Assignment 1

Randomization in Numerical Linear Algebra (PCMI)

1. Sherman-Morrison-Woodbury Formula

For $A \in \mathbb{R}^{n\times n}$ nonsingular, and $U, V \in \mathbb{R}^{m\times n}$ show: If $I + VA^{-1}U^T$ is nonsingular then

$$\left(A + U^T V\right)^{-1} = A^{-1} - A^{-1}U^T \left(I + VA^{-1}U^T\right)^{-1} VA^{-1}.$$

2. More Exercises

(**a**) **(Frobenius norm of outer products)** For $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ show $\|xy^T\|_F = \|x\|_2\|y\|_2$.

(**b**) **(Orthogonal matrices)** Show that all singlular values of $A \in \mathbb{R}^{n\times n}$ are equal to 1 if and only if A is orthogonal matrix.

(**c**) **(Appending a row to a tall and skinny matrix)** Show that if $A \in \mathbb{R}^{m\times n}$ with $m \geq n$, $z \in \mathbb{R}^n$, $B^T = (A^T \quad z) \in \mathbb{R}^{n\times(m+1)}$, then

$$\sigma_n(B) \geq \sigma_n(A), \qquad \sigma_1(A) \leq \sigma_1(B) \leq \sqrt{\sigma_1^2(A) + \|z\|_2^2}.$$

(**d**) Show that if $A \in \mathbb{R}^{m\times n}$ with $\text{rank}(A) = n$ then $\|(A^T A)^{-1}\|_2 = \|A^\dagger\|_2^2$.

(**e**) Show that if $A = BC$ where $B \in \mathbb{R}^{m\times n}$ has $\text{rank}(B) = n$ and $C \in \mathbb{R}^{n\times n}$ is nonsingular then $A^\dagger = C^{-1}B^\dagger$.

(**f**) Suppose $y \in \mathbb{R}^n$ is the minimal ℓ_2 norm solution to $\min_x \|Ax - b\|_2$, where $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$ and $A^T b = \mathbf{0}$, then what can you say about y ?