Fundamentals of Information Transmission and Coding (a.k.a. Communication Theory)

Signals and functions

Elementary operation of communication: send signal on medium from A to B.

- media—copper wire, optical fiber, air/space, . . .
- signals—voltage and currents, light pulses, radio waves, microwaves, . . .
  → electromagnetic wave

Signal can be viewed as a time-varying function $s(t)$. 
If $s(t)$ is “sufficiently nice”—Dirichlet conditions—then $s(t)$ can be represented as a linear combination of complex sinusoids:
Simple example:

\[ s(t) = \sin(t) + \sin(2t) + \sin(3t) \]

\[ s(t) = \sin(t) \]

\[ s(t) = \sin(2t) \]

\[ s(t) = \sin(3t) \]

\[ \rightarrow \text{sinusoids form basis for other signals} \]
Analogous to basis in linear algebra:

other elements can be expressed as linear combinations of elements in the basis set

Ex.: in 3-D, \{ (1, 0, 0), (0, 1, 0), (0, 0, 1) \} form a basis.

\[ (7, 2, 4) = 7 \cdot (1, 0, 0) + 2 \cdot (0, 1, 0) + 4 \cdot (0, 0, 1) \]

\[ \rightarrow \text{ coefficients: } 7, 2, 4 \]

\[ \rightarrow \text{ spectrum} \]

How many elements are there in a basis?
Vector spaces:

- finite dimensional
- infinite dimensional: signals

→ subject of functional analysis

Given an arbitrary element in the vector space, how to find the coefficient of basis elements?

→ e.g., given \((7, 2, 4)\), coefficient of \((0, 1, 0)\)?
Fourier expansion and transform:

\[ s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)e^{i\omega t} d\omega, \]

\[ S(\omega) = \int_{-\infty}^{\infty} s(t)e^{-i\omega t} dt. \]

\[ \rightarrow \text{ recall: } e^{i\omega t} = \cos \omega t + i \sin \omega t \]

\[ \rightarrow S(\omega): \text{ coefficient of basis elements} \]

\[ \rightarrow \text{ time domain vs. frequency domain} \]

Frequency \( \omega \): cycles per second (Hz)

\[ \rightarrow \omega = 1/T \]

\( T \): period of sinusoid
Example: square wave
Example: audio (e.g., speech) signal

Source: Dept. of Linguistics and Phonetics, Lund University
Random function (i.e., white noise) has “flat-looking” spectrum.

→ unbounded bandwidth

Why bother with frequency domain representation?

→ contains same information . . .

→ i.e., invertible
Luckily, most “interesting” functions arising in practice are “special”:

→ bandlimited

→ i.e., $S(\omega) = 0$ for $\omega$ sufficiently large

→ when $S(\omega) \approx 0$, can treat as $S(\omega) = 0$

→ approximation property

→ e.g., square wave
Ex.: human auditory system

\[\rightarrow 20 \text{ Hz–}20 \text{ kHz}\]

\[\rightarrow \text{speech is intelligible at } 300 \text{ Hz–}3300 \text{ Hz}\]

\[\rightarrow \text{broadcast quality audio; CD quality audio}\]

Telephone systems: engineered to exploit this property

\[\rightarrow \text{bandwidth } 3000 \text{ Hz}\]

\[\rightarrow \text{copper medium: various grades}\]
Digital data vs. analog data

Digital data: bits.

→ discrete signal

→ both in time and amplitude

Analog data: audio/voice, video/image

→ continuous signal

→ both in time and amplitude

→ analog data is often digitized

→ digital signal processing

How to digitize such that digital representation is faithful?

→ sampling
**Sampling theorem (Nyquist):** Given continuous bandlimited signal $s(t)$ with $S(\omega) = 0$ for $|\omega| > W$, $s(t)$ can be reconstructed from its samples if

$$\nu > 2W$$

where $\nu$ is the sampling rate.

$\rightarrow \nu$: samples per second

Quantization issue ignored

$\rightarrow$ amplitude must also be digitized
Slowly vs. rapidly varying signal:

\[ \nu_1 = \frac{1}{T_1} < \nu_2 = \frac{1}{T_2} \]
Information transmission over noiseless channel

Sender wants to communicate information to receiver over noiseless channel.

→ idealized case

Set-up:

- source $s$ emits symbols from finite alphabet set $\Sigma$
  $\rightarrow$ e.g., $\Sigma = \{0, 1\}$; $\Sigma =$ ASCII character set
- symbol $a \in \Sigma$ is generated with probability $p_a > 0$
  $\rightarrow$ e.g., books have known distribution for ‘e’, ‘x’ . . .
• code book $F$ assigns code word $w_a = F(a)$ for each $a \in \Sigma$

$\rightarrow w_a$ is a binary string; $|w_a|$ length of $w_a$

• $F$ is invertible

$\rightarrow$ receiver $d$ can recover $a$ from $w_a$

Ex.: $\Sigma = \{A, C, G, T\}$; need at least two bits

• $F^1$: $w_A = 00$, $w_C = 01$, $w_G = 10$, $w_T = 11$

• $F^2$: $w_A = 0$, $w_C = 10$, $w_G = 110$, $w_T = 1110$

$\rightarrow$ pros & cons?
Design problem: find a “good” code book $F$

$\rightarrow$ lots of possible code books

$\rightarrow$ when is $F$ good?

Performance measure: average code length $L$

$$L = \sum_{a \in \Sigma} p_a|w_a|$$

Ex.: If DNA sequence is 10000 letters long, then require on average $10000 \cdot L$ bits to be transmitted.

$\rightarrow$ good to have code book with small $L$

Optimization problem: Given $\langle \Sigma, \mathbf{p} \rangle$ where $\mathbf{p}$ is a probability vector, find a code book $F$ with least $L$. 
A fundamental result on what is achievable:

Entropy $H$ of $\langle \Sigma, p \rangle$ is defined as

$$H = \sum_{a \in \Sigma} p_a \log \frac{1}{p_a}$$


$\rightarrow$ when is it minimum?

**Source Coding Theorem (Shannon):** For all $F$, $H \leq L$.

Moreover, there exist $F$ such that

$$L \leq H + \epsilon$$

where $0 < \epsilon < 1$. 
Remarks:

- Small $\epsilon$ holds for extension codes: blocks of $k$ symbols
- Note: entropy is property of source $s$
- Another view of source coding?
- Ensemble limitation
  - e.g., sending number $\pi = 3.1415927\ldots$
  - better way?
Information transmission over noisy channel

Uncertainty introduced by noise:

\[ \text{encoding/decoding: } a \mapsto w_a \mapsto w \mapsto [?] \]

\[ \text{if } w \text{ is not a valid code word, error detection} \]

\[ \text{if } w = w_b, \text{ incorrectly concludes symbol is } b \]

Would like: If received code word \( w = w_a \), then probability that actual symbol sent is indeed \( a \) is high.

\[ \text{Pr\{symbol sent }= a \mid w = w_a \} \approx 1 \]

\[ \text{noiseless channel: special case } = 1 \]
Channel capacity $C$: maximum achievable reliable data transmission rate (bps) over a noisy channel with bandwidth $W$ (Hz).

\[ \rightarrow \text{assuming analog signal, band-limited by } W \]

**Channel Coding Theorem (Shannon):** Given bandwidth $W$, signal power $P_S$, noise power $P_N$, channel subject to white noise (detailed conditions omitted),

\[ C = W \log(1 + \frac{P_S}{P_N}) \text{ bps.} \]

Here $P_S/P_N$: *signal-to-noise ratio*.

\[ \rightarrow \text{why bps from Hz?} \]
Increasingly important for modern day networking:

- Power control (e.g., pocket PCs)
  - trade-off w.r.t. multi-user interference
  - trade-off w.r.t. battery power
- Recent trend: software radio
  - hardware-to-software migration

Signal-to-noise ratio (SNR) is expressed as

\[ \text{dB} = 10 \log_{10}(P_S/P_N). \]
Example: Assuming a decibel level of 30, what is the channel capacity of a telephone line?

Answer: First, \( W = 3000 \ \text{Hz} \), \( P_S/P_N = 1000 \). Using Channel Coding Theorem,

\[
C = 3000 \log 1001 \approx 30 \ \text{kbps}.
\]

→ compare against 28.8 kbps modems
→ what about 56 kbps modems?
→ DSL lines?

How to increase reliable throughput to approximate channel capacity?