A Note on Order and Lattices

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In this note, when we say a relation R, we mean a binary relation over a certain non-empty set X, i.e., $R \subseteq X \times X$. When $\langle x, y \rangle \in R$, we also write $R(x, y)$; when $\langle x, y \rangle \notin R$, we also write $\neg R(x, y)$. We use $\rightarrow$ to denote logical implication and $\leftrightarrow$ to denote logical equivalence.

Definition. A relation R is **transitive** if $\forall x \forall y \forall z (R(x, y) \wedge R(y, z) \rightarrow R(x, z))$.

Definition. A relation R is **reflexive** if $\forall x R(x, x)$.

Definition. A relation R is **irreflexive** if $\forall x (\neg R(x, x))$.

Exercise. Give a relation that is neither reflexive nor irreflexive.

Definition. A relation R is **symmetrical** if $\forall x \forall y (R(x, y) \leftrightarrow R(y, x))$.

Definition. A relation R is **asymmetrical** if $\forall x \forall y (R(x, y) \rightarrow \neg R(y, x))$.

Definition. A relation R is **anti-symmetrical** if $\forall x \forall y (R(x, y) \wedge R(y, x) \rightarrow x = y)$.

Classroom Exercise. Prove that an asymmetrical relation is irreflexive.

Classroom Exercise. Draw the relationship of the symmetrical relation, asymmetrical, anti-symmetrical relations. Use a big circle for all relations, and three small circles inside the big one to denote the three kinds of relations.

Exercise. Give an example for each of the case in the graph.

Definition. A **strict partial order** is irreflexive, transitive, and asymmetrical.

Ex1: the less than relation ($<$) over any subset of real numbers.

Ex2: the subset relation ($\subset$) over a set of sets, e.g., the powerset of a fixed set.

Definition. A **partial order** is reflexive, transitive, and anti-symmetric.

Ex1: the less than and equal to relation over any subset of real numbers.

Ex2: the $\subseteq$ relation over a set of sets, e.g., the powerset of a fixed set.

Fact. If R is a partial ordering of a set X, then $R - \{\langle x, x \rangle : x \in X\}$ is a strict partial ordering of X.

Definition. A relation R is a **strict total order** if it is a strict partial order and $\forall xy (x \neq y \rightarrow (R(x, y) \vee R(y, x))$.

Definition. A relation R is a **total order** if it is a partial order and $\forall xy(R(x,y) \vee R(y,x))$.

Notation. We often use $\leq$ to denote a partial order R. When $(x,y) \in R$, we write $x \leq y$ or $y \geq x$. Note that while $\leq$ is often used to denote the numerical less than or equal to relation over numbers, we use $\leq$ to denote other partial orders as well.

Definition. A **poset** (partially ordered set) $\langle X, \leq \rangle$ is a set X with a partial ordering $\leq$.

Definition. Given a poset $\langle X, \leq \rangle$, and $x_1, x_2, y \in X$, y is an **upper bound** of x_1, x_2 if $x_1 \leq y$ and $x_2 \leq y$; y is the **least upper bound (lub)** of x_1, x_2 if y is an upper bound of x_1, x_2 and for every z that is an upper bound of x_1, x_2, $y \leq z$. The lub of x_1, x_2 is often denoted $x_1 \vee x_2$, and is also known as the **join** of x_1, x_1 and the **supremum (sup)** of x_1, x_2. (Note that we overload $\vee$ to denote both logical or and the join operator; its meaning should be clear from context.)

Exercise. Give an example poset such that two elements in it have upper bounds, but no least upper bound.

Exercise. Prove that $x_1 \vee x_2$, if exists, is unique.

Definition. A **upper semilattice** is a poset $\langle X, \leq \rangle$ such that for every $x_1, x_2 \in X$, $x_1 \vee x_2$ exists.

Exercise. Given a poset $\langle X, \leq \rangle$, one can extend naturally the definition of upper bound and least upper bound (i.e. join, supremum, sup) to any subset of X.

Prove that in an upper semilattice $\langle X, \leq \rangle$, for every *finite* non-empty subset X' of X, the sup of X' uniquely exists. **Hint: Use induction.**

Definition. Given a poset $\langle X, \leq \rangle$, and $x_1, x_2, y \in X$, y is a **lower bound** of x_1, x_2 if $x_1 \geq y$ and $x_2 \geq y$; y is the **greatest lower bound (glb)** of x_1, x_2 if y is an lower bound of x_1, x_2 and for every z that is a lower bound of x_1, x_2, $y \geq z$. The glp of x_1, x_2 is often denoted $x_1 \wedge x_2$, and is also known as the **meet** of x_1, x_1 and the infimum (inf) of x_1, x_2.

Given $x_1, x_2 \in X$, $x_1 \wedge x_2$ may not exist; but is unique if exists.

Definition. A **lower semilattice** is a poset $\langle X, \leq \rangle$ such that for every $x_1, x_2 \in X$, $x_1 \wedge x_2$ exists.

Definition. A **lattice** is a poset $\langle X, \leq \rangle$ that is both an upper semilattice and a lower semilattice.

Exercise. Given a lattice $\langle X, \leq \rangle$ and $x_0 \in X$. Let $X' = \{x \in X \mid x \geq x_0\}$. Prove that $\langle X', \leq \rangle$ is a lattice.

Definition. A **complete** upper (resp. lower) semilattice is an upper (resp. lower) semilattice $\langle X, \leq \rangle$ where every sub X' of X (not just finite ones) has a sup (resp. inf). We write these subs and infs as $\bigvee X'$ and $\bigwedge X'$.

A **complete lattice** is poset $\langle X, \leq \rangle$ that is both a complete upper semilattice and a complete lower semilattice.

Fact. A finite lattice is a complete lattice.

Notation. Given a complete lattice $\langle X, \leq \rangle$, we use 1 (or $\top$) to denote the element $\bigvee X$ and 0 (or $\bot$) to denote the element $\bigwedge X$.

Definition. A **monotone function** from a poset $\langle A, \leq_X \rangle$ to a poset $\langle B, \leq_B \rangle$ is a function $f : A \longrightarrow B$ such that $\forall xy(x \leq_A y \rightarrow f(x) \leq_B f(y))$.

Theorem. *(Tarski-Knaster Theorem) Let $\langle X, \leq \rangle$ be a complete lattice and $f : X \longrightarrow X$ be a monotone function. Then f has a fixed point, i.e., there exists $x \in X$ such that $f(x) = x$.*

Proof: Let $A = \{x \in X \mid f(x) \leq x\}$. A is nonempty because $\bigvee X \in A$. Let $a = \bigwedge A$. For every $x \in A$, we know that $f(x) \leq x$. As $a \leq x$ and f is monotone, $f(a) \leq f(x)$. By transitivity, $f(a) \leq x$. Therefore, $f(a)$ is also a lower bound of A. Because a is the greatest lower bound of A, $f(a) \leq a$. Therefore, $a \in A$. Furthermore, by monotonicity, $f(f(a)) \leq f(a)$; thus $f(a) \in A$. As $a = \bigwedge A$, $a \leq f(a)$. Therefore $a = f(a)$.

Classroom Exercise. Show that the fixpoint constructed in the proof above is the least fixpoint of f.