CS590U
Access Control: Theory and Practice

Lecture 14 (February 24)
Basics of Logic and Logic Programming
What is Logic?

- Formulas
- Syntactical approach
  - define how to derive new formulas from existing ones
  - \( \Gamma |\rightarrow \varphi \)
- Semantic approach
  - define when a formula is a logical implication of other formulas
  - \( \Gamma |\models \varphi \)
Example Logic Formulas

From
- $\forall X \forall Y (mother(X) \land child_of(Y,X) \Rightarrow loves(X,Y))$
- mother(mary)
- child_of(tom,mary)

Conclude
- loves(mary,tom)
Kinds of Logic

- Propositional logic
  - classical, intuitionistic
- First order logic (predicate logic)
  - classical, intuitionistic
- Second order logic
- Modal logic
Propositional Logic

- $\land$ AND
- $\lor$ OR $p \lor q$ equivalent with $\neg(\neg p \land \neg q)$
- $\neg$ NOT
- $\rightarrow$ $p \rightarrow q$ equivalent with $\neg p \lor q$
- $\leftrightarrow$ $p \leftrightarrow q$ means $p \rightarrow q \land q \rightarrow p$

Well formed formulas
- a variable is a wff
- $(\varphi \land \psi)$, $(\varphi \lor \psi)$, $(\neg \varphi)$, $(\varphi \rightarrow \psi)$, $(\varphi \leftrightarrow \psi)$ are wff’s
Semantics of Propositional Logic

- A valuation of a formula $\varphi$ is a truth assignment for every variable in $\varphi$
  - one can then evaluate $\varphi$
- A valuation of a formula $\varphi$ is a model of $\varphi$ if $\varphi$ evaluates to true
- $\varphi_1, \ldots, \varphi_n \models \psi$ iff $\psi$ is true in every model of $\varphi_1, \ldots, \varphi_n$
- A formula $\varphi$ is satisfiable iff it has one model
- A formula $\varphi$ is valid iff every valuation of it is a model for it
Conjunctive Normal Form

- Conjunctive Normal Form
- A formula is represented as conjunctions of disjunctions

- Checking validity of formulas in CNF is easy, but checking satisfiability is NP-complete
Horn Clauses

- A formula in CNF where each conjunct has 0 or 1 positive literal
  - $p_1$ a fact
  - $p_1 \lor \neg p_2 \lor \ldots \lor \neg p_n$
    - i.e., $p_1 \iff p_2 \land \ldots \land p_n$ a rule
  - $\neg p_1 \lor \neg p_2 \lor \ldots \lor \neg p_n$
    - i.e., $\iff p_2 \land \ldots \land p_n$ a query

- Satisfiability of a formula in horn clauses can be decided in linear time
Predicate Logic

- An alphabet consists of predicates, constants, and variables
  - in $\forall X \forall Y \ (\text{mother}(X) \land \text{child}_\text{of}(Y,X) \Rightarrow \text{loves}(X,Y))$
  - $\land \text{mother}(\text{mary}) \land \text{child}_\text{of}(\text{tom},\text{mary})$
  - mother, child_of, loves are predicates
  - mary and tom are constants
  - X and Y are variables
  - $\forall$ and $\exists$ are quantifiers

- We ignore function symbols for this lecture
Closed Formulas (Sentences)

- Given a formula $\varphi$, the occurrence of a variable $X$ is **bound** if it is inside the scope of a quantifier $\forall X$ or $\exists X$. Otherwise, the occurrence is **free**.
- A formula with no free occurrences of variables is said to be **closed**.
  - A closed formula is also known as a **sentence**.
- A formula with no variable is said to be **ground**.
A formula is just a string that can be parsed to a syntax tree.

A structure is a domain together with a set of relations (ignoring functions).

An interpretation $I$ of an alphabet maps:
- each constant to an element in the domain.
- each $n$-ary predicate to a relation.
Semantics of Predicate Logic

- Given an interpretation $I$, 

  - $\forall X \varphi$ evaluates to true iff for every element in $I$, $\varphi$ with every free occurrence of $X$ replaced by evaluates to true
  
  - $\exists X \varphi$ evaluates to true iff there exists an element in $I$, $\varphi$ with every free occurrence of $X$ replaced by evaluates to true
  
  - $p(c_1,c_2,...,c_n)$ evaluates to true iff $(c_1,c_2,...,c_n)$ is in the relation that $p$ maps to
Models and Logical Consequence

- Given a set $P$ of closed formulas, an interpretation $I$, $I$ is a model of $P$ iff every formula in $P$ is true in $I$.
- A formula is unsatisfiable if it doesn’t have a model.
- Logical consequence: $\varphi$ is a logical consequence of $P$ iff $\varphi$ is true in every model of $P$, written as $P \models \varphi$.
- Proving $P \models \varphi$ may be difficult, one way is to prove that $P \cup \neg \varphi$ is unsatisfiable.
Logical Inference

- Using rules to manipulate formulas to determine whether $\varphi$ follows from $P$.
- Soundness and completeness
Logic Programming

- Rooted in Automated Theorem Proving
  - see an example
- The program consists of clauses
  - how to express grandchild in terms of child
Definite Clauses

- A definite clause has the form
  - $A_0 \leftarrow A_1 \land \ldots \land A_n$ where $n \geq 0$
  - When $n=0$, it is a fact
  - Otherwise, it is a rule

- Logical atoms, literals, clauses
- A define program is a finite set of definite clauses
- A definite goal has the form $\leftarrow A_1 \land \ldots \land A_n$
- The programmer has an intended model; the program describes features of the model. The programmer wants to know properties of the intended model
- The evaluation engine must be sound
The Least Herbrand Model

- Given an alphabet A, the Herbrand universe consists of all ground terms that can be constructed using symbols from A
  - when A doesn’t contain any function symbols, the Herbrand universe is simply the set of all constants in A
- The Herbrand base consists of all ground atoms over A
Herbrand Model

- Herbrand interpretation
  - essentially a subset of the Herbrand base, saying which ground atoms are true

- A Herbrand model of a program is a Herbrand interpretation such that every clause is true in it
Why Herbrand Model?

- Theorem: Let P be a definite program and G a definite goal, if \( P \cup \{G\} \) has a model, then \( P \cup \{G\} \) has a Herbrand model.
- Corollary: if \( P \cup \{G\} \) does not have a Herbrand model, then \( P \models \neg G \).
- Thus, one only need to check whether G is false in all Herbrand models of P to determine whether \( \neg G \) is true.
- Theorem: Given two Herbrand model of a definite program, their intersection is also a Herbrand model.
The Least Herbrand Model

- Theorem: There exists a unique least Herbrand model.
- Theorem: The least Herbrand model is the set of all ground atomic logical consequences of the program.
Construction of the Least Herbrand Model

- Using the immediate consequence operator
- The least fixpoint of the immediate consequence operator is the least Herbrand Model.
Next Lecture

- Overview of Trust Management