3.17) holds when *at least one* of the vectors in the product $\mathcal{H} \times_1 x \times_2 y \cdots$ belongs to N_k . Thus,

$$\mathbb{P} \left(\|\mathcal{H} \times_1 x \times_2 y \cdots \times_d z\|_2 \geq \alpha + t\sqrt{2} \frac{1}{2^{\max\{k, \dots, k'\}/2} \sqrt{\lambda n}} \alpha \right) \leq e^{-t^2} \quad (3.23)$$

holds for any $x \in N_{B_{2,0}}$, $y \in N_k, \dots$, and $z \in N_{k'}$. We apply a union bound by noting that from Lemma 12 the cardinalities of N_k are upper bounded by $e^{2^k \lambda n \ln(e/2^{k-1} \lambda)} \leq e^{2^k \lambda n \ln(2e/\lambda)}$. Combining with eqn. (3.20) we get that the total number of possible vectors over which the *sup* of eqn. (3.23) is computed does not exceed

$$\left(\frac{5e}{\lambda} \right)^{\lambda n} \underbrace{e^{2^k \lambda n \ln(2e/\lambda)} \dots e^{2^{k'} \lambda n \ln(2e/\lambda)}}_{(d-2) \text{ terms}} \leq e^{(d-1) 2^{\max\{k, \dots, k'\}} \lambda n \ln(5e/\lambda)}.$$

We can now use a standard union bound over all $x \in N_{B_{2,0}}$, $y \in N_k, \dots$, and $z \in N_{k'}$ to get

$$\mathbb{P} \left(\sup \|\mathcal{H} \times_1 x_i \times_2 y \cdots\|_2 \geq \alpha + t\sqrt{2} \frac{1}{2^{\max\{k, \dots, k'\}/2} \sqrt{\lambda n}} \alpha \right) \leq e^{-t^2 + (d-1) 2^{\max\{k, \dots, k'\}} \lambda n \ln(5e/\lambda)}.$$

We are now ready to apply Lemma 3 with $a = \alpha$, $b = \frac{\sqrt{2}}{2^{\max\{k, \dots, k'\}/2} \sqrt{\lambda n}} \alpha$, and $h = (d-1) 2^{\max\{k, \dots, k'\}} \lambda n \ln(5e/\lambda)$ to get

$$\mathbb{E} \sup_{x \in N_{B_{2,0}}, y \in N_k, \dots} \|\mathcal{H} \times_1 x \times_2 y \cdots\|_2^q \leq 3\sqrt{q} \left(a + b\sqrt{h} + b\sqrt{q/2} \right)^q.$$

Combining with eqns. (3.21) and (3.22) we get

$$\mathbb{E} \sup_{x \in B_{2,\infty}, y \in B_{2,0}} \|\mathcal{H} \times_1 x \times_2 y \cdots\|_2^q \leq 3\sqrt{q} \left(2^{d-1} (2M)^{d-2} \left(a + b\sqrt{h} + b\sqrt{q/2} \right) \right)^q.$$

The proof follows by substituting the values of a , b , and h in the above equation together with the fact that $M = \lceil 2 + \log_2 \frac{1}{\sqrt{\lambda}} \rceil \leq 3 + \log_2 \frac{1}{\sqrt{\lambda}} \leq 2 \log_2 \frac{1}{\sqrt{\lambda}} = \log_2 1/\lambda$. □

3.2.5 *Concluding the proof of Theorem 2.* Given the results of the preceding sections we can now conclude the proof of Theorem 2. We combine Lemmas 8, 9, and 14 in order to bound terms (3.5), (3.6), (3.7), and (3.8). First,

$$\begin{aligned} (\mathbb{E} \|\mathcal{H}\|_2^q)^{1/q} &\leq (3\sqrt{q})^{1/q} 2^{d-1} \left(\alpha + \beta \sqrt{2d\lambda n \ln(5e/\lambda)} + \beta \sqrt{q} \right) \\ &\quad + (3\sqrt{q})^{1/q} 4^{d-1} (\log_2 1/\lambda)^{d-1} \left(\alpha + \alpha \sqrt{2d \ln(5e/\lambda)} + \alpha \sqrt{\frac{q}{\lambda n}} \right) \\ &\quad + \left(2^{d-1} - 2 \right) \times (3\sqrt{q})^{1/q} 4^{d-1} (\log_2 1/\lambda)^{d-2} \left(\alpha + \alpha \sqrt{2d \ln(5e/\lambda)} + \alpha \sqrt{\frac{q}{\lambda n}} \right). \end{aligned}$$

In the above bound we leveraged the observation that the right-hand side of the bound in Lemma 14 is also an upper bound for the right-hand side of the bound in Lemma 9 for all $\lambda \leq 1$. It is also crucial to note that the constant $2^{d-1} - 2$ that appears in the second term of the above inequality emerges since for general order- d tensors we would have to account for a total of 2^{d-1} terms in the last inequality of Section 3.2.1. Clearly, for order-3 tensors, this inequality has a total of four terms. Simplifying the right-hand side via the assumption $q \leq 2d\lambda n \ln(5e/\lambda)$ and the fact that $q^{1/q}$ is bounded by e , we obtain

$$(\mathbb{E} \|\mathcal{H}\|_2^q)^{1/q} \leq c_1 8^{d-1} \left(\alpha [\log_2 1/\lambda]^{d-1} + \beta \sqrt{\lambda n} \right) \sqrt{2d \ln(5e/\lambda)}, \quad (3.24)$$

where c_1 is a small constant. We now remind the reader that the entries \mathcal{H}_{ijk} of the tensor \mathcal{H} are equal to $g_{ijk} \mathcal{A}_{ijk}$, where the g_{ijk} 's are standard Gaussian random variables. Thus,

$$\mathbb{E} \|\mathcal{H}\|_2^q = \mathbb{E}_g \left\| \sum_{i,j,k} g_{ijk} \mathcal{A}_{ijk} \cdot e_i \otimes e_j \otimes e_k \right\|_2^q.$$

Substituting eqn. (3.24) to eqn. (3.2) yields

$$\begin{aligned} \left(\mathbb{E}_{\mathcal{A}} \left\| \mathcal{A} - \widetilde{\mathcal{A}} \right\|_2^q \right)^{1/q} &\leq \sqrt{2\pi} \left(\mathbb{E}_{\mathcal{A}} \left[c_1 8^{d-1} \left(\alpha [\log_2 1/\lambda]^{d-1} + \beta \sqrt{\lambda n} \right) \right]^q \right)^{1/q} \\ &= c_2 8^d \sqrt{2d \ln \left(\frac{5e}{\lambda} \right)} \left[\mathbb{E}_{\mathcal{A}} \left(\alpha [\log_2 1/\lambda]^{d-1} + \beta \sqrt{\lambda n} \right)^q \right]^{1/q} \\ &\leq c_3 8^d \sqrt{2d \ln \left(\frac{5e}{\lambda} \right)} \left([\log_2 1/\lambda]^{d-1} (\mathbb{E}_{\mathcal{A}} \alpha^q)^{1/q} + \sqrt{\lambda n} (\mathbb{E}_{\mathcal{A}} \beta^q)^{1/q} \right), \end{aligned} \quad (3.25)$$

where the last inequality follows from Lemma 11. Finally, we rewrite the $\mathbb{E}_{\mathcal{A}} \alpha^q$ as

$$\begin{aligned} \mathbb{E}_{\mathcal{A}} \alpha^q &= \mathbb{E}_{\mathcal{A}} \max \left\{ \max_{i,j} \left(\sum_{k=1}^n \mathcal{A}_{ijk}^2 \right)^{q/2}, \max_{i,k} \left(\sum_{j=1}^n \mathcal{A}_{ijk}^2 \right)^{q/2}, \max_{j,k} \left(\sum_{i=1}^n \mathcal{A}_{ijk}^2 \right)^{q/2} \right\} \\ &\leq \mathbb{E}_{\mathcal{A}} \max_{i,j} \left(\sum_{k=1}^n \mathcal{A}_{ijk}^2 \right)^{q/2} + \mathbb{E}_{\mathcal{A}} \max_{i,k} \left(\sum_{j=1}^n \mathcal{A}_{ijk}^2 \right)^{q/2} + \mathbb{E}_{\mathcal{A}} \max_{j,k} \left(\sum_{i=1}^n \mathcal{A}_{ijk}^2 \right)^{q/2}. \end{aligned}$$

More generally, for any order- d tensor, we get

$$\mathbb{E}_{\mathcal{A}} \alpha^q \leq \sum_{j=1}^d \mathbb{E}_{\mathcal{A}} \max_{i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_d} \left(\sum_{i_j=1}^n \mathcal{A}_{i_1 \dots i_{j-1} i_j i_{j+1} \dots i_d}^2 \right)^{q/2}.$$

Combining the above inequality and eqn. (3.25) concludes the proof of Theorem 2.

4. Proving Theorem 1

The main idea underlying our proof is the application of a divide-and-conquer-type strategy in order to decompose the tensor $\mathcal{A} - \widetilde{\mathcal{A}}$ as a sum of tensors whose entries are bounded. Then, we will apply Theorem 2 and Corollary 2 to estimate the spectral norm of each tensor in the summand independently.



To formally present our analysis, let $\mathcal{A}^{[1]} \in \mathbb{R}^{n \times \dots \times n}$ be a tensor containing all entries $\mathcal{A}_{i_1 \dots i_d}$ of \mathcal{A} that satisfy $\mathcal{A}_{i_1 \dots i_d}^2 \geq 2^{-1} \frac{\|\mathcal{A}\|_F^2}{s}$; the remaining entries of $\mathcal{A}^{[1]}$ are set to zero. Similarly, we let $\mathcal{A}^{[k]} \in \mathbb{R}^{n \times \dots \times n}$ (for all $k > 1$) be tensors that contain all entries $\mathcal{A}_{i_1 \dots i_d}$ of \mathcal{A} that satisfy $\mathcal{A}_{i_1 \dots i_d}^2 \in \left[2^{-k} \frac{\|\mathcal{A}\|_F^2}{s}, 2^{-k+1} \frac{\|\mathcal{A}\|_F^2}{s}\right)$; the remaining entries of $\mathcal{A}^{[k]}$ are set to zero. Finally, the tensors $\widetilde{\mathcal{A}}^{[k]}$ (for all $k = 1, 2, \dots$) contain the (rescaled) entries of the corresponding tensor $\mathcal{A}^{[k]}$ that were selected after applying the sparsification procedure of Algorithm 1 to \mathcal{A} . Given these definitions,

$$\mathcal{A} = \sum_{k=1}^{\infty} \mathcal{A}^{[k]} \quad \text{and} \quad \widetilde{\mathcal{A}} = \sum_{k=1}^{\infty} \widetilde{\mathcal{A}}^{[k]}.$$

Let $\ell \triangleq \lfloor \log_2(n^{d/2}/\ln^d n) \rfloor$. Then,

$$\begin{aligned} \left\| \mathcal{A} - \widetilde{\mathcal{A}} \right\|_2 &= \left\| \sum_{k=1}^{\infty} (\mathcal{A}^{[k]} - \widetilde{\mathcal{A}}^{[k]}) \right\|_2 \\ &\leq \left\| \mathcal{A}^{[1]} - \widetilde{\mathcal{A}}^{[1]} \right\|_2 + \sum_{k=2}^{\ell} \left\| \mathcal{A}^{[k]} - \widetilde{\mathcal{A}}^{[k]} \right\|_2 + \left\| \sum_{k=\ell+1}^{\infty} (\mathcal{A}^{[k]} - \widetilde{\mathcal{A}}^{[k]}) \right\|_2. \end{aligned}$$

Using the inequality $(\mathbb{E}(x+y)^q)^{1/q} \leq (\mathbb{E}x^q)^{1/q} + (\mathbb{E}y^q)^{1/q}$, we conclude that

$$\left(\mathbb{E} \left\| \mathcal{A} - \widetilde{\mathcal{A}} \right\|_2^q \right)^{1/q} \leq \left(\mathbb{E} \left\| \mathcal{A}^{[1]} - \widetilde{\mathcal{A}}^{[1]} \right\|_2^q \right)^{1/q} \tag{4.1}$$

$$+ \sum_{k=2}^{\ell} \left(\mathbb{E} \left\| \mathcal{A}^{[k]} - \widetilde{\mathcal{A}}^{[k]} \right\|_2^q \right)^{1/q} \tag{4.2}$$

$$+ \left(\mathbb{E} \left\| \sum_{k=\ell+1}^{\infty} (\mathcal{A}^{[k]} - \widetilde{\mathcal{A}}^{[k]}) \right\|_2^q \right)^{1/q}. \tag{4.3}$$

The remainder of the section will focus on the derivation of bounds for terms (4.1), (4.2), and (4.3) of the above equation.

4.1 Term (4.1): Bounding the spectral norm of $\mathcal{A}^{[1]} - \widetilde{\mathcal{A}}^{[1]}$

The main result of this section is summarized in the following lemma.

Lemma 15 Let $q \leq \frac{5n}{8}$. Then,

$$\left(\mathbb{E} \left\| \mathcal{A}^{[1]} - \widetilde{\mathcal{A}}^{[1]} \right\|_2^q \right)^{1/q} \leq c_1 48^d d^{1/q+1/2} \sqrt{\frac{n \|\mathcal{A}\|_F^2}{s}},$$

where c_1 is a small numerical constant.

Proof. For notational convenience, let $\mathcal{B} = \mathcal{A}^{[1]} - \widetilde{\mathcal{A}}^{[1]}$ and let $\mathcal{B}_{i_1 \dots i_d}$ denote the entries of \mathcal{B} . Recall that $\mathcal{A}^{[1]}$ only contains entries of \mathcal{A} whose squares are greater than or equal to $2^{-1} \frac{\|\mathcal{A}\|_F^2}{s}$. Also, recall that $\widetilde{\mathcal{A}}^{[1]}$ only contains the (rescaled) entries of $\mathcal{A}^{[1]}$ that were selected after applying the sparsification procedure of Algorithm 1 to \mathcal{A} . Using these definitions, $\mathcal{B}_{i_1 \dots i_d}$ is equal to:

$$\mathcal{B}_{i_1 \dots i_d} = \begin{cases} 0 & \text{,if } \mathcal{A}_{i_1 \dots i_d}^2 < 2^{-1} \frac{\|\mathcal{A}\|_F^2}{s} \\ 0 & \text{,if } \mathcal{A}_{i_1 \dots i_d}^2 \geq \frac{\|\mathcal{A}\|_F^2}{s} \quad (\text{since } p_{i_1 \dots i_d} = 1) \\ \left(1 - p_{i_1 \dots i_d}^{-1}\right) \mathcal{A}_{i_1 \dots i_d} & \text{,with probability } p_{i_1 \dots i_d} = \frac{s \mathcal{A}_{i_1 \dots i_d}^2}{\|\mathcal{A}\|_F^2} < 1 \\ \mathcal{A}_{i_1 \dots i_d} & \text{,with probability } 1 - p_{i_1 \dots i_d} \end{cases}$$

It is easily seen from the formula of $\mathcal{B}_{i_1 \dots i_d}$ that $\mathcal{B}_{i_1 \dots i_d}^2 \leq \frac{\mathcal{A}_{i_1 \dots i_d}^2}{p_{i_1 \dots i_d}^2} \leq \frac{\|\mathcal{A}\|_F^4}{s^2 \mathcal{A}_{i_1 \dots i_d}^2} \leq \frac{\|\mathcal{A}\|_F^2}{s}$, which leads to

$$\left(\mathbb{E} \mathcal{B}_{i_1 \dots i_d}^q \right)^{1/q} \leq \sqrt{\frac{\|\mathcal{A}\|_F^2}{s}}.$$



+

In addition, we have for any j , $\max_{i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_d} \sum_{i_j=1}^n \mathcal{B}_{i_1 \dots i_{j-1} i_{j+1} \dots i_d}^2 \leq \frac{n \|\mathcal{B}\|_F^2}{s}$, which leads to

$$\begin{aligned} \left(\sum_{j=1}^d \mathbb{E} \max_{i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_d} \left(\sum_{i_j=1}^n \mathcal{B}_{i_1 \dots i_{j-1} i_{j+1} \dots i_d}^2 \right)^{q/2} \right)^{1/q} &\leq \left(\sum_{j=1}^d \mathbb{E} \max_{i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_d} \left(\sum_{i_j=1}^n \frac{\|\mathcal{A}\|_F^2}{s} \right)^{q/2} \right)^{1/q} \\ &\leq \left(d \left(n \frac{\|\mathcal{A}\|_F^2}{s} \right)^{q/2} \right)^{1/q} = d^{1/q} \sqrt{\frac{n \|\mathcal{A}\|_F^2}{s}}. \end{aligned}$$

We will estimate the quantity $(\mathbb{E} \|\mathcal{B}\|_2^q)^{1/q}$ via Corollary 2 as follows:

$$(\mathbb{E} \|\mathcal{B}\|_2^q)^{1/q} \leq c 8^d \sqrt{2d \ln \left(\frac{5e}{\lambda} \right)} \left(\left[\log_2 \frac{1}{\lambda} \right]^{d-1} d^{1/q} \sqrt{\frac{n \|\mathcal{A}\|_F^2}{s}} + \sqrt{\lambda n} \sqrt{\frac{\|\mathcal{A}\|_F^2}{s}} \right). \quad (4.4)$$

The proof follows by setting $\lambda = \frac{1}{64}$. □

4.2 Term (4.2): Bounding the spectral norm of $\mathcal{A}^{[k]} - \widetilde{\mathcal{A}}^{[k]}$ for small k

We now focus on estimating the spectral norm of the tensors $\mathcal{A}^{[k]} - \widetilde{\mathcal{A}}^{[k]}$ for $2 \leq k \leq \left\lfloor \log_2 \left(n^{d/2} / \ln^{d/2} n \right) \right\rfloor$. The following lemma summarizes the main result of this section.

Lemma 16 Assume that $q \leq 2d\lambda_k n \ln 5e/\lambda_k$; for all $2 \leq k \leq \left\lfloor \log_2 \left(n^{d/2} / \ln^{d/2} n \right) \right\rfloor$ and $\lambda_k \leq 1/64$,

$$\left(\mathbb{E} \left\| \mathcal{A}^{[k]} - \widetilde{\mathcal{A}}^{[k]} \right\|_2^q \right)^{1/q} \leq c_2 8^d \sqrt{2d \ln \left(\frac{5e}{\lambda_k} \right)} \left(\left[\log_2 \frac{1}{\lambda_k} \right]^{d-1} (2d)^{\frac{1}{2q}} \sqrt{5n + (d \ln n + q) 2^{k+1}} + \sqrt{\lambda_k 2^k n} \right) \sqrt{\frac{\|\mathcal{A}\|_F^2}{s}},$$

where c_2 is a small numerical constant.

Proof. For notational convenience, we let $\widetilde{\mathcal{A}}_{i_1 \dots i_d}^{[k]}$ denote the entries of the tensor $\widetilde{\mathcal{A}}^{[k]}$. Then,

$$\widetilde{\mathcal{A}}_{i_1 \dots i_d}^{[k]} = \frac{\delta_{i_1 \dots i_d} \mathcal{A}_{i_1 \dots i_d}}{p_{i_1 \dots i_d}}, \quad (4.5)$$

for those entries $\mathcal{A}_{i_1 \dots i_d}$ of \mathcal{A} satisfying $\mathcal{A}_{i_1 \dots i_d}^2 \in \left[\frac{\|\mathcal{A}\|_F^2}{2^k s}, \frac{\|\mathcal{A}\|_F^2}{2^{k-1} s} \right)$. All the entries of $\widetilde{\mathcal{A}}^{[k]}$ that correspond to entries of \mathcal{A} outside this interval are set to zero. The indicator function $\delta_{i_1 \dots i_d}$ is defined as

$$\delta_{i_1 \dots i_d} = \begin{cases} 1 & \text{with probability } p_{i_1 \dots i_d} = \frac{s \mathcal{A}_{i_1 \dots i_d}^2}{\|\mathcal{A}\|_F^2} \leq 1 \\ 0 & \text{with probability } 1 - p_{i_1 \dots i_d} \end{cases}$$

Notice that $p_{i_1 \dots i_d}$ is always in the interval $\left[2^{-k}, 2^{-(k-1)} \right)$ from the constraint on the size of $\mathcal{A}_{i_1 \dots i_d}^2$. It is now easy to see that $\mathbb{E} \widetilde{\mathcal{A}}^{[k]} = \mathcal{A}^{[k]}$. Thus, by applying Theorem 2 with the parameter λ_k ,

$$\left(\mathbb{E} \left\| \mathcal{A}^{[k]} - \widetilde{\mathcal{A}}^{[k]} \right\|_2^q \right)^{1/q} \leq c 8^d \sqrt{2d \ln \left(\frac{5e}{\lambda_k} \right)} \left(\left[\log_2 \frac{1}{\lambda_k} \right]^{d-1} \left(\sum_{j=1}^d \mathbb{E} \alpha_j^q \right)^{\frac{1}{q}} + \sqrt{\lambda_k n} (\mathbb{E} \beta^q)^{\frac{1}{q}} \right), \quad (4.6)$$

where

$$\alpha_j^2 \triangleq \max_{i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_d} \left(\sum_{i_j=1}^n \widetilde{\mathcal{A}}_{i_1 \dots i_{j-1} i_{j+1} \dots i_d}^2 \right) \quad \text{and} \quad \beta = \max_{i_1, \dots, i_d} |\widetilde{\mathcal{A}}_{i_1 \dots i_d}^{[k]}|. \quad +$$

+

20 of 28

We now follow the same strategy as in Section 4.1 in order to estimate the expectation terms in the right-hand side of the above inequality (i.e., we focus on the first term ($j = 1$) only). First, note that

$$\mathbb{E} \max_{i_2, \dots, i_d} \left(\sum_{i_1=1}^n \widetilde{\mathcal{A}}_{i_1 \dots i_d}^2 \right)^{q/2} \leq \sqrt{\mathbb{E} \max_{i_2, \dots, i_d} \left(\sum_{i_1=1}^n \widetilde{\mathcal{A}}_{i_1 \dots i_d}^2 \right)^q}.$$

Let $S_{i_2 \dots i_d} = \sum_{i_1} \widetilde{\mathcal{A}}_{i_1 \dots i_d}^2$. Then, using eqn. (4.5), the definition of $p_{i_1 \dots i_d}$, and $\delta_{i_1 \dots i_d}^2 = \delta_{i_1 \dots i_d}$, we get

$$S_{i_2 \dots i_d} = \sum_{i_1=1}^n \frac{\delta_{i_1 \dots i_d}}{p_{i_1 \dots i_d}^2} \mathcal{A}_{i_1 \dots i_d}^2 = \sum_{i_1=1}^n \delta_{i_1 \dots i_d} \frac{\|\mathcal{A}\|_F^4}{\mathcal{A}_{i_1 \dots i_d}^4 s^2} \mathcal{A}_{i_1 \dots i_d}^2 = \sum_{i_1=1}^n \delta_{i_1 \dots i_d} \frac{\|\mathcal{A}\|_F^4}{s^2 \mathcal{A}_{i_1 \dots i_d}^2}.$$

Using $\mathcal{A}_{i_1 \dots i_d}^2 \geq \frac{2^{-k} \|\mathcal{A}\|_F^2}{s}$, we get $S_{i_2 \dots i_d} \leq \frac{2^k \|\mathcal{A}\|_F^2}{s} (\sum_{i_1} \delta_{i_1 \dots i_d})$, which leads to

$$\mathbb{E} \max_{i_2, \dots, i_d} \sum_{i_1=1}^n \left(\widetilde{\mathcal{A}}_{i_1 \dots i_d}^2 \right)^q = \mathbb{E} \max_{i_2, \dots, i_d} S_{i_2 \dots i_d}^q \leq \left(\frac{2^k \|\mathcal{A}\|_F^2}{s} \right)^q \mathbb{E} \max_{i_2, \dots, i_d} \left(\sum_{i_1=1}^n \delta_{i_1 \dots i_d} \right)^q. \quad (4.7)$$

We now seek a bound for the expectation $\mathbb{E} \max_{i_2, \dots, i_d} \sum_{i_1} (\delta_{i_1 \dots i_d})^q$. The following lemma, whose proof may be found in the Appendix, provides such a bound.

Lemma 17 For any $q \geq 1$, we have

$$\mathbb{E} \max_{i_2, \dots, i_d} \left(\sum_{i_1=1}^n \delta_{i_1 \dots i_d} \right)^q \leq 2 \left(5n2^{-k} + 2d \ln n + 2q \right)^q.$$

Combining Lemma 17 and equation (4.7), we obtain

$$\mathbb{E} \left(\max_{i_2, \dots, i_d} \sum_{i_1=1}^n \widetilde{\mathcal{A}}_{i_1 \dots i_d}^2 \right)^q \leq 2 \left(\frac{(5n + 2(d \ln n + q)2^k) \|\mathcal{A}\|_F^2}{s} \right)^q.$$

The same bound can be derived for all other terms in the first summand of eqn. (4.6). Thus,

$$\left(\sum_{j=1}^d \mathbb{E} \alpha_j^q \right)^{\frac{1}{q}} \leq \left(\sum_{j=1}^d \mathbb{E} \left(\max_{i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_d} \sum_{i_j=1}^n \widetilde{\mathcal{A}}_{i_1 \dots i_{j-1} i_j i_{j+1} \dots i_d}^2 \right)^q \right)^{\frac{1}{2q}} \leq (2d)^{\frac{1}{2q}} \sqrt{\frac{(5n + (d \ln n + q)2^{k+1}) \|\mathcal{A}\|_F^2}{s}}.$$

In addition, we have $\widetilde{\mathcal{A}}_{i_1 \dots i_d}^2 \leq \frac{\mathcal{A}_{i_1 \dots i_d}^2}{p_{i_1 \dots i_d}^2} \leq \frac{\|\mathcal{A}\|_F^4}{s^2 \mathcal{A}_{i_1 \dots i_d}^2} \leq \frac{2^k \|\mathcal{A}\|_F^2}{s}$. Thus,

$$(\mathbb{E} \beta^q)^{\frac{1}{q}} \leq \sqrt{\frac{2^k \|\mathcal{A}\|_F^2}{s}}.$$

Substituting these two inequalities into eqn. (4.6) we get the claim of the lemma. \square

4.3 Term (4.3): bounding the tail

We now focus on values of k that exceed $\ell = \lfloor \log_2(n^{d/2}/\ln^d n) \rfloor$ and prove the following lemma, which immediately provides a bound for term (4.3).

Lemma 18 Using our notation,

$$\left\| \sum_{k=\ell+1}^{\infty} (\mathcal{A}^{[k]} - \widetilde{\mathcal{A}}^{[k]}) \right\|_2 \leq \sqrt{\frac{n^{d/2} \ln^d n}{s}} \|\mathcal{A}\|_F.$$

+

Proof. Intuitively, by the definition of $\mathcal{A}^{[k]}$, we can observe that when k is larger than $\ell = \lfloor \log_2(n^{d/2}/\ln^d n) \rfloor$, the entries of $\mathcal{A}^{[k]}$ are very small, whereas the entries of $\widetilde{\mathcal{A}}^{[k]}$ are all set to zero during the second step of our sparsification algorithm. Formally, consider the sum

$$\mathcal{D} = \sum_{k=\ell+1}^{\infty} (\mathcal{A}^{[k]} - \widetilde{\mathcal{A}}^{[k]}).$$

For all $k \geq \ell + 1 \geq \log_2(n^{d/2}/\ln^d n)$, notice that the squares of all the entries of $\mathcal{A}^{[k]}$ are at most $\frac{\ln^d n \|\mathcal{A}\|_F^2}{n^{d/2} s}$ (by definition) and thus the tensors $\widetilde{\mathcal{A}}^{[k]}$ are all-zero tensors. The above sum now reduces to

$$\mathcal{D} = \sum_{k=\ell+1}^{\infty} \mathcal{A}^{[k]},$$

where the squares of all the entries of \mathcal{D} are at most $\frac{\ln^d n \|\mathcal{A}\|_F^2}{n^{d/2} s}$. Since $\mathcal{D} \in \mathbb{R}^{n \times \dots \times n}$, using $\|\mathcal{D}\|_2 \leq \|\mathcal{D}\|_F$, we immediately get

$$\|\mathcal{D}\|_2 = \left\| \sum_{k=\ell+1}^{\infty} (\mathcal{A}^{[k]} - \widetilde{\mathcal{A}}^{[k]}) \right\|_2 \leq \sqrt{\sum_{i_1, i_2, \dots, i_d=1}^n \mathcal{D}_{i_1 \dots i_d}^2} \leq \sqrt{\frac{n^{d/2} \ln^d n}{s}} \|\mathcal{A}\|_F.$$

□

4.4 Completing the proof of Theorem 1

Theorem 1 emerges by substituting Lemmas 15, 16, and 18 to bound terms (4.1), (4.2), and (4.3). We have

$$\begin{aligned} \left(\mathbb{E} \left\| \mathcal{A} - \widetilde{\mathcal{A}} \right\|_2^{2d \ln n} \right)^{\frac{1}{2d \ln n}} &\leq c_1 48^d d^{1/q+1/2} \sqrt{n} \frac{\|\mathcal{A}\|_F}{\sqrt{s}} \\ &+ \sum_{k=2}^{\lfloor \log_2(n^{d/2}/\ln^d n) \rfloor} c_2 8^d \sqrt{2d \ln \frac{5e}{\lambda_k}} \left(\log_2 \frac{1}{\lambda_k} \right)^{d-1} (2d)^{\frac{1}{2q}} \sqrt{5n + (d \ln n + q)2^{k+1}} \frac{\|\mathcal{A}\|_F}{\sqrt{s}} \\ &+ \sum_{k=2}^{\lfloor \log_2(n^{d/2}/\ln^d n) \rfloor} c_2 8^d \sqrt{2d \ln \frac{5e}{\lambda_k}} \sqrt{2^k \lambda_k n} \frac{\|\mathcal{A}\|_F}{\sqrt{s}} \\ &+ \sqrt{n^{d/2} \ln^d n} \frac{\|\mathcal{A}\|_F}{\sqrt{s}} \triangleq (M_1 + M_2 + M_3 + M_4) \frac{\|\mathcal{A}\|_F}{\sqrt{s}}. \end{aligned} \quad (4.8)$$

While the first term M_1 and the last term M_4 on the right-hand side are fixed, the second and third terms largely depends on the choice of parameters λ_k . We would want to select λ_k 's such that the right-hand side is as small as possible. For this task, we set

$$\lambda_k \triangleq \frac{1}{n} \quad \text{for } k = 2, 3, \dots, \log_2 \frac{n^{d/2}}{\ln^d n}.$$

Clearly, $\lambda_k \leq \frac{1}{64}$ as required by Theorem 2. In addition, the requirement $q \leq 2d \lambda_k n \ln \frac{5e}{\lambda_k}$ is always satisfied as long as $q \leq 2d \ln n$. We set $q \triangleq 2d \ln n$. This immediately implies that the quantity $d^{1/q}$ is bounded by a constant. Let $N \triangleq \lfloor \log_2 \frac{n^{d/2}}{\ln^d n} \rfloor$ to get

$$\begin{aligned} \sum_{k=2}^N \sqrt{2^k} &= \sum_{k=1}^{\lfloor N/2 \rfloor} 2^k + 2^{1/2} \sum_{k=1}^{\lfloor (N-1)/2 \rfloor} 2^k \leq (2^{\lfloor N/2 \rfloor + 1} - 1) + 2^{1/2} (2^{\lfloor (N-1)/2 \rfloor + 1} - 1) \\ &\leq 4 \times 2^{N/2} \leq 4 \sqrt{\frac{n^{d/2}}{\ln^d n}} \end{aligned} \quad (4.9)$$

Using the fact that $\sqrt{5n + (d \ln n + q)2^{k+1}} \leq \sqrt{5n} + \sqrt{3d2^{k+1} \ln n}$ and $\log_2 n \leq \ln n$, we can get the upper bound of M_2 as follows:



$$\begin{aligned}
M_2 &\leq \sum_{k=2}^{\log_2(n^{d/2}/\ln^d n)} c_4 8^d (2d \ln(5en))^{1/2} (\ln n)^{d-1} (\sqrt{5n} + \sqrt{3d2^{k+1} \ln n}) \\
&\leq c_4 8^d \sqrt{d} \log_2 \left(\frac{n^{d/2}}{\ln^d n} \right) (\ln n)^{d-1/2} \sqrt{n} + c_5 8^d d \ln^d n \sum_{k=2}^{\log_2(n^{d/2}/\ln^d n)} 2^{k/2} \\
&\leq c_5 8^d d^{3/2} n^{1/2} (\ln n)^{d+1/2} + c_6 8^d d (\ln n)^d \sqrt{\frac{n^{d/2}}{\ln^d n}},
\end{aligned}$$

where the last inequality is due to eqn. (4.9). For $d \geq 3$, we derive an upper bound for M_2 as follows:

$$M_2 \leq c_8 8^d d^{3/2} \sqrt{n^{d/2} \ln^d n} \sqrt{\max \left\{ 1, \frac{\ln^{d+1} n}{n^{d/2-1}} \right\}}.$$

A similar bound can be derived for M_3 :

$$M_3 = \sum_{k=2}^{\lfloor \log_2(n^{d/2}/\ln^d n) \rfloor} c_2 8^d (2d \ln(5en))^{1/2} \sqrt{2^k} \leq c_9 8^d \sqrt{d} \sqrt{\frac{n^{d/2}}{\ln^{d-1} n}}.$$

Combining the above results and substituting into (4.8), we get

$$\left(\mathbb{E} \left\| \mathcal{A} - \widetilde{\mathcal{A}} \right\|_2^{2d \ln n} \right)^{\frac{1}{2d \ln n}} \leq c_{10} 20^d d^{3/2} \sqrt{\max \left\{ 1, \frac{\ln^{d+1} n}{n^{d/2-1}} \right\}} \sqrt{\frac{n^{d/2} \ln^d n}{s}} \|\mathcal{A}\|_F,$$

where the bound is due to the fact that $48^d \leq 20^d \sqrt{\ln^d n}$ for any $n \geq 320$. Applying Markov's inequality, we conclude that

$$\left\| \mathcal{A} - \widetilde{\mathcal{A}} \right\|_2 \leq c'_{10} 20^d \sqrt{\max \left\{ 1, \frac{\ln^{d+1} n}{n^{d/2-1}} \right\}} \sqrt{\frac{d^3 n^{d/2} \ln^d n}{s}} \|\mathcal{A}\|_F$$

holds with probability at least $1 - n^{-2d}$. The first part of Theorem 1 now follows by setting s to the appropriate value. For $d = 2$, the upper bound for M_2 can be simplified:

$$M_2 \leq c_7 8^d d \sqrt{n \ln^5 n}.$$

Following the same steps as above, we also derive that

$$\left\| \mathcal{A} - \widetilde{\mathcal{A}} \right\|_2 \leq c'_{10} 20^d d \sqrt{\frac{n \ln^5 n}{s}} \|\mathcal{A}\|_F$$

holds with probability at least $1 - n^{-4}$. Theorem 1 now follows by setting s to the appropriate value.

5. Conclusions and open problems

We presented the first provable bound for tensor sparsification with respect to the spectral norm. The main technical difficulty that we had to address in our work was the lack of measure concentration inequalities (analogous to the matrix-Bernstein and matrix-Chernoff bounds) for random tensors. To overcome this obstacle, we developed such an inequality using the so-called *entropy-concentration tradeoff*. To the best of our knowledge, this is the first bound of its kind in the literature.

An interesting open problem would be to investigate whether there exist algorithms that, either deterministically or probabilistically, select elements of \mathcal{A} to include in $\widetilde{\mathcal{A}}$ and achieve much better accuracy than existing schemes. For example,

notice that our algorithm, as well as prior ones, sample entries of \mathcal{A} with respect to their magnitudes; better sampling schemes might be possible. Improved accuracy will probably come at the expense of increased running time. Such algorithms would be very interesting from a mathematical and algorithmic viewpoint, since they will allow a better quantification of properties of a matrix/tensor in terms of its entries.

Acknowledgements. We would like to thank Prof. Dimitris Achlioptas for bringing [21] to our attention, as well as Tasos Zouzias for numerous useful discussions regarding our results. We also would be grateful to Prof. Roman Vershynin for the clarification regarding his papers.

References

- [1] D. Achlioptas and F. McSherry. Fast computation of low rank matrix approximations. In *Proceedings of the 33rd Annual ACM Symposium on Theory of Computing*, pages 611–618, 2001.
- [2] D. Achlioptas and F. McSherry. Fast computation of low rank matrix approximations. *Journal of the ACM*, 54(2), 2007.
- [3] Dimitris Achlioptas, Zohar Shay Karnin, and Edo Liberty. Near-optimal entrywise sampling for data matrices. In *Advances in Neural Information Processing Systems*, pages 1565–1573, 2013.
- [4] R. Ahlswede and A. Winter. Strong converse for identification via quantum channels. *IEEE Transactions on Information Theory*, 48(3):569–579, 2002.
- [5] N. Alon, W.F. de la Vega, R. Kannan, and M. Karpinski. Random sampling and approximation of MAX-CSPs. *Journal of Computer and System Sciences*, 67:212–243, 2003.
- [6] S. Arora, E. Hazan, and S. Kale. Fast algorithms for approximate semidefinite programming using the multiplicative weights update method. In *Proceedings of the 46th Annual IEEE Symposium on Foundations of Computer Science*, pages 339–348, 2005.
- [7] S. Arora, E. Hazan, and S. Kale. A Fast Random Sampling Algorithm for Sparsifying Matrices. In *APPROX-RANDOM*, pages 272–279, 2006.
- [8] Y. Azar, A. Fiat, A.R. Karlin, F. McSherry, and J. Saia. Spectral analysis of data. In *Proceedings of the 33rd Annual ACM Symposium on Theory of Computing*, pages 619–626, 2001.
- [9] J. Batson, D. A. Spielman, and N. Srivastava. Twice-ramanujan sparsifiers. In *Proceedings of the 41th Annual IEEE Symposium on Foundations of Computer Science*, pages 255–262, 2009.
- [10] E. J. Candès and B. Recht. Exact matrix completion via convex optimization. *Foundations of Computational Mathematics*, 9(3):717–772, 2009.
- [11] E. J. Candès and T. Tao. The power of convex relaxation: Near-optimal matrix completion. *IEEE Transaction on Information Theory*, 56(5):2053–2080, 2010.
- [12] A. d’Aspremont. Subsampling Algorithms for Semidefinite Programming. *arXiv:0803.1990v5*, 2009.
- [13] W.F. de la Vega, R. Kannan, M. Karpinski, and S. Vempala. Tensor decomposition and approximation schemes for constraint satisfaction problems. In *Proceedings of the 37th Annual ACM Symposium on Theory of Computing*, pages 747–754, 2005.
- [14] P. Drineas, R. Kannan, and M. W. Mahoney. Sampling subproblems of heterogeneous max-cut problems and approximation algorithms. *Random Structures and Algorithms*, 32(3):307–333, 2008.
- [15] P. Drineas, M. W. Mahoney, and S. Muthukrishnan. Relative-error CUR matrix decompositions. *SIAM Journal on Matrix Analysis and Applications*, 30(2):844–881, 2008.
- [16] P. Drineas and M.W. Mahoney. A randomized algorithm for a tensor-based generalization of the Singular Value Decomposition. *Linear Algebra and its Applications*, 420(2):553–571, 2007.



- [17] P. Drineas and A. Zouzias. A note on element-wise matrix sparsification via matrix-valued chernoff bounds. *Information Processing Letters*, 111:385–389, 2011.
- [18] C. Faloutsos, T. G. Kolda, and J. Sun. Tutorial: Mining Large Time-evolving Data Using Matrix and Tensor Tools. In *Proceedings of the ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, 2007.
- [19] A. Frieze, R. Kannan, and S. Vempala. Fast Monte-Carlo algorithms for finding low-rank approximations. In *Proceedings of the 39th Annual IEEE Symposium on Foundations of Computer Science*, pages 370–378, 1998.
- [20] Z. Füredi and J. Komlós. The eigenvalues of random symmetric matrices. *Combinatorica*, 1(3):233–241, 1981.
- [21] A. Gittens and J. Tropp. Error bounds for random matrix approximation schemes. Preprint, 2009.
- [22] T. G. Kolda. Orthogonal tensor decompositions. *SIAM Journal on Matrix Analysis and Applications*, 23(1):243–255, 2001.
- [23] T. G. Kolda and B. W. Bader. Tensor Decompositions and Applications. *SIAM Review*, 51(3):455–500, 2009.
- [24] R. Latała. Some estimates of norms of random matrices. *Proceeding of the American Mathematical Society*, 133(5):1273–1282, 2004.
- [25] L. De Lathauwer, B. De Moor, and J. Vandewalle. An introduction to independent component analysis. *Journal of Chemometrics*, 14:123–149, 2000.
- [26] L. De Lathauwer, B. De Moor, and J. Vandewalle. A multilinear singular value decomposition. *SIAM Journal on Matrix Analysis and Applications*, 21(4):1253–1278, 2000.
- [27] L. De Lathauwer, B. De Moor, and J. Vandewalle. On the best rank-1 and rank- (R_1, R_2, \dots, R_N) approximation of higher-order tensors. *SIAM Journal on Matrix Analysis and Applications*, 21(4):1324–1342, 2000.
- [28] M. Ledoux. *The Concentration of Measure Phenomenon*. American Mathematical Society, 2001.
- [29] M. Ledoux and M. Talagrand. *Probability in Banach Space: Isoperimetry and Processes*. Springer, 1991.
- [30] M. W. Mahoney and P. Drineas. CUR matrix decompositions for improved data analysis. *Proceedings of the National Academy of Sciences*, 106(3):697–702, 2009.
- [31] M. W. Mahoney, M. Maggioni, and P. Drineas. Tensor-CUR decompositions for tensor-based data. In *Proceedings of the Twelfth ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, pages 327–336, 2006.
- [32] M. W. Mahoney, M. Maggioni, and P. Drineas. Tensor-CUR decompositions and data applications. *SIAM Journal on Matrix Analysis and Applications*, 30(2):957–987, 2008.
- [33] N.H. Nguyen, T.T. Do, and T.D. Tran. A fast and efficient algorithm for low-rank approximation of a matrix. In *Proceedings of the 41st Annual ACM Symposium on Theory of Computing*, pages 215–224, 2009.
- [34] R. I. Oliveira. Sums of random hermitian matrices and an inequality by Rudelson. *Electronic Communications in Probability*, 15:203–212, 2002.
- [35] M. Rudelson and R. Vershynin. The smallest singular value of a random rectangular matrix. *Communications on Pure and Applied Mathematics*, 62:1707–1739, 2009.
- [36] T. Sarlos. Improved approximation algorithms for large matrices via random projections. In *IEEE Symposium on Foundations of Computer Science (FOCS)*, 2006.
- [37] D. A. Spielman and N. Srivastava. Graph sparsification by effective resistances. In *Proceedings of the 40th Annual IEEE Symposium on Foundations of Computer Science*, pages 563–568, 2008.
- [38] J. A. Tropp. User-friendly tail bounds for sums of random matrices. *Foundations of Computational Mathematics*, 1:2–20, 2012.



- [39] J. A. Tropp. An introduction to matrix concentration inequalities. *Preprint*, 2015.
- [40] C. E. Tsourakakis. Mach: Fast randomized tensor decompositions. In *SIAM international conference on Data Mining*, pages 689–700, 2010.
- [41] R. Vershynin. Spectral norm of products of random and deterministic matrices. *Probability Theory and Related Fields*, 150:471–509, 2011.
- [42] R. Vershynin. Introduction to the non-asymptotic analysis of random matrices. *Compressed sensing: Theory and Applications*, pages 210–268, 2012. Cambridge University Press, Cambridge.

Appendix

Proof of Lemma 3.

Proof. (a) From our assumption,

$$\mathbb{P}(X \geq a + b(t + h)) \leq e^{-t}.$$

Let $s = a + b(t + h)$. For any $q \geq 1$,

$$\begin{aligned} \mathbb{E}X^q &= \int_0^\infty \mathbb{P}(X \geq s) ds^q = q \int_0^\infty \mathbb{P}(X \geq s) s^{q-1} ds \\ &\leq q \int_0^{a+bh} s^{q-1} ds + q \int_{a+bh}^\infty s^{q-1} e^{-\frac{(s-a-bh)}{b}} ds. \end{aligned}$$

The first term in the above sum is equal to $(a + bh)^q$. The second term is somewhat harder to compute. We start by letting $g = a + bh$ and changing variables, thus getting

$$\int_{a+bh}^\infty s^{q-1} e^{-\frac{(s-a-bh)}{b}} ds = b \int_0^\infty (g + bt)^{q-1} e^{-t} dt = b \sum_{i=0}^{q-1} \binom{q-1}{i} b^{q-1-i} g^i \int_0^\infty t^{q-1-i} e^{-t} dt.$$

We can now integrate by parts and get

$$\int_0^\infty t^{q-1-i} e^{-t} dt = (q-1-i)! \leq q^{q-1-i} \quad \text{for all } i = 0, \dots, q-1.$$

Combining the above,

$$q \int_{a+bh}^\infty s^{q-1} e^{-\frac{(s-a-bh)}{b}} ds \leq qb \sum_{i=0}^{q-1} \binom{q-1}{i} (bq)^{q-1-i} g^i = qb(bq + g)^{q-1}.$$

Finally,

$$\mathbb{E}X^q \leq (a + bh)^q + qb(bq + g)^{q-1} \leq 2(a + bh + bq)^q,$$

which concludes the proof of the first part.

(b) From our assumption and since t and h are non-negative, we get

$$\mathbb{P}(X \geq a + b(t + \sqrt{h})) \leq e^{-(t+\sqrt{h})^2+h} \leq e^{-t^2}.$$

Let $s = a + b\sqrt{h} + tb$. For any $q \geq 1$,

$$\begin{aligned} \mathbb{E}X^q &= \int_0^\infty \mathbb{P}(X \geq s) ds^q = q \int_0^\infty \mathbb{P}(X \geq s) s^{q-1} ds \\ &\leq q \int_0^{a+b\sqrt{h}} s^{q-1} ds + q \int_{a+b\sqrt{h}}^\infty s^{q-1} e^{-\frac{(s-a-b\sqrt{h})^2}{b^2}} ds. \end{aligned}$$





The first term in the above sum is equal to $(a + b\sqrt{h})^q$. We now evaluate the second integral. Let $g = a + b\sqrt{h}$ and perform a change of variables to get

$$\begin{aligned} \int_{a+b\sqrt{h}}^{\infty} s^{q-1} e^{-\frac{(s-a-b\sqrt{h})^2}{b^2}} ds &= b \int_0^{\infty} (g+bt)^{q-1} e^{-t^2} dt \\ &= b \sum_{i=0}^{q-1} \binom{q-1}{i} b^{q-1-i} g^i \int_0^{\infty} t^{q-1-i} e^{-t^2} dt. \end{aligned}$$

By integrating by parts we get (see below for a proof of eqn. (5.1)):

$$\int_0^{\infty} t^{q-1-i} e^{-t^2} dt \leq \sqrt{\frac{\pi}{2}} \left(\frac{q-1-i}{2} \right)^{(q-1-i)/2} \leq \sqrt{\frac{\pi}{2}} \left(\frac{q}{2} \right)^{(q-1-i)/2}. \quad (5.1)$$

Thus, using $g = a + b\sqrt{h}$,

$$\int_{a+b\sqrt{h}}^{\infty} s^{q-1} e^{-\frac{(s-a-b\sqrt{h})^2}{b^2}} ds \leq b \sqrt{\frac{\pi}{2}} \sum_{i=0}^{q-1} \binom{q-1}{i} \left(b \sqrt{\frac{q}{2}} \right)^{q-1-i} g^i \leq \sqrt{2} b \left(a + b\sqrt{h} + b \sqrt{\frac{q}{2}} \right)^{q-1}.$$

Finally, we conclude that

$$\begin{aligned} \mathbb{E}X^q &\leq (a + b\sqrt{h})^q + \sqrt{2} b q \left(a + b\sqrt{h} + b \sqrt{\frac{q}{2}} \right)^{q-1} \\ &\leq \left(a + b\sqrt{h} + b \sqrt{\frac{q}{2}} \right)^q + \sqrt{4q} \left(a + b\sqrt{h} + b \sqrt{\frac{q}{2}} \right)^q \\ &\leq 3\sqrt{q} \left(a + b\sqrt{h} + b \sqrt{\frac{q}{2}} \right)^q, \end{aligned}$$

which is the claim of the lemma. In the above we used the positivity of a, b , and h as well as the fact that $1 + \sqrt{4q} \leq 3\sqrt{q}$ for all $q \geq 1$. \square

Proof of eqn. (5.1).

Proof. We now compute the integral $\int_0^{\infty} t^q e^{-t^2} dt$. Integrating by parts, we get

$$\begin{aligned} \int_0^{\infty} t^q e^{-t^2} dt &= \frac{1}{2} \int_0^{\infty} -t^{q-1} d e^{-t^2} \\ &= -\frac{1}{2} t^{q-1} e^{-t^2} \Big|_0^{\infty} + \frac{1}{2} \int_0^{\infty} e^{-t^2} dt^{q-1} \\ &= \frac{1}{2} (q-1) \int_0^{\infty} t^{q-2} e^{-t^2} dt. \end{aligned}$$

When q is even, we get

$$\int_0^{\infty} t^q e^{-t^2} dt = \left(\frac{1}{2} \right)^{q/2} (q-1)!! \int_0^{\infty} e^{-t^2} dt = \sqrt{\frac{\pi}{2}} \left(\frac{1}{2} \right)^{q/2} (q-1)!!.$$

where $q!! = q(q-2)(q-4)\dots$. If q is odd, then

$$\int_0^{\infty} t^q e^{-t^2} dt = \left(\frac{1}{2} \right)^{\lfloor q/2 \rfloor} (q-1)!! \int_0^{\infty} t e^{-t^2} dt = \left(\frac{1}{2} \right)^{\lfloor q/2 \rfloor + 1} (q-1)!!.$$

We thus conclude

$$\int_0^{\infty} t^q e^{-t^2} dt \leq \sqrt{\frac{\pi}{2}} \left(\frac{1}{2} \right)^{\lfloor q/2 \rfloor} (q-1)!! \leq \sqrt{\frac{\pi}{2}} \left(\frac{q-1}{2} \right)^{\lfloor q/2 \rfloor} \leq \sqrt{\frac{\pi}{2}} \left(\frac{q-1}{2} \right)^{q/2}.$$



+

□

Proof of Lemma 4.

Proof. We start by noting that every vector $z \in B$ can be written as $z = x + h$, where x lies in \mathbb{N} and $h \in \mathcal{E}B$. Using the triangle inequality for the tensor spectral norm, we get

$$\sup_{z \in B} \|\mathcal{A} \times_1 z\|_2 \leq \sup_{x \in \mathbb{N}} \|\mathcal{A} \times_1 x\|_2 + \sup_{h \in \mathcal{E}B} \|\mathcal{A} \times_1 h\|_2.$$

It is now easy to bound the second term in the right-hand side of the above equation by $\varepsilon \sup_{z \in B} \|\mathcal{A} \times_1 z\|_2$. Thus,

$$\sup_{z \in B} \|\mathcal{A} \times_1 z\|_2 \leq \frac{1}{1 - \varepsilon} \sup_{x \in \mathbb{N}} \|\mathcal{A} \times_1 x\|_2.$$

Repeating the same argument recursively for the tensor $\mathcal{A} \times_1 x$ etc. we obtain the lemma. □

Proof of Lemma 17.

Proof. Let $S = \max_{i_2, \dots, i_d} \sum_{i_1=1}^n \delta_{i_1 \dots i_d}$. We will first estimate the probability $\mathbb{P}(S \geq t)$ and then apply Lemma 3 in order to bound the expectation $\mathbb{E}S^q$. Recall from the definition of $\delta_{i_1 \dots i_d}$ that $\mathbb{E}(\delta_{i_1 \dots i_d} - p_{i_1 \dots i_d}) = 0$ and let

$$X = \sum_{i_1=1}^n (\delta_{i_1 \dots i_d} - p_{i_1 \dots i_d}).$$

We will apply Bennett's inequality in order to bound X . Clearly $|\delta_{i_1 \dots i_d} - p_{i_1 \dots i_d}| \leq 1$ and

$$\begin{aligned} \mathbf{Var}(X) &= \sum_{i_1=1}^n \mathbf{Var}(\delta_{i_1 \dots i_d} - p_{i_1 \dots i_d}) = \sum_{i_1=1}^n \mathbb{E}(\delta_{i_1 \dots i_d} - p_{i_1 \dots i_d})^2 \\ &= \sum_{i_1=1}^n (p_{i_1 \dots i_d} - p_{i_1 \dots i_d}^2) \leq \sum_{i_1=1}^n p_{i_1 \dots i_d}. \end{aligned}$$

Recalling the definition of $p_{i_1 \dots i_d}$ and the bounds on the $\mathcal{A}_{i_1 \dots i_d}$'s, we get

$$\mathbf{Var}(X) \leq \sum_{i_1=1}^n \frac{s \mathcal{A}_{i_1 \dots i_d}^2}{\|\mathcal{A}\|_F^2} \leq n 2^{-(k-1)}.$$

We can now apply Bennett's inequality in order to get

$$\mathbb{P}(X > t) = \mathbb{P}\left(\sum_{i_1=1}^n \delta_{i_1 \dots i_d} > \sum_{i_1=1}^n p_{i_1 \dots i_d} + t\right) \leq e^{-t/2},$$

for any $t \geq 3n 2^{-(k-1)}/2$. Thus, with probability at least $1 - e^{-t/2}$,

$$\sum_{i_1=1}^n \delta_{i_1 \dots i_d} \leq n 2^{-(k-1)} + t,$$

since $\sum_{i_1=1}^n p_{i_1 \dots i_d} \leq n 2^{-(k-1)}$. Setting $t = (3n 2^{-(k-1)}/2) + 2\tau$ for any $\tau \geq 0$ we get

$$\mathbb{P}\left(\sum_{i_1=1}^n \delta_{i_1 \dots i_d} \geq \frac{5}{2} n 2^{-(k-1)} + 2\tau\right) \leq e^{-\tau}.$$

Taking a union bound yields

$$\mathbb{P}\left(\max_{i_2, \dots, i_d} \sum_{i_1=1}^n \delta_{i_1 \dots i_d} \geq 5n 2^{-k} + 2\tau\right) \leq n^{d-1} e^{-\tau} = e^{-\tau + (d-1)\ln n},$$

+



where the n^{d-1} term appears because of all possible choices for the indices i_2, \dots, i_d . Applying Lemma 3 with $a = 5n2^{-k}$, $b = 2$, and $h = (d-1)\ln n$, we get

$$\mathbb{E} \left(\max_{i_2, \dots, i_d} \sum_{i_1=1}^n \delta_{i_1 \dots i_d} \right)^q \leq 2 \left(5n2^{-k} + 2(d-1)\ln n + 2q \right)^q \leq 2 \left(5n2^{-k} + 2d\ln n + 2q \right)^q. \quad (5.2)$$

The proof is completed.

□

