CS590U
Access Control: Theory and Practice

Lecture 14 (February 24)
Basics of Logic and Logic Programming
What is Logic?

- Formulas
- Syntactical approach
  - define how to derive new formulas from existing ones
  - $\Gamma \vdash \phi$
- Semantic approach
  - define when a formula is a logical implication of other formulas
  - $\Gamma \models \phi$
Example Logic Formulas

- From
  - $\forall X \forall Y \ (\text{mother}(X) \land \text{child}_\text{of}(Y,X) \Rightarrow \text{loves}(X,Y))$
  - $\text{mother}(\text{mary})$
  - $\text{child}_\text{of}(\text{tom},\text{mary})$

- Conclude
  - $\text{loves}(\text{mary},\text{tom})$
Kinds of Logic

- Propositional logic
  - classical, intuitionistic
- First order logic (predicate logic)
  - classical, intuitionistic
- Second order logic
- Model logic
Propositional Logic

- $\land$ AND
- $\lor$ OR $p \lor q$ equivalent with $\neg(\neg p \land \neg q)$
- $\neg$ NOT
- $\Rightarrow$ $p \Rightarrow q$ equivalent with $\neg p \lor q$
- $\Leftarrow$ $p \Leftarrow q$ means $p \Rightarrow q \land q \Rightarrow p$

Well formed formulas
- a variable is a wff
- $(\phi \land \psi), (\phi \lor \psi), (\neg \phi), (\phi \Rightarrow \psi), (\phi \Leftarrow \psi)$ are wff’s
Semantics of Propositional Logic

- A valuation of a formula $\varphi$ is a truth assignment for every variable in $\varphi$
  - one can then evaluate $\varphi$
- A valuation of a formula $\varphi$ is a model of $\varphi$ if $\varphi$ evaluates to true
- $\varphi_1, \ldots, \varphi_n \models \psi$ iff $\psi$ is true in every model of $\varphi_1, \ldots, \varphi_n$
- A formula $\varphi$ is satisfiable iff it has one model
- A formula $\varphi$ is valid iff every valuation of it is a model for it
Conjunctive Normal Form

- Conjunctive Normal Form
- A formula is represented as conjunctions of disjunctions

- Checking validity of formulas in CNF is easy, but checking satisfiability is NP-complete
Horn Clauses

- A formula in CNF where each conjunct has 0 or 1 positive literal
  - $p_1$
  - $p_1 \lor \neg p_2 \lor \ldots \lor \neg p_n$
    - i.e., $p_1 \leftarrow p_2 \land \ldots \land p_n$ a rule
  - $\neg p_1 \lor \neg p_2 \lor \ldots \lor \neg p_n$
    - i.e., $\leftarrow p_2 \land \ldots \land p_n$ a query

- Satisfiability of a formula in horn clauses can be decided in linear time
Predicate Logic

- An alphabet consists of predicates, constants, and variables
  - in $\forall X \ \forall Y \ (\text{mother}(X) \land \text{child\_of}(Y,X) \implies \text{loves}(X,Y))$
    - $\land \text{mother}(\text{mary}) \land \text{child\_of}(\text{tom},\text{mary})$
  - mother, child\_of, loves are predicates
  - mary and tom are constants
  - $X$ and $Y$ are variables
  - $\forall$ and $\exists$ are quantifiers
- We ignore function symbols for this lecture
Closed Formulas (Sentences)

- Given a formula $\varphi$, the occurrence of a variable $X$ is bound if it is inside the scope of a quantifier $\forall X$ or $\exists X$. Otherwise, the occurrence is free.
- A formula with no free occurrences of variables is said to be closed.
  - A closed formula is also known as a sentence.
- A formula with no variable is said to be ground.
Semantics of Predicate Logic

- A formula is just a string that can be parsed to a syntax tree.
- A structure is a domain together with a set of relations (ignoring functions).
- An interpretation $I$ of an alphabet maps
  - each constant to an element in the domain
  - each $n$-ary predicate to a relation
Semantics of Predicate Logic

- Given an interpretation $I$,
  - $\forall X \phi$ evaluates to true iff for every element in $I$, $\phi$ with every free occurrence of $X$ replaced by evaluates to true
  - $\exists X \phi$ evaluates to true iff there exists an element in $I$, $\phi$ with every free occurrence of $X$ replaced by evaluates to true
  - $p(c_1,c_2,...,c_n)$ evaluates to true iff $(c_1,c_2,...,c_n)$ is in the relation that $p$ maps to
Models and Logical Consequence

- Given a set $P$ of closed formulas, an interpretation $I$, $I$ is a model of $P$ iff every formula in $P$ is true in $I$.
- A formula is unsatisfiable if it doesn’t have a model.
- Logical consequence: $\varphi$ is a logical consequence of $P$ iff $\varphi$ is true in every model of $P$, written as $P \models \varphi$.
- Proving $P \models \varphi$ may be difficult, one way is to prove that $P \cup \neg \varphi$ is unsatisfiable.
Logical Inference

- Using rules to manipulate formulas to determine whether $\varphi$ follows from P.
- Soundness and completeness
Logic Programming

- Rooted in Automated Theorem Proving
  - see an example
- The program consists of clauses
  - how to express grandchild in terms of child
Definite Clauses

- A definite clause has the form
  - $A_0 \leftarrow A_1 \land \ldots \land A_n$ where $n \geq 0$
  - When $n=0$, it is a fact
  - Otherwise, it is a rule

- Logical atoms, literals, clauses
- A define program is a finite set of definite clauses
- A definite goal has the form $\leftarrow A_1 \land \ldots \land A_n$
- The programmer has an intended model; the program describes features of the model. The programmer wants to know properties of the intended model
- The evaluation engine must be sound
The Least Herbrand Model

- Given an alphabet $A$, the Herbrand universe consists of all ground terms that can be constructed using symbols from $A$.
  - When $A$ doesn’t contain any function symbols, the Herbrand universe is simply the set of all constants in $A$.
- The Herbrand base consists of all ground atoms over $A$. 
Herbrand Model

- Herbrand interpretation
  - essentially a subset of the Herbran base, saying which ground atoms are true

- A Herbrand model of a program is a Herbrand interpretation such that every clause is true in it
Why Herbrand Model?

- Theorem: Let $P$ be a definite program and $G$ a definite goal, if $P \cup \{G\}$ has a model, then $P \cup \{G\}$ has a Herbrand model.
- Corollary: if $P \cup \{G\}$ does not have a Herbrand model, then $P \models \neg G$.
- Thus, one only need to check whether $G$ is false in all Herbrand models of $P$ to determine whether $\neg G$ is true.
- Theorem: Given two Herbrand model of a definite program, their intersection is also a Herbrand model.
The Least Herbrand Model

- Theorem: There exists a unique least Herbrand model.
- Theorem: The least Herbrand model is the set of all ground atomic logical consequences of the program.
Construction of the Least Herbrand Model

- Using the immediate consequence operator
- The least fixedpoint of the immediate consequence operator is the least Herbrand Model.
Next Lecture

- Overview of Trust Management