Assignment 3

Randomization in Numerical Linear Algebra (PCMI)

1. Let $\mathbf{A}$ be an $n \times d$ matrix with $n \gg d$. **(i)** Give an example of a matrix A whose row leverage scores are all equal. **(ii)** Give an example of a matrix $\mathbf{A}$ whose row leverage scores are equal to either zero or one; to be precise, d of the n row leverage scores should be equal to one and the remaining $n-d$ row leverage scores should be equal to zero.

2. Let $\mathbf{U}$ be an $n \times d$ matrix with $n \gg d$, such that $\mathbf{U}^T\mathbf{U} = I$ (i.e., the columns of $\mathbf{U}$ are pairwise orthogonal and normal). Sample r rows of $\mathbf{U}$ with probability proportional to their leverage scores (in r independent, identically distributed trials); let $\mathbf{S}$ be the corresponding sampling and rescaling matrix (recall the notation of slide 16). Prove that, with probability at least 0.9,

$$\left\|\mathbf{U}^T\mathbf{S}\mathbf{S}^T\mathbf{U}^T - \mathbf{U}\mathbf{U}^T\right\|_2^2 \leq \varepsilon$$

assuming that

$$r = O\left(\frac{d}{\epsilon^2}\ln\left(\frac{d}{\epsilon^2\sqrt{\delta}}\right)\right).$$

Hint. Use results from the randomized matrix multiplication lecture and homework.

3. In this problem, we will prove the first of the two inequalities (the first structural result) of slide 76, subject to the two assumptions stated in slide 76.

1. First, let $\mathbf{U_A}$ be the matrix of the left singular vectors of the tall-and-thin matrix $\mathbf{A}$. Prove that

$$\min_{\mathbf{x}\in\mathbb{R}^d} \|\mathbf{X}\mathbf{b} - \mathbf{X}\mathbf{A}\mathbf{x}\|_2^2 = \min_{\mathbf{z}\in\mathbb{R}^d} \left\|\mathbf{X}\mathbf{b}^{\perp} - \mathbf{X}\mathbf{U_A}\mathbf{z}\right\|_2^2.$$

2. Write an explicit formula for the vector

$$\mathbf{z}_{opt} = \arg\min_{\mathbf{z}\in\mathbb{R}^d} \left\|\mathbf{X}\mathbf{b}^{\perp} - \mathbf{X}\mathbf{U_A}\mathbf{z}\right\|_2^2$$

   and prove that $\mathbf{z}_{opt} = \mathbf{U_A}^T\mathbf{A}\left(\mathbf{x}_{opt} - \tilde{\mathbf{x}}_{opt}\right)$. (See slides 72 and 73 for definitions of $\mathbf{x}_{opt}$ and $\tilde{\mathbf{x}}_{opt}$.

3. Using the above facts as well as the assumptions in slide 76, prove that

$$\|\mathbf{z}_{opt}\|_2^2 \leq \varepsilon \mathcal{Z}_2^2.$$

   (See slide 70 for the definition of $\mathcal{Z}_2^2$.)

4. Now prove that

$$\|\mathbf{b} - \mathbf{A}\mathbf{x}_{opt}\|_2 \leq \|\mathbf{b} - \mathbf{A}\tilde{\mathbf{x}}_{opt}\|_2 \leq \left(1+\sqrt{\epsilon}\right)\|\mathbf{b} - \mathbf{A}\mathbf{x}_{opt}\|_2.$$