Assignment 5

Randomization in Numerical Linear Algebra (PCMI)

1. Recall the definition of the edge-incidence matrix $\mathbf{B}$ and the weight matrix $\mathbf{W}$ (positive weights only) from slide 151. Recall that the Laplacian matrix of a graph can be expressed as $\mathbf{L} = \mathbf{B}^T\mathbf{W}\mathbf{B}$, and recall that the effective resistances of the edges of the graph are defined to be the diagonal entries of the matrix $\mathbf{B}\mathbf{L}^\dagger\mathbf{B}^T$. Prove that the effective resistances are equal (up to rescaling) to the row leverage scores of the matrix $\mathbf{W}^{1/2}\mathbf{B}$.

2. Let $\mathbf{A}$ and $\tilde{\mathbf{A}}$ be two matrices with the same dimensions. Prove:

$$\left\|\mathbf{A} - \left(\tilde{\mathbf{A}}\right)_k\right\|_2 \leq \|\mathbf{A} - \mathbf{A}_k\|_2 + 2\left\|\mathbf{A} - \tilde{\mathbf{A}}\right\|_2.$$

In the above, $\mathbf{A}_k$ is the best rank-k approximation to $\mathbf{A}$ (similarly for $\tilde{\mathbf{A}}$).

3. We will prove a bound for the spectral norm of the matrix $\mathbf{A} - \mathcal{S}_\Omega(\mathbf{A})$, where $\mathcal{S}_\Omega(\mathbf{A})$ is constructed using the element-wise sampling algorithm of slide 165 and the hybrid sampling probabilities of slide 175.

1. Let $\mathbf{M}_t = \frac{\mathbf{A}_{i_tj_t}}{p_{i_tj_t}}\mathbf{e}_{i_t}\mathbf{e}_{j_t}^T - \mathbf{A}$. Prove:

   $$\mathbf{A} - \mathcal{S}_\Omega(\mathbf{A}) = -\frac{1}{s}\sum_{t=1}^{s}\mathbf{M}_t.$$

   Also compute $\mathbf{E}\left[\mathbf{M}_t\right]$.

2. Compute the expectation $\mathbf{E}\left[\mathbf{M}_t\mathbf{M}_t^T\right]$.

3. Use the sampling probabilities of slide 175 to bound $\left\|\mathbf{E}\left[\mathbf{M}_t\mathbf{M}_t^T\right]\right\|_2$ and $\max_{i,j}\left\|\frac{\mathbf{A}_{ij}}{p_{ij}}\mathbf{e}_i\mathbf{e}_j^T - \mathbf{A}\right\|_2$.

4. Apply the matrix-Bernstein inequality of slide 177 to bound the spectral norm of the matrix $\mathbf{A} - \mathcal{S}_\Omega(\mathbf{A})$ with high probability.